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ERROR ESTIMATES OF A NUMERICAL METHOD FOR A CLASS OF NONLINEAR EVOLUTION EQUATIONS

MIRCEA SOFONEA

Université Blaise Pascal, Clermont-Ferrand

§1. INTRODUCTION

Let H be a real Hilbert space and let X,Y be two orthogonal subspaces of H such that $H = X \oplus Y$. Let $T > 0$, $A \in \mathcal{L}(H)$ and $B : [0,T] \times X \times Y \to H$ be a nonlinear operator. We are interested in the following evolution problem :

$$\dot{y}(t) = A\dot{x}(t) + B(t, x(t), y(t)) \qquad \text{for all} \qquad t \in [0,T] \tag{1.1}$$

$$x(0) = x_0 \quad , \quad y(0) = y_0 \ , \tag{1.2}$$

in which the unknowns are the functions $x : [0,T] \to X$ and $y : [0,T] \to Y$. In (1.1) and everywhere in this paper the dot above represents the derivative with respect to the time variable. Such type of problems arise in the study of quasistatic processes for elastic-viscoplastic materials (see for instance [1],[2] and the example presented in section 3). In this case the unknowns x and y are the small deformation tensor and the stress tensor and (1.1) involves the constitutive law of the material.

Problems of the form (1.1), (1.2) were already studied in papers [3] and [4]. So, the existence and uniqueness of the solution was proved in [3] using a technique based on the equivalence between (1.1),(1.2) and a Cauchy problem for an ordinary differential equation in the product Hilbert space $X \times Y$. In the same paper an explicit Euler method and an internal approximation technique is considered, in order to approximate the solution. A more general existence result is given in [4] using again a Cauchy-Lipschitz technique and monotonicity arguments.

The purpose of this paper is to investigate (1.1), (1.2) by a different method. So, in section 2 we prove an existence and uniqueness result using standard arguments for elliptic equations followed by a fixed point technique (theorem 2.1) and in section 3 we give a concrete example of such type of problems. In section 4 we present a semi-discretisation method and we give an estimation of the error (theorem 4.1). A final algorithm in the study of the problem

(1.1), (1.2) is proposed in section 5 and some numerical results are presented in section 6.

§2. An existence and uniqueness result

Everywhere in this paper we use the following notations: $|\cdot|_H$ the norm on H ; $< \cdot,\cdot >_H$ the inner product of H ; $C^0(0,T,H)$ the space of continuous functions on $[0,T]$ with values in H; $C^1(0,T,H)$ the space of differentiable functions with continuous derivative on $[0,T]$ with values in H ; $|\cdot|_{0,T,H}$ the norm on the space $C^0(0,T,H)$ i.e.

$$|z|_{0,T,H} = \max_{t\in[0,T]} |z(t)|_H \text{ for all } z \in C^0(0,T,H) \ ;$$

$|\cdot|_{1,T,H}$ the norm on the space $C^1(0,T,H)$ i.e.

$$|z|_{1,T,H} = |z|_{0,T,H} + |\dot{z}|_{0,T,H} \text{ for all } z \in C^1(0,T,H) \ .$$

The spaces $C^1(0,T,X)$ and $C^1(0,T,Y)$ are defined in a similar way.

Let us suppose that A is a positive definite symmetric operator, i.e.

$$\text{there exists } m>0 \text{ such that } < Ax,x >_H \geq m|x|_H^2 \text{ for all } x \in H \tag{2.1}$$

$$< Ax,y >_H = < x,Ay >_H \text{ for all } x,y \in H \ . \tag{2.2}$$

We shall also suppose that B satisfies

$$\begin{cases} \text{there exists } L>0 \text{ such that } |B(t,x_1,y_1)-B(t,x_2,y_2)|_H \\ L(|x_1-x_2|_H + |y_1-y_2|_H) \text{ for all } t\in[0,T],\ x_1,x_2\in X,\ y_1,y_2\in Y \end{cases} \tag{2.3}$$

$$t \mapsto B(t,x,y) : [0,T] \to H \text{ is a continuous function for all } x\in X \text{ and } y \in Y \tag{2.4}$$

and for the initial data we consider the following assumptions:

$$x_0 \in X \ , \ y_0 \in Y \ . \tag{2.5}$$

The main result of this section is the following:

Theorem 2.1. *Let (2.1)-(2.5) hold. Then the problem(1.1), (1.2) has a unique solution $x \in C^1(0,T,X)$, $y \in C^1(0,T,Y)$.*

Proof. Let $\eta \in C^0(0,T,H)$ and let $z_\eta \in C^1(0,T,H)$ be the function defined by

$$z_\eta(t) = \int_0^t \eta(s)ds + z_0 \text{ for all } t \in [0,T] \tag{2.6}$$

where

$$z_0 = y_0 - Ax_0 \ . \tag{2.7}$$

Using standard arguments for elliptic equations we obtain the existence and uniqueness of two functions $x_\eta \in C^1(0,T,X)$, $y_\eta \in C^1(0,T,Y)$ such that

$$y_\eta(t) = Ax_\eta(t) + z_\eta(t) \text{ for all } t \in [0,T]\ . \tag{2.8}$$

Moreover, the function x_η is characterized by the equality

$$a(x_\eta(t), x') + < z_\eta(t), x' >_H = 0 \text{ for all } x' \in X \text{ and } t \in [0,T] \tag{2.9}$$

where a is the bilinear, continuous, symmetric and coercive form on H defined by

$$a(u,v) = < Au, v >_H \quad \text{for all } u, v \in H \tag{2.10}$$

(see (2.1) and (2.2)). Let us also remark that by (2.6)-(2.8) and the orthogonality of the spaces X and Y we obtain

$$x_\eta(0) = x_0\ ,\ y_\eta(0) = y_0 \tag{2.11}$$

and by (2.1), (2.2) it results

$$|x_\eta(t)|_H + |y_\eta(t)|_H \le C|z_\eta(t)|_H \text{ for all } t \in [0,T] \tag{2.12}$$

$$|x_\eta|_{1,T,H} + |y_\eta|_{1,T,H} \le C|z_\eta|_{1,T,H} \text{ for all } j = 0,1 \tag{2.13}$$

where C is a strictly positive constant which depends only on the operator A.

Using now (2.3) and (2.4) we obtain that $t \mapsto B(t, x_\eta(t), y_\eta(t))$ is a continuous function on $[0,T]$ with values in H, hence we can define the operator $\Lambda : C^0(0,T,H) \to C^0(0,T,H)$ in the following way:

$$\Lambda\eta(t) = B(t, x_\eta(t), y_\eta(t)) \text{ for all } \eta \in C^0(0,T,H) \text{ and } t \in [0,T]. \tag{2.14}$$

We shall prove that Λ has a unique fixed point. Indeed, let η_1, $\eta_2 \in C^0(0,T,H)$; using (2.3), (2.12) and (2.6) we get

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_H \le CL \int_0^t |\eta_1(s) - \eta_2(s)|_H ds \text{ for all } t \in [0,T]. \tag{2.15}$$

By recurrence, denoting by Λ^p the powers of the operator Λ, (2.15) implies

$$|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)|_H \le (CL)^p \underbrace{\int_0^t \int_0^s \cdots \int_0^q}_{p\ integrals} |\eta_1(r) - \eta_2(r)|_H dr \ldots ds$$

for all $t \in [0,T]$ and $p \in \mathbb{N}$. It results

$$|\Lambda^p\eta_1 - \Lambda^p\eta_2|_{0,T,H} \le \frac{(CLT)^p}{p!} |\eta_1 - \eta_2|_{0,T,H} \text{ for all } \ p \in \mathbb{N} \tag{2.16}$$

and since $\lim\limits_p \dfrac{(CLT)^p}{p!} = 0$, (2.16) implies that for p large enough the operator Λ^p is a contraction in $C^0(0,T,H)$. Then there exists a unique $\eta^* \in C^0(0,T,H)$ such that $\Lambda^p\eta^* = \eta^*$. Moreover η^* is the unique fixed point of Λ. Using now (2.6), (2.8), (2.11) and (2.14) we obtain that $x_{\eta^*} \in C^1(0,T,X)$, $y_{\eta^*} \in C^1(0,T,Y)$ is a solution of (1.1), (1.2).

The uniqueness part of the theorem follows from the uniqueness of the fixed point of Λ or by standard arguments for evolutions equations.

§3. A CONCRETE EXAMPLE

Concrete evolution problems of the form (1.1), (1.2) arise in the study of quasistatic processes for rate-type elastic-viscoplastic materials. For simplicity we present here only a pure displacement problem in the one dimensional case, referring to [1], [2] for more complexe examples.

Let $\Omega = (0,1)$ and $T > 0$; we consider the following mixed problem:

$$\dot{\sigma}(t,x) = A\dot{\varepsilon}(t,x) + F(\varepsilon(t,x), \sigma(t,x)) \tag{3.1}$$

$$\varepsilon(t,x) = \frac{\partial u}{\partial x}(t,x) \tag{3.2}$$

$$\frac{\partial \sigma}{\partial x}(t,x) = 0 \tag{3.3}$$

for $t \in [0,T]$ and $x \in (0,1)$,

$$u(t,0) = 0 \ , \ u(t,1) = f(t) \text{ for } t \in [0,T] \tag{3.4}$$

$$u(0,x) = u_0(x) \ , \ \sigma(0,x) = \sigma_0 \text{ for } x \in (0,1) \tag{3.5}$$

in which the unknowns u, ε and σ represent the displacement, the deformation and the stress function.

This problem models the time evolution of an elastic-viscoplastic bar which at the moment $t = 0$ occupies the interval $(0,1) \subset \mathbb{R}$ and whose constitutive equation is given by (3.1). In this equation $A > 0$ is the Young modulus and $F : \mathbb{R}^2 \to \mathbb{R}$ is a given function. Concrete examples of constitutive equations of the form (3.1), as well as various mechanical interpretations on this subject, can be found, for instance, in the paper of Cristescu and Suliciu [5], chapter II. Equation (3.2) defines the strain tensor while equation (3.3) represents the equilibrium equation in the absence of body forces. The boundary conditions (3.4) show that the bar is fixed at the extremity $x = 0$ and the displacement at the extremity $x = 1$ is prescribed in time. Finally, the function u_0 and the scalar σ_0 in (3.5) represent the initial data.

In the study of problem (3.1)-(3.5) we consider the following assumptions:

$$\begin{cases} \text{there exists } L > 0 \text{ such that } |F(\varepsilon_1, \sigma_1) - F(\varepsilon_2, \sigma_2)| \\ \leq L(|\varepsilon_1 - \varepsilon_2| + |\sigma_1 - \sigma_2|) \text{ for all } \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2 \in \mathbb{R} \end{cases} \tag{3.6}$$

$$F(0,0) = 0 \tag{3.7}$$

$$f \in C^1(0,T,\mathbb{R}) \tag{3.8}$$

$$u_0 \in H^1(\Omega) \text{ and } u_0(0) = 0 \ , \ u_0(1) = f(0) \ . \tag{3.9}$$

Let us denote by τ the function defined by

$$\tau(t,x) = \varepsilon(t,x) - f(t) \text{ for } t \in [0,T] \text{ and } x \in (0,1). \tag{3.10}$$

In order to eliminate the displacement field from (3.1)-(3.5) we introduce the following notations:

$$H = L^2(\Omega) \ , \ X = \{ \ \frac{dv}{dx} \mid v \in H_0^1(\Omega)\} = \{ \ \theta \in L^2(\Omega) \mid \int_0^1 \theta(x)dx = 0 \ \} \ ,$$

$$Y = \{ \ \theta \in L^2(\Omega) \mid \theta \ = \text{constant a.e in } \Omega \ \}.$$

The following result can be easily obtained:

Lemma 3.1. *Let (3.8) and (3.9) hold; then (u, ε, σ) is a solution of the problem (3.1)-(3.5) having the regularity*

$$u \in C^1(0,T,H^1(\Omega)) \ , \ \varepsilon \in C^1(0,T,L^2(\Omega)) \ , \ \sigma \in C^1(0,T,H^1(\Omega)) \tag{3.11}$$

if and only if

$$\tau \in C^1(0,T,X) \ , \ \sigma \in C^1(0,T,Y) \tag{3.12}$$

$$\dot{\sigma}(t) = A\dot{\tau}(t) + F(\tau(t) + f(t), \sigma(t)) + A\dot{f}(t) \ \text{for all} \ t \in [0,T] \ , \ \text{a.e. in } \Omega \tag{3.13}$$

$$\tau(0) = \frac{du_0}{dx} - f(0) \ , \ \sigma(0) = \sigma_0 \ \text{a.e. in } \Omega \tag{3.14}$$

and u is defined by

$$u(t,x) = \int_0^x \tau(t,x)dx + f(t)x \ \text{for all} \ t \in [0,T] \ \text{and} \ x \in (0,1).$$

Therefore, the study of the mechanical problem (3.1)-(3.5) is equivalent to the study of the problem (3.13), (3.14) which is of the form (1.1), (1.2) and for which theorem 2.1 can be applied. More precisely, we have:

Theorem 3.2. *Under the hypotheses (3.6)-(3.9) the problem (3.13), (3.14) has a unique solution (τ, σ) having the regularity (3.12).*

Proof. The orthogonal decomposition $H = X \oplus Y$ follows by standard arguments; moreover, using (3.6)-(3.8) we can define the operator $B : [0,T] \times X \times Y \to H$ by the equality

$$B(t, \tau, \sigma) = F(\tau + f(t), \sigma) + A\dot{f}(t)$$

and we obtain that (2.3), (2.4) are satisfied. Using now (3.9) it results that the element τ_0 defined by

$$\tau_0 = \frac{du_0}{dx} - f(0)$$

belongs to X. Theorem 3.2 is now a consequence of theorem 2.1.

Remark 3.1. Using lemma 3.1 and theorem 3.1 we obtain that if (3.6)-(3.9) are satisfied then the problem (3.1)-(3.5) has a unique solution (u, ε, σ) having the regulatity (3.11).

We finish this section by proving that problem (3.13), (3.14) can model the *relaxation phenomenon* for viscoelastic materials. For this, let us define the function F by the equality

$$F(\varepsilon, \sigma) = -\lambda(\sigma - G(\varepsilon)) \text{ for all } \varepsilon, \sigma \in \mathbb{R} \tag{3.15}$$

where $\lambda > 0$ and $G : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that $G(0) = 0$. We also take

$$u_0(x) = \alpha x \ , \ f(t) = \alpha \text{ for all } x \in (0,1) \text{ and } t \in [0,T] \tag{3.16}$$

where α is a given constant; using (3.14) we obtain $\tau(0) = 0$ hence the solution of the problem (3.13)-(3.15) is given by

$$\tau(t) = 0 \tag{3.17}$$

$$\dot{\sigma}(t) = -\lambda(\sigma - G(\alpha)) \tag{3.18}$$

for all $t \in [0,T]$, a.e. in Ω,

$$\sigma(0) = \sigma_0 \text{ a.e. in } \Omega. \tag{3.19}$$

Using now (3.10), (3.16)-(3.19) we get

$$\varepsilon(t,x) = \alpha \ , \ \sigma(t,x) = (\sigma_0 - G(\alpha))e^{-\lambda t} + G(\alpha) \tag{3.20}$$

for all $t \in [0,T]$, a.e. in Ω. Therefore, if $\sigma_0 > G(\alpha)$, then the function ε, σ given by (3.20) describe the relaxation phenomenon; indeed, the deformation is constant in time while the stress decreases in time up to the limit $G(\alpha)$, a.e. in the bar.

§4. A NUMERICAL APPROACH

The existence and uniqueness result of theorem 2.1 was obtained in two steps: the study of the elliptic problem (2.6)-(2.9) defined for every $\eta \in C^0(0,T,H)$ and the fixed point property of the operator Λ defined by (2.14). So, in order to obtain a numerical approximation of the solution for the problem (1.1),(1.2), we start by presenting an approximation in space of the problem (2.6)-(2.9).

Let us suppose in the following that (2.1)-(2.5) hold and let X_h be a closed subspace included in X. For every $\eta \in C^0(0,T,H)$ we consider the problem

$$a(x_\eta^h(t), x'_h) + < z_\eta(t), x'_h >_H = 0 \text{ for all } x'_h \in X_h \text{ and } t \in [0,T] \tag{4.1}$$

$$y_\eta^h(t) = Ax_\eta^h(t) + z_\eta(t) \text{ for all } t \in [0,T] \tag{4.2}$$

where z_η is defined by (2.6), (2.7) and a is the bilinear form defined by (2.10). Using standard arguments we obtain that (4.1), (4.2) has a unique solution $x_\eta^h \in C^1(0,T,X_h)$, $y_\eta^h \in C^1(0,T,H)$; moreover

$$|x_\eta^h(t)|_H + |y_\eta^h(t)|_H \leq C|z_\eta(t)|_H \text{ for all } t \in [0,T] \tag{4.3}$$

$$|x_\eta^h|_{j,T,H} + |y_\eta^h|_{j,T,H} \leq C|z_\eta|_{j,T,H} \text{ for all } j = 0,1 \tag{4.4}$$

where C is the strictly positive constant of (2.12), (2.13).

Let us denote by (x_η, y_η) the solution of (2.8), (2.9) and let $S_\eta^h(j,T)$ be the quantities defined by

$$S_\eta^h(0,T) = \sup_{t\in[0,T]} \big(\inf_{x'_h \in X_h} |x_\eta(t) - x'_h|_H \big) \tag{4.5}$$

$$S_\eta^h(1,T) = \sup_{t\in[0,T]} \big(\inf_{x'_h \in X_h} |x_\eta(t) - x' -_h|_H \big) + \sup_{t\in[0,T]} \big(\inf_{x'_h \in X_h} |\dot{x}_\eta(t) - x'_h|_H \big) \, . \tag{4.6}$$

The distance between the couples (x_η^h, y_η^h) and $(x_\eta, y_\eta)$is given by the following result:

Lemma 4.1. *There exists $\tilde{C}$ which depends only on A such that*

$$|x_\eta^h - x_\eta|_{j,T,H} + |y_\eta^h - y_\eta|_{j,T,H} \leq \tilde{C} S_\eta^h(j,T) \textit{ for all } j = 0,1 \, . \tag{4.7}$$

Proof. Using the first Strang lemma (see for instance [6], p.43, or [7], p.186), from (2.9) and (4.1) we get

$$|x_\eta^h(t) - x_\eta(t)|_H C_1 \inf_{x'_h \in X_h} |x_\eta(t) - x'_h|_H \text{ for all } t \in [0,T] \tag{4.8}$$

where $C_1 > 0$ depends only on the operator A. In a similar way, taking the derivative with respect to the time variable in (2.9) and (4.1) it follows

$$|\dot{x}_\eta^h(t) - \dot{x}_\eta(t)|_H C_1 \leq \inf_{x'_h \in X_h} |\dot{x}_\eta(t) - x'_h|_H \text{ for all } t \in [0,T] \, . \tag{4.9}$$

Using now the notations (4.5), (4.6), from (4.8), (4.9) it results

$$|x_\eta^h - x_\eta|_{j,T,H} \leq C_1 S_\eta^h(j,T) \text{ for } j = 0,1. \tag{4.10}$$

Moreover, from (4.2),(2.8) and the continuity of the operator A we obtain

$$|y_\eta^h - y_\eta|_{j,T,H} \leq C_2 |x_\eta^h - x_\eta|_{j,T,H} \text{ for all } j = 0,1 \tag{4.11}$$

where C_2 depends only on A and using (4.10) it results

$$|y_\eta^h - y_\eta|_{j,T,H} \leq C_1 C_2 S_\eta^h(j,T) \text{ for all } j = 0,1. \tag{4.12}$$

The inequality (4.7) is now a consequence of (4.10),(4.12).

We now study the discrete version of the fixed point property of theorem 2.1. As in the continuous case let us now define the operator $\Lambda_h : C^0(0,T,H) \to C^0(0,T,H)$ by the equality:

$$\Lambda_h \eta(t) = B(t, x_\eta^h(t), y_\eta^h(t)) \text{ for all } \eta \in C^0(0,T,H) \text{ and } t \in [0,T]. \tag{4.13}$$

Let $\eta_1,\ \eta_2 \in C^0(0,T,H)$; using (2.3), (4.3) and (2.6) we obtain

$$|\Lambda_h \eta_1(t) - \Lambda_h \eta_2(t)|_H \leq CL \int_0^t |\eta_1(s) - \eta_2(s)|_H ds \text{ for all } t \in [0,T] \tag{4.14}$$

and, by recurrence, denoting by Λ_h^p the powers of the operator Λ_h, we get

$$|\Lambda_h^p \eta_1 - \Lambda_h^p \eta_2|_{0,T,H} \leq \frac{(CLT)^p}{p!} |\eta_1 - \eta_2|_{0,T,H} \text{ for all } p \in \mathbb{N}. \tag{4.15}$$

The inequality (4.15) shows that for p large enough the operator Λ_h^p is a contraction in $C^0(0,T,H)$, hence the operator Λ_h has a unique fixed point $\eta_h^* \in C^0(0,T,H)$.

Now let η^* be the fixed point of the operator Λ defined by (2.14); as it results from the proof of theorem 2.1, the solution (x_{η^*}, y_{η^*}) of (2.6)-(2.9) is the solution of the problem (1.1), (1.2), i.e.

$$x_{\eta^*} = x\ ,\ y_{\eta^*} = y\ . \tag{4.16}$$

For this reason we are interested to examining the distance between the couples $(x_{\eta_h^*}^h, y_{\eta_h^*}^h)$ and (x_{η^*}, y_{η^*}):

Lemma 4.2. *Let C and $\tilde{C}$ be the constants of (2.12) and (4.7) and $K = CLT$. Then*

$$\begin{cases} |x_{\eta_h^*}^h - x_{\eta^*}|_{j,T,H} + |y_{\eta_h^*}^h - y -_{\eta^*}|_{j,T,H} \\ \leq C(T+j)L\tilde{C}e^K S_{\eta^*}^h(0,T) + \tilde{C} S_{\eta^*}^h(j,T) \text{ for } j = 0,1. \end{cases} \tag{4.17}$$

Proof. Since $\eta_h^* = \Lambda_h \eta_h^*$ and $\eta^* = \Lambda \eta^*$ using (2.14), (4.13), (4.14) and (2.3) we obtain

$$|\eta_h^*(t) - \eta^*(t)|_H |\Lambda_h \eta_h^*(t) - \Lambda_h \eta^*(t)|_H + +|\Lambda_h \eta^*(t) - \Lambda \eta^*(t)|_H$$

$$\leq CL\int_0^t |\eta_h^*(s)-\eta^*(s)|_H ds + L(|x_{\eta^*}^h(t)-x_{\eta^*}(t)|_H + |y_{\eta^*}^h(t)-y_{\eta^*}(t)|_H)$$

for all $t \in [0,T]$.

If we apply (4.7) for $\eta = \eta^*$ and $j = 0$, the previous inequality becomes

$$|\eta_h^*(t)-\eta^*(t)|_H \leq L\tilde{C}S_{\eta^*}^h(0,T) + CL\int_0^t |\eta_h^*(s)-\eta^*(s)|_H ds \text{ for all } t \in [0,T]$$

and using a Gronwall-type inequality we get

$$|\eta_h^*(t)-\eta^*(t)| \leq L\tilde{C}e^K S_{\eta^*}^h(0,T) \text{ for all } t \in [0,T]\ . \tag{4.18}$$

Let us also remark that from (2.6) we obtain

$$|z_{\eta_h^*} - z_{\eta^*}|_{j,T,H} \leq (T+j)|\eta_h^* - \eta^*|_{0,T,H} \text{ for all } j = 0,1$$

hence by (4.18) it results

$$|z_{\eta_h^*} - z_{\eta^*}|_{j,T,H} \leq (T+j)L\tilde{C}e^K S_{\eta^*}^h(0,T) \text{ for all } j = 0,1\ . \tag{4.19}$$

Using now (4.4) for $\eta = \eta_h^* - \eta^*$ and the linearity of the problem (4.1), (4.2) with respect to η we obtain

$$|x_{\eta_h^*}^h - x_{\eta^*}^h|_{j,T,H} + |y_{\eta_h^*}^h - y_{\eta^*}^h|_{j,T,H} \leq C|z_{\eta_h^*} - z_{\eta^*}|_{j,T,H} \text{ for all } j = 0,1 \tag{4.20}$$

and using again (4.7) for $\eta = \eta^*$ it results

$$|x_{\eta^*}^h - x_{\eta^*}|_{j,T,H} + |y_{\eta^*}^h - y_{\eta^*}|_{j,T,H} \leq \tilde{C}S_{\eta^*}^h(j,T) \text{ for all } j = 0,1. \tag{4.21}$$

The inequality (4.17) is now a consequence of (4.19)-(4.21).

We now consider the iterative part of the method. Let η_0 be an arbitrary element of $C^0(0,T,H)$ and $(\eta_h^n) \subset C^0(0,T,H)$ be the sequence defined by

$$\eta_h^0 = \eta_0\ ,\ \eta_h^{n+1} = \Lambda_h \eta_h^n \text{ for all } n \in \mathbb{N}. \tag{4.22}$$

Let $(x_{\eta_h^n}^h, y_{\eta_h^n}^h)$ be the solution of (4.1), (4.2) for $\eta = \eta_h^n$ and recall that $(x_{\eta_h^*}^h, y_{\eta_h^*}^h)$ is the solution of (4.1), (4.2) for $\eta = \eta_h^*$. The distance between the couples $(x_{\eta_h^n}^h, y_{\eta_h^n}^h)$ and $(x_{\eta_h^*}^h, y_{\eta_h^*}^h)$ is given by:

Lemma 4.3. *Let C be the strictly positive constant defined in (2.12), (2.13) and let $K = CLT$. Then*

$$\begin{cases} |x_{\eta_h^n}^h - x_{\eta_h^*}^h|_{j,T,H} + |y_{\eta_h^n}^h - y_{\eta_h^*}^h|_{j,T,H} \\ \leq C(T+j)e^K \frac{K^n}{n!}|\Lambda_h\eta_0 - \eta_0|_{0,T,H} \text{ for all } j = 0,1 \text{ and } n \in \mathbb{N} \end{cases} \tag{4.23}$$

Proof. We start by estimating the distance between η_h^n and η_h^*; we remark that for every $m, n \in \mathbb{N}$, $m \geq n$, from (4.22) and (4.15) we deduce

$$|\eta^n -_h -\eta_h^m|_{0,T,H} \leq |\eta_h^n - \eta_h^{n+1}|_{0,T,H} + \cdots + |\Lambda_h^{m-n-1}\eta_h^n - \Lambda_h^{m-n-1}\eta_h^{n+1}|_{0,T,H}$$

$$\leq (1 + \frac{K}{1!} + \cdots + \frac{K^{m-n-1}}{(m-n-1)!})|\eta^- n_h - \eta_h^{n+1}|_{0,T,H} .$$

This inequality implies

$$|\eta_h^n - \eta_h^m|_{0,T,H} \leq e^K |\eta_h^n - \eta_h^{n+1}|_{0,T,H}$$

and passing to the limit when $m \to +\infty$ since $\eta_h^m \to \eta_h^*$ in $C^0(0,T,H)$ (consequence of (4.22) and (4.15))we get

$$|\eta_h^n - \eta_h^*|_{0,T,H} \leq e^K |\eta_h^n - \eta_h^{n+1}|_{0,T,H} .$$

Since by (4.22) we get $\eta_h^n = \Lambda_h^n \eta_0$, $\eta_h^{n+1} = \Lambda_h^{n+1}\eta_0$ using again (4.15) the last inequality leeds to

$$|\eta_h^n - \eta_h^*|_{0,T,H} \leq e^K \frac{K^n}{n!} |\Lambda_h \eta_0 - \eta_0|_{0,T,H} . \tag{4.24}$$

Let us denote by $z_{\eta_h^n}$ the element defined by (2.6) for $\eta = \eta_h^n$. We have

$$|z_{\eta_h^n} - z_{\eta_h^*}|_{j,T,H} \leq (T + j)|\eta_h^n - \eta_h^*|_{0,T,H} \text{ for all } j = 0,1 \text{ and } n \in \mathbb{N} \tag{4.25}$$

and using (4.4) for $\eta = \eta_h^n - \eta_h^*$ and the linearity of the problem (4.1), (4.2) with respect to η we get

$$|x_{\eta_h^n}^h - x_{\eta_h^*}^h|_{j,T,H} + |y_{\eta_h^n}^h - y_{\eta_h^*}^h|_{j,T,H} \leq C|z_{\eta_h^n} - z_{\eta_h^*}|_{j,T,H} , \tag{4.26}$$

for all $j = 0,1$ and $n \in \mathbb{N}$.

The estimation (4.23) follows now from (4.24)-(4.26).

In order to conclude we use (4.16), (4.17), (4.23) and we obtain the following estimation of the difference beween the solution (x, y) of the problem (1.1), (1.2) and the solution $(x_{\eta_h^n}^h, y_{\eta_h^n}^h)$ of the approximate problem (4.1), (4.2) for $\eta = \eta_h^n$:

Theorem 4.1. *There exist C, $\tilde{C}$ which depend only on A such that*

$$\begin{cases} |x_{\eta_h^n}^h - x|_{j,T,H} + |y^- h_{\eta_h^n} - y|_{j,T,H} \leq C(T+j)L\tilde{C}e^K S_{\eta^*}^h(0,T) + \\ +\tilde{C}S_{\eta^*}^h(j,T) + C(T+j)e^K \frac{K^n}{n!}|\Lambda_h \eta_0 - \eta_0|_{0,T,H} \end{cases} \tag{4.27}$$

for all $j = 0, 1$, $n \in \mathbb{N}$ where $K = CLT$.

§5. THE FINAL ALGORITHM

We are now interested in the numerical solution of the evolution problem (1.1), (1.2). For this purpose we approximate the unknowns x and y in space and time and we purposea numerical algorithm which can be directely run on a computer.

As it results from section 4, the approximation in space is realized by considering a closed subspace X_h of X and replacing problem (1.1), (1.2) by the following sequence of linear problems:

Find $x_h^n : [0,T] \to X_h$, $y_h^n : [0,T] \to H$ such that

$$a(x_h^n(t), x'_h) + < z_h^n(t), x'_h >_H = 0 \text{ for all } x'_h \in X_h \text{ and } t \in [0,T] \tag{5.1}$$

$$y_h^n(t) = Ax_h^n(t) + z_h^n(t) \text{ for all } t \in [0,T] \tag{5.2}$$

where

$$z_h^n(t) = \int_0^t \eta_h^n(s)ds + z_0 \text{ for all } t \in [0,T] \tag{5.3}$$

and η_h^n is recursively defined by the equalities

$$\eta_h^n(t) = \Lambda_h \eta_h^{n-1}(t) = B(t, x_h^{n-1}(t), y_h^{n-1}(t)) \text{ for all } t \in [0,T] \text{ and } n \in \mathbb{N}. \tag{5.4}$$

In (5.4) $\eta_h^0 = \eta_0$ is an arbitrary element of the space $C^0(0,T,H)$.

In practice X_h is a finite dimensional subspace of X (constructed for instance using the finite element method) hence (5.1),(5.2) reduces to a linear algebraic system.

Let us now consider $M \in \mathbb{N}$ and let $k = T/M$ be the time step. The approximation in space and time must enable us to compute the elements $x_h^n(mk)$, $y_h^n(mk)$ for every $n \in \mathbb{N}$ and $m = \overline{0,M}$. For this let us denote by $P_h(n,m)$ the set defined by

$$P_h(n,m) = \{\eta_h^n(mk),\ z_h^n(mk),\ x_h^n(mk),\ y_h^n(mk)\} \tag{5.5}$$

for all $n \in \mathbb{N}$ and $m = \overline{0,M}$ and let us split the computing of $P_h(n,m)$ in the following steps:

a) Computing the set $P_h(n,0)$.
For every $\eta_h^0 = \eta_0 \in C^0(0,T,H)$ by (5.3) we get $z_h^n(0) = z_0$ for every $n \in \mathbb{N}$; hence by (5.1), (5.2), (5.4) we obtain $x_h^n(0)$, $y_h^n(0)$ and $\eta_h^n(0)$ for all $n \in \mathbb{N}$.

b) Computing the set $P_h(0,m)$.
Since η_h^0 is given, the values $\eta_h^0(mk)$ are known for all $m = \overline{0,M}$. The elements $z_h^0(mk)$ can be obtained using the trapezoidal rule in order to approximate (5.3):

$$z_h^0(0) = z_0 \text{ , } z_h^0(mk) = z_h^0((m-1)k) + \frac{k}{2}(\eta_h^0(mk) + \eta_h^0((m-1)k) \tag{5.6}$$

for all $m = \overline{1, M}$ and finally $x_h^0(mk)$, $y_h^0(mk)$ are determined from (5.1), (5.2) and (5.6), for all $m = \overline{0, M}$.

c) Computing the set $P_h(n+1, m)$.
Let us suppose that the sets $P_h(n+1, m-1)$,$P_h(n, m)$ are known for a given $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $1mM$. Using (5.4) we get

$$\eta_h^{n+1}(mk) = B(mk, x_h^n(mk), y_h^n(mk))$$

and using again the trapezoidal rule, from (5.3) we obtain

$$z_h^{n+1}(mk) = z_h^{n+1}((m-1)k) + \frac{k}{2}(\eta_h^{n+1}(mk) + \eta_h^{n+1}((m-1)k).$$

Finally $x_h^{n+1}(mk)$, $y_h^{n+1}(mk)$ can be obtained by (5.1) and (5.2).

Using now the steps a), b), c) we compute the set $P_h(n, m)$ for all $n \in \mathbb{N}$ and $m = \overline{0, M}$; in this way the approximative solution $x_h^n(t)$, $y_n^h(t)$ is computed for all $t = mk$, $m = \overline{0, M}$.

§6. Numerical examples

The above algorithm was applied to the relaxation problem (3.13)-(3.16) with the following data: $A = 20$, $\lambda = 1$, $\sigma_0 = 15$, $\alpha = 4$,

$$G(\varepsilon) = \begin{cases} 10\varepsilon & \text{for } \varepsilon 2 \\ 5\varepsilon + 30 & \text{for } 2 < -\varepsilon < 4 \\ 10\varepsilon - 30 \text{for } \varepsilon \geq 4 \ . \end{cases}$$

As it results from (3.20) in this case the stress an strain function can be computed and we have

$$\varepsilon(t, x) = 4 \ , \ \sigma(t, x) = 5e^{-t} + 10 \text{ for all } t \in [0, T], \text{ a.e. in } \Omega = (0, 1). \quad (6.1)$$

The space X_h was defined by $X_h = \{ \dfrac{dv_h}{dx} \mid v_h \in V_h \}$ where $V_h \subset H_0^1(\Omega)$ is the finite element space constructed with polynomial function of degre 1, Ω being divided into 50 finite elements. The initial value considered for η_0 is $\eta_0 = 5$ and the number of iterations made was $n = 10$ (the numerical experiments show that for $n \geq 10$ the numerical solution sabilized). The appoximative solution ε_h^n, σ_h^n were calculated in all the points $x = i/50$ ($i \in \mathbb{N}, 0i50$) and for differents values of t. For a given t the following notation were used

$$\varepsilon_{min}(t) = \min_{i=\overline{0,50}} \varepsilon_h^n(t, i/50) \ , \ \varepsilon_{max}(t) = \max_{i=\overline{0,50}} \varepsilon_h^n(t, i/50)$$

$$\sigma_{min}(t) = \min_{i=\overline{0,50}} \sigma_h^n(t, i/50) \ , \ \sigma_{min}(t) = \min_{i=\overline{0,50}} \sigma_h^n(t, i/50) \ .$$

The time steps choosen were $k = 0.1$ and $k = 0.05$. The results obtained are represented in the following table :

		t = 1	t = 2	t = 3	t = 5
k = 0.1	$\varepsilon_{min}(t)$ $\varepsilon_{max}(t)$	3.99990 4.00018	3.99985 4.00024	3.99981 4.00027	3.99971 4.00034
	$\sigma_{min}(t)$ $\sigma_{max}(t)$	11.8371 11.8405	10.6751 10.6786	10.2481 10.2514	10.0338 10.0373
k = 0.05	$\varepsilon_{min}(t)$ $\varepsilon_{max}(t)$	3.99990 4.00018	3.99985 4.00023	3.99980 4.00027	3.99971 4.00034
	$\sigma_{min}(t)$ $\sigma_{max}(t)$	11.8382 11.8416	10.6759 10.6793	10.2485 10.2519	10.0339 10.0374

The numerical approximation of the solution (6.1).

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UNIVERSITÉ "BLAISE PASCAL" AND CNRS (URA 1501), CLERMONT-FERRAND – FRANCE