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IRREDUCIBLE $C(X)$–MODULES ARE ONE–DIMENSIONAL: A BUNDLE–THEORETICAL PROOF

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ABSTRACT. Using the theory of Banach bundles, we present a proof of the well known fact that, when X is a compact Hausdorff space, irreducible $C(X)$-modules are one-dimensional

Let X be a compact Hausdorff space, and let $M \neq 0$ be an irreducible Banach $C(X)$-module, i.e. the only $C(X)$-submodules of M are itself and 0. It is well known that M is one-dimensional; in fact, if M is an irreducible Banach module over the commutative Banach algebra A, then Helemskii remarks [2, p.41] that even in this more general case M is one-dimensional.

We present here as a couriosity what to our knowledge is a new proof of this fact when $A = C(X)$. Our proof relies on some basic facts about $C(X)$-locally convex modules, for which we refer the reader to [3]; function modules, for basic about which see [1]; and the canonical bundle of a $C(X)$-module, for which see [4]. We recall (see [3]) that a $C(X)$-module M is said to be $C(X)$-locally convex iff whenever $a, b \in C(X)$ and $ab = 0$, then

$$\|(a+b)m\| = max\{\|am\|, \|bm\|\} \text{ for all } m \in M.$$

Every $C(X)$-locally convex module M is isometrically $CX)$-isomorphic to the section space $\Gamma(\pi)$ of the canonical bundle $\pi : E \to X$ of M, where the fibers $E_x = \pi^{-1}(\{x\})(x \in X)$ of the bundle are themselves isometrically isomorphic to $\dfrac{M}{I_x M}$ and $I_x M$ is the closed span in M of the set $\{am \mid a \in C(X), a(x) = 0, m \in M\}$. The isomorphism $\wedge : M \to \Gamma(\pi)$, called the Gelfand map, satisfies the equation $\widehat{(am)}(x) = a(x)\widehat{m}(x)$ for all $a \in C(X), m \in M$, and $x \in X$. In particular $\Gamma(\pi)$ may be regarded as a space of functions $\sigma : X \to \dot{\bigcup}\{E_x : x \in X\}$ (where ($\dot{\bigcup}$ denotes disjoint union), such that $\sigma(x) \in E_x$ for each $x \in X$; this function space is a $C(X)$-module under the pointwise operations.

Key words and phrases. $C(X)$-locally convex Banach module, function module, irreducible.

Proposition. *Let X be a compact Hausdorff space, and suppose that $M \neq 0$ is an irreducible Banach module over $C(X)$. Then M is one-dimensional, and there exist an $x \in X$ such that the module multiplication is given by*

$$am = a(x)m$$

for all $a \in C(X)$ and $m \in M$.

Proof. Suppose for a moment that M is $C(X)$-locally convex, i.e., that for all $a, b, \in C(X)$ such that $ab = 0$, we have

$$||(a+b)m|| = max\{||am||, ||bm||\} \text{ for all } m \in M.$$

We let $\pi : E \to X$ be the canonical bundle of M, so that $M \simeq \Gamma(\pi)$ as a module of functions $\sigma : X \to \dot{\bigcup}\{E_x : x \in X\}$. If there exist $x \neq y \in X$ such that $E_x \neq 0 \neq E_y$, then M is not irreducible. For, closed disjoint neighborhoods U and V of x and y, respectively, and functions $a, b \in C(X)$ such that $a(x) = b(y) = 1$ and $a(X \backslash U) = b(X \backslash V) = 0$. If we let M_1 be closure of $\{a\sigma : \sigma \in \Gamma(\pi)\}$ and M_2 be the closure of $\{b\sigma : \sigma \in \Gamma(\pi)\}$, then M_1 and M_2 are submodules, and it is clear that

$$M_1 \neq 0 \neq M_2.$$

Thus, if M is irreducible, there exist $x_0 \in X$ such that $E_x = 0$ iff $x \neq x_0$.

Now, the Banach space E_{x_0} may be made into a $C(X)$-module by setting $az = a(x_0)z$ for all $a \in C(x)$ and $z \in E_{x_0}$. We then have

$$M \simeq \Gamma(\pi) \simeq E_{x_0}$$

as a $C(X)$-modules. In particular, subspaces of E_{x_0} correspond to submodules, so that if dim $E_{x_0} \geq 2$, then $\Gamma(\pi)$ (and hence M) has non-trivial submodules. Hence, if M is $C(X)$-locally convex and irreducible, then M is isomorphic (but not necessarily isometric) to $\mathbb{C}$ as a $C(X)$-module, and the multiplication is given by $am = a(x_0)m$, for $a \in C(X)$ and $m \in M$.

We now show that every irreducible $C(x)$-module M is $C(X)$-locally convex. Note first that if $U, V \subseteq X$are closed, with $U \cup V = X$, then $I_U I_V = 0$, where $I_U = \{a \in C(X) : a(U) = 0\}$. Then at least one of $I_U M$ or $I_V M$ is 0. For, if both

$$I_U M = I_V M = M$$

(the only alternative, since M is irreducible) we then have, say,

$$0 = I_V I_U M = (I_V)^2 M = I_V M = M,$$

an evident contradiction.

Suppose, now, that the irreducible module M is not $C(X)$-locally convex. We may then find $a, b \in C(X)$ and $m \in M$ such that

$$||(a+b)m|| \neq max\{||am||, ||bm||\}$$

and such that $ab = 0$. Letting $U = a^{-1}(\{0\})$ and $V = b^{-1}(\{0\})$, we see that $U \cup V = X$.

Assume that $||(a+b)m|| > max\{||am||, ||bm||\}$. Then $(a+b)m \neq 0$, so that at least one of am and bm is non-zero, say $am \neq 0$; but this forces $I_U M = M$, so that $I_V M = 0$, and thus $bm = 0$. But the the inequality does not hold.

In a similar fashion, suppose that $max\{||am||, ||bm||\} > ||(a+b)m||$. Then again at least one of am and bm is not zero, and the rest of the proof follows as a above.

Hence, M is $C(X)$-locally convex, and it follows that M is one-dimensional. In particular, there exist $x \in X$ such that

$$am = a(x)m \text{ for all } a \in C(X) \text{ and } m \in M .$$

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