Revista Colombiana de Matemáticas
Volumen XXVII (1993), págs.223–230

EXTREME POINTS OF NUMERICAL RANGES OF QUASIHYPONORMAL OPERATORS

JAIME RODRÍGUEZ MONTES

Universidad Nacional de Colombia, Santafé de Bogotá

ABSTRACT. It is shown that a quasihypnormal operator on a Hilbert space having 0 as a boundary point of its numerical range is hyponormal. A necessary and sufficient condition is given for the extreme points of the numerical range of a quasihyponormal operator to be eigenvalues. It is also established that if T is bounded and there is $||x|| = 1$ such that $||Tx|| = ||T||$ and that $< Tx, x >$ is a boundary point of the numerical range of T, then T has eigenvalues. Finally, an example is included of a paranormal operator which is not convexoid and such that $T - \alpha I$ is not paranormal for certain values of α.

§1. BASIC NOTIONS

In what follows, H will be a complex Hilbert space with inner product $<,>$ and norm $|| \; ||$, and $B(H)$ will be the algebra of bounded linear operators on H. An operator $T \in B(H)$ is said to be in the class $C(N,k)$ (see [3]) if

$$||Tx||^k \leq ||T^k x||, \tag{1-1}$$

for all $x \in H$ with $||x|| = 1$. $C(N,2)$ will be the class of *paranormal* operators, a class that generalizes that of *quasihyponormal* operators, characterized by

$$||T^*Tx|| \leq ||T^2 x||, \; x \in H, \tag{1-2}$$

where T^* is the adjoint operator of T (i.e.,$< T^*x, y > = < x, Ty >$, for all x, y in H). The latter class contains in its own turn the class of hyponormal operators, determined by

$$||T^*x|| \leq ||Tx||, \; x \in H. \tag{1-3}$$

AMS Subject classification. Primary 47B20, Secondary 47A12..

Key words and phrases. Hilbert space; bounded operator; normal, paranormal, hyponormal, quasihyponormal operator; adjoint of an operator; self–adjoint operator; eigenvalue; spectrum of an operator..

If $T \in B(H)$ then

$$T = T_1 + iT_2, \quad T_1 = \frac{1}{2}(T + T^*), \quad T_2 = \frac{1}{2i}(T - T^*) \tag{1-4}$$

We observe that both T_1 and T_2 are self-adjoint operators on H, i.e., $T_i = T_i^*$, $i = 1, 2$. Representation $T = T_1 + iT_2$ is called the cartesian representation of T. We have

$$< Tx, x >=< T_1x, x > +i < T_2x, x >, \qquad x \in H, \tag{1-5}$$

and $< T_ix, x >$, $i = 1, 2$, is always a real number.

The *numerical range* of an operator T is the set

$$W(T) = \{< Tx, x >: x \in H, \ \|x\| = 1\}, \tag{1-6}$$

and $\partial W(T)$ will be its boundary in $\mathbb{C}$, the field of complex numbers. With $\mathbb{R}$ we will denote the set of real numbers. As usual, $\sigma(T)$, $\sigma_c(T)$, $\sigma_p(T)$ *and* $\sigma_r(T)$ will denote, respectively, the *spectrum* and the *continuous*, *point* and *residual spectra*, and $\rho(T)$ will stand for the *resolvent*.

§2. Main results

J. G. Stampfly ([6],Theorem 1) has shown that for a hyponormal operator T, the extreme points in W(T) belong to $\sigma_p(T)$. This is well known for normal operators and trivial for $T = \alpha I + \beta T'$, α, $\beta \in \mathbb{C}$ and T' self-adjoint.

In this section we will give necessary and sufficient conditions for the non zero extreme points of the numerical range of a quasihyponormal operator to have the same property (Theorem 2.2). We will also prove that $0 \in \operatorname{Int} W(T)$, the interior in $\mathbb{C}$ of $W(T)$, whenever T is quasihyponormal and not hyponormal (Corollary 2.4).

Theorem 2.1. *If $T \in B(H)$ and $< Ty, y >\in \partial W(T) \cap W(T)$, then*

$$\begin{aligned} | < (T- < Ty, y >)x, y > |^2 = | < (T_1- < T_1y, y >)x, y > |^2 \\ + | < (T_2- < T_2y, y >)x, y > |^2 , \end{aligned} \tag{2-1}$$

for all $x \in H$.

Proof. Arguing as in the proof of [5] Theorem 2.1 we see that since $< Ty, y >$ is not interior to $W(T)$, $J_T(x + y, y) = 0$ for all x linearly independent of y. Hence (2.1) follows at once from [5], 2.17. □

Corollary 2.1. *If $T \in B(H)$ and $< Ty, y > \in \partial W(T) \cap W(T)$, there are $\alpha,\ \theta \in \mathbb{R}$ such that*

$$e^{i\theta} Ty = \alpha\, y + iT_2' y, \quad T_2' = \sin\theta\, T_1 + \cos\theta\, T_2; \tag{2-2}$$

i.e., if, $T_1' = \cos\theta\, T_1 - \sin\theta\, T_2$, then $T_1'\, y = \alpha\, y$.

Proof. The assertion is clear if y is an eigenvalue of either T_1 or T_2. If that is not the case, it follows, from Theorem 2.1, and from [5], 2.19, Lemma 2.6, that for some $\beta \in \mathbb{R}$

$$T_1 y - < T_1 y, y > y = \beta(T_2 y - < T_2 y, y > y),$$

and choosing θ such that $\beta = \tan\theta$ and $\alpha = < T_1 y, y > \cos\theta - < T_2 y, y > sin\theta$, the assertion follows.

Corollary 2.2. *If T is hyponormal and $W(T) \cap \partial W(T) \neq \varnothing$, there is a closed subspace M of H reducing T and such that $T|M$ is normal.*

Proof. Since hypnormality is invariant under scalar multiplication, we can assume, in view of Corollary 2.1, that, for $y \in H$ with $||y|| = 1$ such that $< Ty, y > \in \partial W(T)$, we have $Ty = \alpha\, y + iT_2\, y$, $\alpha \in \mathbb{R}$. Then $T^* y = \alpha\, y - iT_2\, y$, and if M is the subspace of $x \in H$ such that $Tx = \alpha\, x + iT_2\, x$, then $||Tx|| = ||T^* x||$, $x \in M$. Since T is hyponormal (so that $T^*T - TT^* \geq 0$) and $< (T^*T - TT^*)x, x > = 0$ for $x \in M$, we have that $T^*Tx = TT^*x$, $x \in M$, so that M reduces T and $T|M$ is normal.

Proposition 2.1, next, is a result of D. K. Rao [4]. We make some corrections to their proof.

Proposition 2.1. *Let $T \in B(H)$, $M = \overline{R(T)}$, $N = N(T^*)$, $A = T \mid M$, $B = T|N$. Then T is quasihyponormal if and only if*

$$||A^* x||^2 + ||B^* x||^2 \leq ||Ax||^2 \qquad x \in M. \tag{2-3}$$

Under such circunstances, A is a bounded hyponormal operator on M.

Proof. T is quasihyponormal if and only if $||T^*Tz|| \leq ||T^2 z||$ for all $z \in H$, i.e., if and only if $||T^* x|| \leq ||Tx||$ for all $x \in M$. Since $H = M \oplus N$ and $Tx = Ax$, $T^* x = A^* x \oplus B^* x$ for $x \in M$, it follows that $||T^* x||^2 = ||A^* x||^2 + ||B^* x||^2 \leq ||Tx||^2 = ||Ax||^2$, $x \in M$, which implies (2.3) and, in particular, that A is hyponormal.

With the same notations as in Proposition 2.1, we have

Theorem 2.2. *If T on H is quasihyponormal and $\lambda = < T(x \oplus y), x \oplus y > \neq 0$, $x \in M$, $y \in N$, is an extreme point of $W(T)$, the following assertions are equivalent:*

(1) $< By, x > = 0$.
(2) $\Re e < Ax, By > \geq 0$.
(3) *λ is an eingenvalue of T with $x \oplus y$ as an eigenvector.*

Proof. Assume either $< By, x >= 0$ or $\Re e < Ax, By >\geq 0$. If $\theta \in \mathbb{R}$ then

$$e^{i\theta}T(x \oplus y) = e^{i\theta}Ax + e^{i\theta}By. \tag{2-4}$$

Hence

$$< e^{i\theta}By, x > = 0 \qquad or \qquad \Re e < e^{i\theta}Ax,\ e^{i\theta}By >\geq 0. \tag{2-5}$$

Because of the rotation invariance of quasihyponormality, we may assume, in view of Corollary 2.1, that

$$T(x \oplus y) = \alpha(x \oplus y) + iT_2(x \oplus y),\ T^*(x \oplus y) = \alpha(x \oplus y) - iT_2(x \oplus y) \tag{2-6}$$

with $\alpha \neq 0$ in $\mathbb{R}$. Also

$$T_1(x \oplus y) = \alpha(x \oplus y), \tag{2-7}$$

Relation (2.7) yields

$$\alpha(x \oplus y) = \frac{1}{2}(Ax + By + A^*x) \oplus \frac{1}{2}B^*x \tag{2-8}$$

so that $B^*x = 2\alpha y$, and thus

$$2\alpha||y||^2 =< B^*x, y > . \tag{2-9}$$

From (2.6),

$$||T(x \oplus y)|| = ||T^*(x \oplus y)|| \tag{2-10}$$

which, together with (2.3), gives

$$\begin{aligned} ||Ax||^2 + ||By||^2 + 2\Re e < Ax, By > &= ||T(x \oplus y||^2 \\ &= ||T^*(x \oplus y)||^2 = ||A^*x||^2 + ||B^*y||^2 \\ &\leq ||Ax||^2 \end{aligned} \tag{2-11}$$

From (2.9) and (2.11) it follows that $y = 0$ provided that $< By, x >= 0$ or $\Re e < Ax, By >\geq 0$, and from Corollary 2.2 and the hyponormality of $A = T|M$, that $M' = \{z \in M : Tz = \alpha z + iT_2 z\}$, which is different from $\{0\}$ as $x \in M'$, reduces A. Also, $S = A|M'$ is normal. Since $W(S) \subseteq W(T)$, $\lambda =< Tx, x >$ is an extreme point of W(S), and thus (as observed at the beginning of this section) an eigenvalue of S, and hence of T, with x an eigenvector.

The proof of the converse is trivial.

Corollary 2.3. *([6]) If T is hyponormal and $\lambda =< Tx, x >$ is an extreme point of $W(T)$, λ is an eigenvalue of T.*

Proof. Since T is hyponormal if and only if $T - \alpha I$ is hyponormal for all $\alpha \in \mathbb{C}$, the assertion follows from proposition 2.1 and Theorem 2.2 applied to $T - \alpha I$, $\alpha \in \rho(T)$, $\alpha \neq \lambda$, instead of T, in which case $B = 0$.

Remark 2.1. If T is quasihyponormal and not hyponormal, then $0 \in IntW(T)$, the interior in $\mathbb{C}$ of $W(T)$. This follows from

Lemma 2.1. *If $T \in B(H)$ is such that $N(T) \neq N(T^*)$, then $0 \in$ Int $W(T)$.*

Proof. With no loss of generality we may assume that there is $y \in H$ with $\|y\| = 1$ and $T^*y = 0$, $Ty \neq 0$. Then $0 = < Ty, y > \in W(T)$, and since $\|Ty\| \neq \|T^*y\|, 0 \in \text{Int } W(T)$ ([5], Corollary 2.1).

Corollary 2.4. *If T in $B(H)$ is quasihyponormal and not hyponormal, then $0 \in \text{Int}W(T)$.*

Proof. The assumption and Proposition 2.1 ensure the existence of y in $N(T^*)$ such that $By \neq 0$. Hence $N(T) \neq N(T^*)$, and the assertion follows from Lemma 2.1.

From Corollary 2.4 it follows at once that

Corollary 2.5. *If T in $B(H)$ is quasihyponormal and $0 \in \partial W(T)$, then T is hyponormal.*

§3. Some additional results

We examine for arbitrary T in $B(H)$ some consequences of the existence of elements $x \in H$ with $\|x\| = 1$ such that $< Tx, x > \in \partial W(T)$ and $\|Tx\| = \|T\|$. For example, for an operator $T = \alpha I + \beta T_1$, α, $\beta \in \mathbb{C}$ and T_1 self-adjoint, we have $\partial W(T) = \overline{W(T)}$. Then it follows for such an operator that $\sigma_p(T) \neq \varnothing$ whenever $\|Tx\| = |T\|$ for some $x \in H$ with $\|x\| = 1$.

The following lemma is well known (see [1], p.9).

Lemma 3.1. *If $T \in B(H)$ and there is $x \in H$ with $\|x\| = 1$ and $\|Tx\| = \|T\|$ then*

$$T^*Tx = \|T\|^2 x \tag{3-1}$$

Proof. We may assume that $\|T\| = 1$. From $< (I - T^*T)y, y > = \|y\|^2 - \|Ty\|^2 \geq 0$ for all $y \in H$ we get that $I - T^*T \geq 0$. Hence, from $< (I - T^*T)x, x > = \|x\|^2 - \|Tx\|^2, T^*Tx = x$, and the assertion follows.

Corollary 3.1. *If T belongs to $C(N, k)$, $k \geq 2$, and there is $x \in H$ with $\|x\| \doteq 1$ and $\|Tx\| = \|T\|$, then T has an invariant proper subspace.*

Proof. We may assume that $\|T\| = 1$. From Lemma 3.1, $T^*Tx = x$. Let $M = \{y \in H : T^*Ty = y\}$. Then M is a closed subspace of H and $\|Ty\| = \|y\|$ for all $y \in M$ with $\|y\| = 1$. From $1 = \|Ty\| = \|Ty\|^k \leq \|T^2y\| \leq 1$ it follows that $\|T(Ty)\| = \|Ty\| = \|T\|$, and from Lemma 3.1 that $T^*T(Ty) = Ty$. Hence $T(M) \subseteq M$, and we can assume that $M \neq H$, as $M = H$ implies that T is isometric, in which case T has a proper invariant subspace.

Remark 3.1. Corollary 3.1 generalizes a result of Stampfly [7] for subnormal operators.

Theorem 3.1. *Assume for T in $B(H)$ there is $x \in H$ such that $\|x\| = 1$, $\|Tx\| = \|T\|$ and $< Tx, x > \in \partial W(T)$. Then, there are α, β, μ in $\mathbb{C}$ with $|\mu| = \|T\|$, such that*

$$T(\alpha x + \beta Tx) = \mu(\alpha x + \beta Tx). \tag{3-2}$$

Proof. We may assume $\|T\| = 1$ and that (Lemma 2.1)

$$Tx = \alpha x + iT_2 x, \qquad T^* x = \alpha x - iT_2 x \tag{3-3}$$

with $\alpha \in \mathbb{R}$ and T_2 self-adjoint. Then

$$\|Tx\| = \|T^* x\| = \|T\| \tag{3-4}$$

and (3.4) and Lemma 3.1 imply that

$$T^* Tx = TT^* x = x \tag{3-5}$$

Let M be the subspace of H spanned by x, Tx, i.e., by $x, T_2 x$. From (3.3) and (3.4)

$$TT_2 x = T_2 Tx, \qquad T^* T_2 x = T_2 T^* x, \tag{3-6}$$

so that M reduces T. Relations (3.3), (3.5) and (3.6) also imply that

$$x = \alpha Tx - iT_2 Tx, \tag{3-7}$$

that

$$< x, Tx >= \alpha - i < T_2 x, x >, \tag{3-8}$$

and that

$$< x, Tx >= \alpha - i < T_2 Tx, Tx > . \tag{3-9}$$

Then

$$< T_2 x, x >=< T_2 Tx, Tx > . \tag{3-10}$$

Since

$$T(\alpha_1 x + \beta_1 T_2 x) = \alpha_1 Tx + \beta_1 T_2 Tx, \quad \alpha_1, \beta_1 \in \mathbb{C}, \tag{3-11}$$

(3.10) ensures that

$$\|T(\alpha_1 x + \beta_1 T_2 x)\| = \|\alpha_1 x + \beta_1 T_2 x\| \tag{3-12}$$

so that $T|M$ is unitary. Since M has *dimension* ≤ 2, (3.2) follows at once.

Corollary 3.2. *Let T in $B(H)$ be self-adjoint, and assume there is x in H with $\|x\| = 1$ and $\|Tx\| = \|T\|$. Then, the following alternative holds:*

(1) *x is an eigenvector of T for an eigenvalue μ with $|\mu| = \|T\|$, or*
(2) *$x + Tx$ and $x - Tx$ are eigenvectors of T for the eigenvalues $\|T\|$ and $-\|T\|$, respectively.*

Proof. Since $\partial W(T) = \overline{W(T)}$, Theorem 3.1 ensures that if x is not an eigenvector of T then $dim(M) = 2$. The assertion then follows from (3.11) and from $T^2x = \|T\|^2x$.

By the way, we also obtain that

Corollary 3.3. *If T in $B(H)$ is positive and there is $x \in H$ with $\|x\| = 1$ and $\|Tx\| = \|T\|$, then x is an eigenvector of T for the eigenvalue $\|T\|$.*

Proof. This follows from Corollary 3.2, recalling that all eigenvalues of T are non-negative.

The above corollary is well known, but the above proof is simpler than the usual ones.

§4. A COUNTEREXAMPLE

We end up giving an example which answers in the negative a question of Stampfly [8] regarding whether the paranormality of T ensures the paranormality of $T - \alpha I$ for all $\alpha \in \mathbb{C}$. We mention that this question does not seem to be clearly answered in [4].

Let $M = l^2, N = \mathbb{C}$ and $A = S + I$ where

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots). \tag{4-1}$$

If $B: N \to M$ is given by

$$By = (y, 0, 0, \dots), \tag{4-2}$$

let $H = M \oplus N$ and $T(x \oplus y) = Ax + By,\ x \in M,\ y \in N$, as in Proposition 2.1. Then T is quasihyponormal; hence, paranormal. Given $y \in N$ with $\|y\| = 1$, condition

$$\|(T - \alpha)y\|^2 \le \|(T - \alpha)^2 y\| \tag{4-3}$$

is equivalent to

$$1 + |\alpha|^2 \le \sqrt{\|(A - 2\alpha)By\|^2 + |\alpha|^4}, \tag{4-4}$$

which does not hold for $\alpha = 1/2$. Thus, $T - \frac{1}{2}I$ is not paranormal. Furthermore, since $N(T) = \{0\} \neq N(T^*) = \mathbb{C}$, Lemma 2.1 above implies that $0 \in Int(W(T))$. On the other hand,

$$Co(\sigma(T)) \subseteq Co(\sigma(A) \cup \{0\}) = \{z \in \mathbb{C} \mid |z - 1| \le 1\},$$

as follows from Proposition 2.2 in [4] and from $\sigma(S) = \{z \in \mathbb{C} \mid |z| \leq 1\}$. This shows that T is not convexoid and answers in the negative a question of Furuta [2] as to the convexoid character of paranormal operators. We mention that the example used in [4] to respond this question does not seem to be correctly handled.

We finally observe that if T is quasihyponormal and not hyponormal, then $\overline{T(H)} \neq H$, as follows readily from (1.2) and (1.3). Hence, if $\alpha \in \rho(T), T - \alpha I$ is not quasihyponormal.

Acknowledgments. I thank Professors J. Charris and A. Takahashi for their careful reading of the manuscript and for suggestions that were helpful for a better presentation of the material.

References

1. Fillmore, P.A., *Notes on operator theory*, Van Nostrand, New York, 1970.
2. Furula T., *On the class of paranormal operators*, Proc.Jap.Acad. **43** (1967), 594–598.
3. Istratescu, I., Istratescu, V., *On some classes of operators, I.*, Proc.Jap.Acad. **43** (1967).
4. Rao, D.K., *Operadores paranormales*, Rev. Colombiana Mat. **21** (1987), 135–140.
5. Rodríguez, M.J., *On two properties of the numerical range of bounded Hilbert space operators*, Pre–print.
6. Stampfly, J.G., *Extreme points of the numerical range of a hyponormal operator*, Michigan Math.J. **13** (1966).
7. Stampfly, J.G, *Hyponormal operators and spectral density*, Trans.Amer.Math.Soc. **117** (1965), 469–476.
8. Stampfly, J.G, *On hyponormal and Toeplitz operators*, Math.Ann. **183** (1969), 328–336.
9. Taylor, A.E. and Lay, D.C., *An introduction to functional analysis*, Wiley, N.Y., 1980.

(*Recibido en agosto de 1992, la versión revisada en junio de 1993*)

DEPARTAMENTO DE MATEMÁTICAS Y ESTADÍSTICA, UNIVERSIDAD NACIONAL DE COLOMBIA, SANTAFÉ DE BOGOTÁ – COLOMBIA