Revista Colombiana de Matemáticas
Volumen XXVII (1993), págs. 203–208

QUASIVARIETIES OF DE MORGAN ALGEBRAS: RCEP

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ABSTRACT. In this note we prove that there is a least strict quasivariety (i.e., a quasivariety which is not a variety) of De Morgan algebras and that such a quasivariety is perhaps the only strict quasivariety enjoying the relative congruence extension property.

§1. INTRODUCTION

For a quasivariety $\mathcal{Q}$ and an algebra $A \in \mathcal{Q}$, let $Con_{\mathcal{Q}}(A) = \{\Theta \in Con(A) : A/\Theta \in \mathcal{Q}\}$, where $Con(A)$ denotes the set of congruence relations on A. The elements of $Con_{\mathcal{Q}}(A)$ are called $\mathcal{Q}$-congruences on A. A is said to have relative (to $\mathcal{Q}$) congruence extension property (further on RCEP) if for every subalgebra B of A, any $\mathcal{Q}$-congruence on B is the restriction of a $\mathcal{Q}$-congruence on A. $\mathcal{Q}$ has RCEP if all of its elements have this property. The purpose of this note is to prove that there is a least strict quasivariety of De Morgan algebras and such a quasivariety is perhaps the only strict quasivariety enjoying RCEP. For more results in this direction we refer the reader to [3] and [7]. Recall that a De Morgan algebra is an algebra $\langle A; \wedge, \vee, ', 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ such that the reduct $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and the following identities are satisfied:

$$x'' = x \quad ; \quad (x \vee y)' = x' \wedge y'.$$

The lattice of subvarieties of De Morgan algebras is a four-element chain $\mathcal{T} \subset \mathcal{B} \subset \mathcal{K} \subset \mathcal{M}$ where $\mathcal{T}$, $\mathcal{B}$, $\mathcal{K}$, and $\mathcal{M}$ denote respectively the varieties of trivial, Boolean, Kleene and De Morgan algebras. There are three non-trivial subdirectly irreducible De Morgan algebras each of which generates one of the non-trivial varieties above: $\mathcal{B}$ is generated by the two- element chain $\mathbf{2} = \{0, 1\}$, $\mathcal{K}$ is generated by the three element chain $\mathbf{3} = \{0, a, 1\}$ in which $a' = a$ and $\mathcal{M}$

Research supported by the CDCHT (project C-507-91) of the University of the Andes, Mérida, Venezuela .

by the four-element complemented lattice $\mathbf{4} = \{0, a, b, 1\}$ in which $a' = a$ and $b' = b$. $\mathcal{B}$ satisfies the identity $x \wedge x' = 0$ and $\mathcal{K}$ sastisfies $x \wedge x' \leq y \vee y'$. For a systematic study of $\mathcal{M}$ see [1].

The strategy of the paper is based on the analysis of the two possible cases concerning the variety generated by the strict quasivariety. Section 2 is devoted to the case in which such a variety is $\mathcal{M}$. It is proved that under this assumption no strict quasivariety has RCEP. In Section 3 it is proved that, except for the quasivariety generated by the four element chain $C_4 = \{0 < a < a' < 1\}$ (which is the least strict quasivariety of De Morgan algebras), no strict quasivariety such that the least variety containing it is $\mathcal{K}$ has RCEP. We still do not know wheter or not $Q(C_4)$ has RCEP. Observe that there are not strict quasivarieties contained in $\mathcal{B}$.

The basic concepts of universal algebra can be found in [2]. We follow the notation of this book, particularly for the operators on classes of algebras. In addition, by $A \leq_{SD} \prod_{i \in I} A_i$ we mean that A is a subdirect product of the family $\{A_i : i \in I\}$. $A \leq B$ means that A is a subalgebra of B. Any class $\mathcal{A}$ of algebras such that $\mathcal{Q}$ is the least quasivariety containing $\mathcal{A}$ is said to generate $\mathcal{Q}$ and in this case we write $\mathcal{Q} = Q(\mathcal{A})$. An algebra $A \in \mathcal{Q}$ is said to be *relatively subdirectly irreducible* or, *$\mathcal{Q}$-subdirectly irreducible*, if it can not be subdirectly embedded in a direct product of algebras of $\mathcal{Q}$ unless the composite of the embedding with one of the projections is an isomorphism. It can be shown that $A \in \mathcal{Q}$ is relatively subdirectly irreducible iff there is a least non-zero $\mathcal{Q}$-congruence on A. Such a congruence is called the *$\mathcal{Q}$-monolith* of A. We denote the class of $\mathcal{Q}$-subdirectly irreducible members of $\mathcal{Q}$ by $\mathcal{Q}_{RSI}$.

§2. Quasivarieties that generate $\mathcal{M}$

In this section we prove the following proposition:

Proposition 2.1. *Let $\mathcal{Q}$ be a strict quasivariety such that $H(\mathcal{Q}) = \mathcal{M}$. Then $\mathcal{Q}$ does not have RCEP.*

We pave the way for the proof of this proposition with two lemmas.

Lemma 2.2. *Let A be a homomorphic image of* $\mathbf{4}$ *such that for all $x \in A$, $x' \neq x$. Then, either the algebra depicted in Figure 1, (i) or the one depicted in 1, (ii) is in $IS(A)$*

Proof. Let $f : A \longrightarrow \mathbf{4}$ be a surjective homomorphism. Fix an element $u \in A$ such that $f(u) \notin \{0, 1\}$. Pick $v \in A$ such that $f(u)' = f(v)$. Let $a = u \wedge u'$ and $b = v \wedge v'$. Clearly, a and b' are not comparable. Let $c = (a \vee b) \wedge a'$ and $d = (a \vee b) \wedge b'$. It is routine to verify the following: c and d are not comparable; $c < c'$; $d < d'$; $c' \vee d = c' \vee d' = c \vee d'$; $c \wedge d = c \wedge d' = c' \wedge d$; $c \vee d < c' \vee d'$; $c \wedge d < c' \wedge d'$. Thus, the subalgebra of A generated by c and d meets the requirements of the lemma.

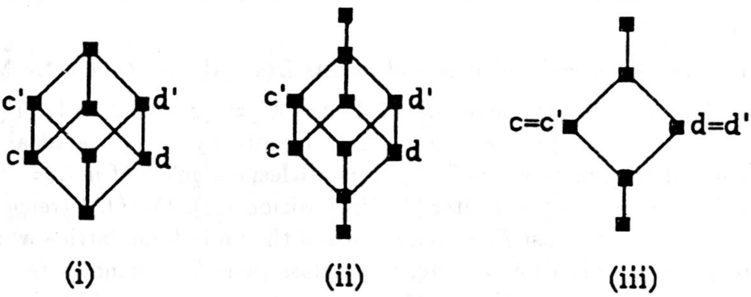

FIGURE 1. $C \leq A$

Lemma 2.3. *Let A be a homorphic image of **4** such that there exists $c \in A$ with $c = c'$. Then either **4** or the algebra depicted in Figure 1, (iii), is isomorphic to a subalgebra of A.*

Proof. Let $f : A \longrightarrow \mathbf{4}$ be a surjective homomorphism. Clearly, $u = f(c) \notin \{0, 1\}$. Let v be the Boolean complement of u. Notice that $u' = u$ and $v' = v$. Fix $b \in A$ such that $f(b) = v$. Put $a = b \wedge b'$ and $d = (c \wedge a') \vee a$. One checks now that $f(d) = v$. Since u and v are not comparable, so are c and d. Using now the hypothesis about c and the fact that $a \leq a'$ one gets $d = d'$. Thus, the subalgebra of A generated by c and d meets the requirements of the lemma. □

Proof of Proposition 2.1. . By Birkhoff's subdirect representation theorem, there exists $A \in \mathcal{Q} - \mathcal{K}$ such that one of its homomorphic images is isomorphic to a subalgebra of **4**. In view of the two previous lemmas, one of the algebras depicted in Figure 1 is in $\mathcal{Q}$. Let us denote such an algebra by C. Let D be the subalgebra of C generated by $t = c \vee d$. The proposition now follows from observing that the $\mathcal{Q}$-congruence on D generated by $(0, t')$ coincides with the congruence on D generated by the same element and this $\mathcal{Q}$-congruence can not be extended to a $\mathcal{Q}$-congruence on C.

§3. QUASIVARIETIES THAT GENERATE $\mathcal{K}$

We start this section proving that there is a least strict quasivariety of De Morgan algebras. Notice that any strict quasivariety contains the variety $\mathcal{B}$ of Boolean algebras.

Proposition 3.1. *C_4 generates the least strict quasivariety of De Morgan algebras.*

Proof. Let $\mathcal{Q}$ be a strict quasivariety of De Morgan algebras. Clearly $\mathcal{B} \subset \mathcal{Q}$, so, there exists $A \in \mathcal{Q}$ and $a \in A$ such that $0 < d = a \wedge a'$. Clearly $d \leq d' = a \vee a' < 1$. If $d < d'$, the subalgebra of A generated by d is isomorphic to C_4, so $C_4 \in \mathcal{Q}$. If $d = d'$, denote by D the subalgebra of A generated by d. As $\mathbf{2} \times D$ has a subalgebra isomorphic to C_4, $C_4 \in \mathcal{Q}$. Hence the proposition is established. □

We now recall some definitions and results from [4]. Let L be a De Morgan algebra. For a non-emty subset X of L, let $X' = \{x' : x \in X\}$. $T(L) \stackrel{def}{=} \{t \in L : t \leq t'\} = \{x \wedge x' : x \in L\}$. Denote by $n(L)$ the ideal of the underlying lattice generated by $T(L)$. L is a Kleene algebra iff $n(L) = T(L)$ iff $T(L)' = \{x \vee x' : x \in L\}$ is a filter [4], Proposition 1.2). $\Theta(n(L))$ (respectively $\Theta(n(L)'))$ denote the least $\mathcal{D}_{0,1}$-congruence of the underlying lattice which has $n(L)$ (respectively $n(L)'$) as a congruence class (here $\mathcal{D}_{0,1}$ denotes the variety of bounded distributive lattices). More precisely, $x \equiv y\Theta(n(L))$ (respectively $\Theta(n(L)')$) iff there exists $j \in n(L)$ (respectively $k \in n(L)'$) such that $x \vee j = y \vee j$ (respectively $x \wedge k = y \wedge k$). Let $\beta(L)$ be the least congruence on the De Morgan algebra L such that the quotient algebra $L/\beta(L)$ is a Boolean algebra. Then $\beta(L) = \Theta(n(L)) \vee \Theta(n(L)')$ [4], Theorem 1.3).

Lemma 3.2. *Let $\mathcal{Q}$ be a strict quasivariety cointained in $\mathcal{K}$. Let $L \in \mathcal{Q}_{RSI}$. Let α be the $\mathcal{Q}$-monolith of L. Then there exist $t, u \in n(L)$ with $t < u$ such that (t, u) generates α.*

Proof. We first claim that one may choose a, b such that $a \equiv b\Theta(n(L))$ or $a \equiv b\Theta(n(L)')$ with the pair (a, b) generating α. To prove the claim, notice that $\beta(L) = \Theta(n(L)) \vee \Theta(n(L)')$ is a $\mathcal{Q}$-congruence, so $\alpha \subseteq \Theta(n(L)) \vee \Theta(n(L)')$. Pick $c, d \in L$ with $c < d$ such that (c, d) generates α. Thus $c \equiv d(\Theta(n(L)) \vee \Theta(n(L)'))$. It follows from this that for some $j \in n(L)$ and $k \in n(L)'$, $(c \vee j) \wedge k = (d \vee j) \wedge k$. If $c \vee j = d \vee j$ then $c \equiv d\Theta(n(L))$. In this case, take $a = c$ and $b = d$. Otherwise, take $a = c \vee j$ and $b = d \vee j$. In this case $a \equiv b\Theta(n(L)')$. This ends the proof of the claim. Assume now that $a \equiv b\Theta(n(L))$. Observe that since L as a lattice is distributive, either $a \wedge a' \neq b \wedge a'$ or $a \wedge b' \neq b \wedge b'$ or $a \wedge a' \neq b \wedge b'$. If $a \wedge b' \neq b \wedge b'$, take $t = a \wedge b' \wedge b$ and $u = (a \wedge b') \vee (b \wedge b')$. From $a \wedge b' \equiv b \wedge b'\Theta(n(L))$ and $b \wedge b' \in n(L)$ it follows that $a \wedge b' \in n(L)$. Then, because $n(L)$ is an ideal, $u, t \in n(L)$. The case $a \wedge a' \neq b \wedge a'$ is taken care of similarly. If $a \wedge a' \neq b \wedge b'$, take $u = (a \wedge a') \wedge (b \wedge b')$ and $t = (a \wedge a') \vee (b \wedge b')$. Whatever the case is, $u, t \in n(L)$ and (u, t) generates α. Finally, if $a \equiv b\Theta(n(L)')$ then $b' \equiv a'\Theta(n(L))$ and α is also generated by (b', a') and in this situation one can argue as above. Now the proof is complete. □

If $A \in \mathcal{Q}$ and $a, b \in A$ then $\Theta_{\mathcal{Q}}^{A}(a, b)$ denotes the least $\mathcal{Q}$-congruence on A which contains the pair (a, b); i.e., the $\mathcal{Q}$-congruence generated by (a, b).

Corollary 3.3. *In the previous lemma, if $\mathcal{Q}$ has RCEP then $t = 0$.*

Proof. Assume $t > 0$. As α is the least non-zero $\mathcal{Q}$-congruence, $t \equiv u\Theta_{\mathcal{Q}}^{L}(0, t)$. Now, since $\mathcal{Q}$ has RCEP, by Proposition 2.4 of [3], $t \equiv u\Theta_{\mathcal{Q}}^{S}(0, t)$ where $S = \{0 < t < u < u' < t' < 1\}$ is the subalgebra of L generated by $\{t, u\}$. Observe next that $\Theta_{\mathcal{Q}}^{S}(0, t) = \Theta^{S}(0, t)$ because $S/\Theta^{S}(0, t) \cong \{0, u, u', 1\} \in \mathcal{Q}$

(Proposition 3.1). But, as it is easily checked, $u \not\equiv t\Theta^S(0,t)$, a contradiction. Then t must be 0. □

Proposition 3.4. *Let $\mathcal{Q} \neq \mathcal{Q}(C_4)$ be a strict quasivariety of Kleene algebras. Then $\mathcal{Q}$ does not have RCEP.*

Proof. Assume on the contrary that $\mathcal{Q}$ has RCEP. Let $L \in \mathcal{Q}_{RSI}$ such that L is not a chain (such an L must exist; otherwise $\mathcal{Q}$ would be the quasivariety generated by C_4) and let α be the $\mathcal{Q}$-monolith of L. By Lemma 3.2 and Corollary 3.3 we may pick $b \in L$ with $b < b'$ such that the pair $(0,b)$ generates α. Pick $a \in L$ non-comparable to b. Now look at the two possibilities: $a \wedge b = c > 0$; $a \wedge b = 0$. In the first one, α is also generated by $(0,c)$ and therefore $b \equiv 0\Theta^L_{\mathcal{Q}}(0,c)$. Since by assumption $\mathcal{Q}$ has RCEP, $b \equiv 0\Theta^S_{\mathcal{Q}}(0,c)$ where $S = \{0 < c < b < b' < c' < 1\}$ is the subalgebra of L generated by $\{c,b\}$. It is evident that $\Theta^S_{\mathcal{Q}}(0,c) = \Theta^S(0,c)$, so $b \not\equiv 0\Theta^S(0,c)$ and this is a contradiction. Let us consider now the possibility $a \wedge b = 0$. Denote by A the subalgebra of L generated by a and b. Without lost of generality we may assume that either $a < a'$ or $a \wedge a' = 0$. All other possibilities about the comparability of a and a' can be reduced to one of these two. Assume first that $a < a'$.

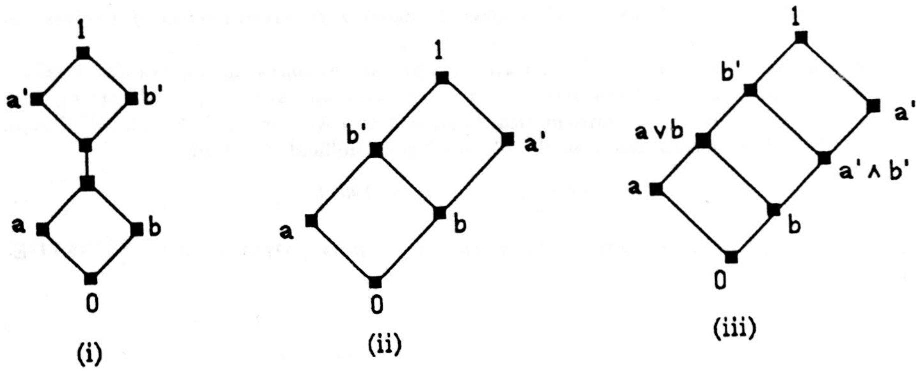

FIGURE 2. $A \leq L$

Observe that $a \vee b < (a \vee b)'$ because $n(L)$ is an ideal ($a \vee b = (a \vee b)'$ is not possible because $\mathcal{Q} \subset \mathcal{K}$). Now, since α is the least non-zero $\mathcal{Q}$-congruence and $\mathcal{Q}$ has RCEP, $b \equiv 0\Theta^A_{\mathcal{Q}}(0,a)$ (see Figure 2, (i)). Next, notice that $A/\Theta^A(0,a) \cong C_4 \in \mathcal{Q}$; so, $\Theta^A_{\mathcal{Q}}(0,a) = \Theta^A(0,a)$ and consequently $b \not\equiv 0\Theta^A(0,a)$ which is a contradiction. Now we consider the possibility $a \wedge a' = 0$. By Corollary 2.5 of [7], $\Theta(x,1) = \Theta_{lat}(x,1)$, $x \in \{a,a'\}$, and these two congruen- ces are complement of each other (see [8], Lemma 3.10). Thus $a' \wedge b > 0$ ($a' \wedge b = 0$ implies $b \equiv 0\Theta(a',1)$). If a' and b are not comparable, we proceed as in the earliest case. If $a' > b$ ($a' \leq b$ is not possible) then A looks like either the algebra depicted in Figure 2, (ii) or the one in 2, (iii). If $a \vee b = b'$ the $\mathcal{Q}$-congruence generated by $(0,a)$ on the subalgebra of A generated by a can not be extended to a $\mathcal{Q}$-congruence on A, which is a contradiction. If $a \vee b < b'$, then $b \equiv 0(\Theta^A_{\mathcal{Q}}(0,a) = \Theta^A(0,a))$, obviously a contradiction. Now the proof of the proposition is complete. □

Question. Does $Q(C_4)$ enjoy RCEP? Proposition 2.9 of [3] can not be used to answer this question in the affirmative because according to Proposition 2.4 of [5], no strict quasivariety of De Morgan algebras is relatively congruence distributive. On the other hand, the method of proof of Fact 2.5 of [3] can not be used to answer it in the negative because $\mathcal{M}$ is not congruence permutable.

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(Recibido en julio de 1991)

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