

SOME RESULTS ON FIXED AND COINCIDENCE POINTS FOR PAIRS OF MAPS IN METRIC SPACES

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ABSTRACT. Following ideas of Dugundji and Granas, some general results are established on the existence of fixed and coincidence points for a pair of maps on metric spaces. Conditions on the continuity or the commutativity of the maps are expressly avoided.

§1. INTRODUCTION

Dugundji and Granas [1] present many useful and interesting results on fixed points of maps on metric spaces. We will follow in this paper ideas of these authors to establish some further results on fixed and coincidence points for pairs of maps. Since no assumptions are made on the continuity or the commutativity of the maps in the pair, our work extends previous results in [1], [2] and [3].

In what follows (X, ρ) and (Y, d) will be metric spaces and f, g will be maps of X into Y . We will say that

- (i) f, g satisfy condition (R), if there is a sequence (x_n) in X such that $g(x_n) = f(x_{n+1})$, $n \geq 1$.
- (ii) f, g satisfy condition (C), if for any sequence (x_n) in X and all $y \in Y$ such that $\lim f(x_n) = \lim g(x_n) = y$, we have that $y \in f(X)$.

When $X = Y$, we will also say that

- (iii) f, g satisfy condition (C,C), if $f(x) = g(x)$ holds for any x in X for which there is a sequence (x_n) in X with $\lim f(x_n) = \lim g(x_n) = x$.

Remark 1.1. We observe that condition (R) and (C) automatically holds if $g(X) \subseteq f(X)$ and $f(X)$ is closed in Y .

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§2. RESULTS

Theorem 2.1. Let f, g be as above and assume that (Y, d) is complete. Further assume that

- (2.1) Given $\epsilon > 0$ there is $\delta(\epsilon) > 0$, $\delta \leq \epsilon$, such that $d(g(y), f(x)) < \epsilon$ whenever $d(g(x), f(x)) < \delta$ and $d(f(y), f(x)) < \epsilon$.
 (2.2) There is a sequence (x_n) in X with $g(x_n) = f(x_{n+1})$ and such that for any $\delta > 0$ there is $N \in \mathbb{N}$ with $d(f(x_N), g(x_N)) < \delta$.

Then, there is $y \in Y$ such that

$$\lim f(x_n) = \lim g(x_n) = y \quad (2.3)$$

Proof. For $\epsilon > 0$ let $\delta > 0$ be as in (2.1), and let (x_n) be a sequence as in (2.2). Then $d(g(x_N), f(x_N)) < \delta$ for some $N \in \mathbb{N}$, and from $g(x_n) = f(x_{n+1})$, also $d(f(x_{N+1}), f(x_N)) < \delta \leq \epsilon$. Thus, again from (2.1) $d(g(x_{N+1}), f(x_N)) < \epsilon$, and therefore $d(f(x_{N+2}), f(x_N)) < \epsilon$. Continuing this way we conclude that $d(f(x_{N+p}), f(x_N)) < \epsilon$, $p = 1, 2, \dots$ which implies that $(f(x_n))$ is a Cauchy sequence. Completeness of (Y, d) then ensures that (2.3) holds for some $y \in Y$.

Corollary 2.1. Under the assumptions of the theorem, and if f, g satisfy condition (C), then $f(x_0) = g(x_0)$ for some x_0 in X .

Proof. Let (x_n) in X and $y \in Y$ be such that (2.2) and (2.3) hold.

By condition (C), there is x_0 in X such that $y = f(x_0)$. For $\epsilon > 0$ choose $\delta > 0$ as in (2.1), and let $N > 0$ be such that $d(g(x_n), f(x_n)) < \delta$ and $d(f(x_0), f(x_n)) < \epsilon$ for $n \geq N$. Then $d(g(x_0), f(x_n)) < \epsilon$ for $n \geq N$. Hence $f(x_0) = g(x_0)$.

For $x \in X$ and $\epsilon > 0$ let $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

Corollary 2.2. (See [1]) Let (X, d) be a complete metric space and let g be a map of X into itself. Assume that

- (2.4) For any $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such $g(B(x, \epsilon)) \subseteq B(x, \epsilon)$ whenever $d(g(x), x) < \delta$.

- (2.5) There is $x_0 \in X$ such that $d(g^n(x_0), g^{n+1}(x_0)) \rightarrow 0$ as $n \rightarrow \infty$.

Then, $g^n(x_0) \rightarrow x_0$ and $g(x_0) = x_0$.

Proof. With $X = Y$, $f = I$ (the identity map of X) and $x_n = g^n(x_0)$, the other assumptions of Corollary 2.1 are easily verified.

Let $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t > 0\}$.

Theorem 2.2. Assume (Y, d) is complete and let f, g be maps of (X, ρ) into (Y, d) . Also assume that f, g satisfy conditions (R) and (C), and that

- (2.6) There is a monotonic non-decreasing function $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ such that $\varphi^n(t) \rightarrow 0$ as $n \rightarrow \infty$ and that

$$d(g(x), g(y)) \leq \varphi(d(f(x), f(y)))$$

for $x, y \in X$.

Then, $f(x_0) = g(x_0)$ for some x_0 in X .

Proof. From the monotonicity of φ and from $\varphi^n(t) \rightarrow 0$ for all $t > 0$, it follows that $\varphi(t) < t$, $t > 0$.

Let $\epsilon > 0$ and let $\delta(\epsilon) = \epsilon - \varphi(\epsilon)$. If $d(f(x), g(x)) < \delta(\epsilon)$ and $d(f(y), f(x)) < \epsilon$, then $d(g(y), f(x)) \leq d(g(y), g(x)) + d(g(x), f(x)) \leq \varphi(d(f(y), f(x))) + \delta(\epsilon) < \varphi(\epsilon) + \delta(\epsilon) = \epsilon$, which ensures that assumption (2.1) of Theorem 2.1 holds. Also, from (2.6), and with (x_n) given by condition (R), we see that

$$d(g(x_n), g(x_{n+1})) \leq \varphi(d(f(x_n), f(x_{n+1}))) = \varphi(d(g(x_{n-1}), g(x_n))),$$

which after iteration yields

$$d(g(x_n), g(x_{n+1})) \leq \varphi^n(d(g(x_0), g(x_1))).$$

Since $\varphi^n(d(g(x_0), g(x_1))) \rightarrow 0$ as $n \rightarrow \infty$, also assumption (2.2) of Theorem 2.1 holds, and recalling that condition (C) is assumed, the conclusion then follows from Corollary 2.1.

Remark 2.1. Under assumption (2.6) of Theorem 2.2, $f(x_0) = g(x_0)$ and $f(x'_0) = g(x'_0)$ imply $f(x_0) = f(x'_0) = g(x_0) = g(x'_0)$. In fact, if we assume $d(f(x_0), f(x'_0)) > 0$ then

$$d(g(x_0), g(x'_0)) \leq \varphi(d(f(x_0), f(x'_0))) < d(f(x_0), f(x'_0)),$$

which is absurd.

Remark 2.2. Suppose granted the assumptions of Theorem 2.2. If for y, y' in Y there are points x_0, x'_0 and sequences $(x_n), (x'_n)$ in X such that

$$(i) \quad g(x_n) = f(x_{n+1}), \quad g(x'_n) = f(x'_{n+1}).$$

$$(ii) \quad \lim f(x_n) = \lim g(x_n) = y = f(x_0), \quad \lim f(x'_n) = \lim g(x'_n) = y' = f(x'_0),$$

then $g(x_0) = f(x_0)$ and $g(x'_0) = f(x'_0)$, as follows from Corollary 2.1. Remark 2.1 then ensures that $f(x_0) = f(x'_0) = g(x_0) = g(x'_0)$.

Corollary 2.3. (See [1]) Let g be a map of a complete metric space (X, d) into itself. Let φ be as in Theorem 2.2, and assume that

$$d(g(x), g(y)) \leq \varphi(d(x, y)) \tag{2-7}$$

for all x, y in X . Then g has a unique fixed point x_0 , and $g^n(x) \rightarrow x_0$ as $n \rightarrow \infty$ for all x in X .

Proof. With $X = Y$, $f = I$, $x \in X$ arbitrary, and $x_n = g^n(x)$, the assertion follows from Remark 2.2.

Corollary 2.4. Let f, g be maps of (X, ρ) into (Y, d) and assume that (Y, d) , is complete. Further assume that there is $0 < \alpha < 1$, such that

$$d(g(x), g(y)) \leq \alpha d(f(x), f(y)), \quad x, y \in X \quad (2-8)$$

and that conditions (R) and (C) hold for f and g . Then $f(x_0) = g(x_0)$ for some x_0 in X .

Proof. Take $\varphi(t) = \alpha t$ in Theorem 2.2.

Theorem 2.3. Let f, g be maps of a complete metric space (X, d) into itself. Assume conditions (2.6) in the Theorem 2.2 holds. Also assume that f, g satisfy conditions (R) and (C,C). Then, f, g have exactly one common fixed point.

Proof. From Theorem 2.2 with $X = Y$ we can ensure the existence of a point \bar{x} and a sequence (x_n) in X such that $\lim f(x_n) = \lim g(x_n) = \bar{x}$, and from condition (C,C), that $f(\bar{x}) = g(\bar{x})$. Now applying condition (C,C) to the sequence $x_n = \bar{x}$, $n \geq 1$, we obtain that

$$f^2(\bar{x}) = g^2(\bar{x}) = g \circ f(\bar{x}) = f \circ g(\bar{x}), \quad (2-9)$$

and if we assume $d(f^2(\bar{x}), f(\bar{x})) > 0$ then

$$d(g \circ f(\bar{x}), g(\bar{x})) \leq \varphi(d(f^2(\bar{x}), f(\bar{x}))) < d(f^2(\bar{x}), f(\bar{x})) = d(g \circ f(\bar{x}), g(\bar{x})),$$

which is absurd. Thus $f^2(\bar{x}) = f(\bar{x})$, and (2.9) then guarantees that $x_0 = f(\bar{x}) = g(\bar{x})$ is a common fixed point of f, g . Now recalling Remark 2.2 we conclude that if x'_0 is another such point then $x'_0 = x_0$, and the proof is complete.

Corollary 2.5. (See [2], [3]) Let f, g be maps of a metric space (X, d) into itself. Assume (2.8) holds and that f, g satisfy conditions (R) and (C,C). Then f, g have exactly one common fixed point.

Proof. Let $\varphi(t) = \alpha t$ in Theorem 2.3.

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