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ON COMMUTING ELEMENTS OF GROUPS ACTING ON TREES

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ABSTRACT. In this paper we extend the problem of commutativity of free products of groups with amalgamation (i.e. given two elements f and g of a group which commute, what can be said about them?) to groups acting on trees in which the action with inversions is possible. This will include the cases of free products of groups and HNN groups.

§1. INTRODUCTION

Magnus, Karras and Solitar [3] showed that if two elements of a free group commute then they are powers of some element in the free group. Also they showed that if two elements of a free product of groups commute then they are in the same conjugate of a factor or they are powers of some element in the free product. Then they genralized the above cases to free products of groups with amalgamation (Theorem 4.5., p. 209).

Free groups, free products of groups, and free products of groups with amalgamation are special cases of groups acting on trees. In this paper we formulate the problem of commutativity of groups acting on trees in general to include the cases of free products of groups and HNN groups.

§2. DEFINITIONS AND NOTATIONS

We begin by giving some preliminary definitions. By a *graph* X we understand a pair of disjoint sets $V(X)$ and $E(X)$, with $V(X)$ non-empty, together with a mapping $E(X) \to V(X) \times V(X)$, $y \to (o(y), t(y))$, and a mapping $E(X) \to E(X)$, $y \to \overline{y}$ satisfying $\overline{\overline{y}} = y$ and $o(\overline{y}) = t(y)$, for all $y \in E(X)$. The case $\overline{y} = y$ is possible for some $y \in E(X)$.

A *path* in a graph X is defined to be either a single vertex $v \in V(X)$ (a trivial path), or a finite sequence of edges $y_1, y_2, \cdots, y_n$, $n \geq 1$, such that $t(y_i) = o(y_{i+1})$ for $i = 1, 2, \cdots, n-1$.

A path $y_1, y_2, \cdots, y_n$ is *reduced* if $y_{i+1} \neq \bar{y}_i$, for $i = 1, 2, \cdots, n-1$. A graph X is *connected*, if for every pair of vertices u and v of $V(X)$ there is a path $y_1, y_2, \cdots, y_n$ in X such that $o(y_1) = u$ and $t(y_n) = v$.

A graph X is called a *tree* if for every pair of vertices of $V(X)$ there is a unique reduced path in X joining them. A *subgraph* Y of a graph X consists of sets $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$ such that if $y \in E(Y)$, then $\bar{y} \in E(Y)$, $o(y)$ and $t(y)$ are in $V(Y)$. We write $Y \subseteq X$. We take any vertex to be a subtree without edges.

A reduced path $y_1, y_2, \cdots, y_n$ is called a *circuit* if $o(y_1) = t(y_n)$, and $o(y_i) \neq o(y_j)$ when $1 < i \neq j < n$. It is clear that a graph X is tree if X is connected and contains no circuits.

If X_1 and X_2 are two grphs, then the map $f : X_1 \to X_2$ is called a *morphism*, if f takes vertices to vertices and edges to edges in such a way that

$$\overline{f(y)} = f(\bar{y}) \, .$$
$$f(o(y)) = o(f(y)), \text{ and}$$
$$f(t(y)) = t(f(y)), \text{ for all } y \in E(X_1) \, ;$$

f is called an *isomorphism* if it is one-to-one and onto. and is called an *automorphism* if it is an isomorphism and $X_1 = X_2$. The automorphisms of X form a group under composition of maps. denoted by Aut(X).

We say that a group G *acts* on a graph X, if there is a group homomorphism $\phi : G \to$ Aut (X). If $x \in X$ is vertex or an edge, we write $g(x)$ for $\phi(g)(x)$. If $y \in E(X)$, the $\overline{g(y)} = g(\bar{y})$, $g(o(y)) = o(g(y))$, and $g(t(y)) = t(g(y))$. The case $g(y) = \bar{y}$ for some $y \in E(x)$ and $g \in G$ may occur, i.e. G acts with inversions on X. If $y \in X$ (vertex or edge), we define $G(y) = \{g(y) : g \in G\}$ and this set is called an *G-orbit* or simply an *orbit*. If $x, y \in X$, we define $G(x, y) = \{g \in G : g(y) = x\}$. and $G(x, x) = G_x$, the *stabilizer* of x. Thus, $G(x, y) \neq \emptyset$ if and only if x and y are in the same G-orbit.

It is clear that if $x \in V(X)$, $y \in E(X)$ and $u \in \{o(y), t(y)\}$, then $G(v, y) = \emptyset$, $G_{\bar{y}} = G_y$ and G_y is a subgroup of G_u. As a result of the action of the group G on a graph X we have the graph $X/G = \{G(x) : x \in X\}$ called the *quotient graph*, defined as follows

$$V(X/G) = \{G(v) : v \in V(X)\}, \qquad E(X/G) = \{G(y) : y \in E(X)\} \, ,$$

and for $y \in E(X)$ we have

$$\overline{G(y)} = G(\bar{y}), \qquad t(G(y)) = G(t(y)), \qquad \text{and} \qquad o(G(y)) = G(o(y)) \, .$$

It is clear that there is an obvious morphism $p : X \to X/G$ given by $p(x) = G(x)$, which is called the *projection*. It can be easily shown that if X is connected, then X/G is connected. For more details, see Mahmud [4] or Serre [6].

Definition 2.1. Let G be a group acting on a connected graph X. A subtree T of X is called a *tree of representatives* for the action of G on X if T contains exactly one vertex from each G-vertex orbit. A subtree Y of X containing a tree T of representatives is called a fundamental domain for the action of G on X if each edge in Y has at least one end point in T, and Y contains exactly one edge, say y, from eacg G-edge orbit such that $G(\bar{y}, y) = \emptyset$ and exactly one pair x and $\bar{x}$ from each G-edge orbit such that $G(\bar{x}, x) \neq \emptyset$.

The following procedures for constructing a tree of representatives for the action of a group G on a connected graph X, and a fundamental domain, are taken from Khanfar and Mahmud [1] .

Let S be the set of all T, where T is a subtree of X containing at most one vertex from each G-vertex orbit and at most one edge from each G-edge orbit. S is not empty, since every vertex of X is subtree, and hence is in S. For T_1 and T_2 in S we define $T_1 \leq T_2$ if T_1 is a subtree of T_2. Hence S becomes a partially ordered set. Let $\{t_i : i \in I\}$ be a linearly ordered subset of S. Define $T^* = \bigcup_{i \in I} T_i$. We need to show that T^* is a subtree of X. It is connected, for if $u, v \in V(T^*)$, then $u \in V(T_i)$ and $v \in V(T_j)$ for some $i, j \in I$. We can suppose by symmetry that $T_i \leq T_j$, so $u, v \in V(T_j)$, and there is a path in T^* from u to v. The path has no circuit, for if $y_1, y_2, \cdots, y_n$ is a circuit, then $y_1 \in E(T_{i_1}), \cdots, y_n \in E(T_{i_n})$, so $y_1, y_2, \cdots, y_n$ will all be edges of T_i, where $T_i = \max\{T_{I_1}, \cdots, T_{i_n}\}$, contradicting the fact that T_i is a subtree. It is clear that T^* has at most one vertex and one edge from each orbit under G. Hence $T^* \in S$ and T^* is an upper bound for $\{T_i : i \in I\}$. By Zorn's lemma, S has a maximal element, say T_0.

Claim. *T_0 contains exactly one vertex from each G-vertex orbit.*

O. n the contrary, suppose that $v \in V(X)$ is such that $V(T_0) \cap G(V) = \emptyset$, where $G(v)$ is the orbit containing v. Since X is connected, there is a shortest path $y_1, y_2, \cdots, y_n$ joining a vertex of T_0 to v. Let Y_i be the first edge of of this path such that $V(T_0) \cap G(o(y_i)) \cap \neq \emptyset$ and $V(T_0) \cap G(t(y_i)) = \emptyset$. Then there exists $g \in G$ such that $o(g(y_i)) \in V(T_0)$, $t(g(v_i)) \notin V(T_0)$ and so $g(y_i \notin E(T_0)$. Let T' be the subgraph with $V(T') = V(T_0) \cup \{t(g(y_i))\}$ and $E(T') = E(T_0) \cup \{g(Y_i), g(\bar{y}_i)\}$. It is clear that T' is a subtree of X that properly contains T_0 and at most one vertex from each G-vertex orbit. This contradicts the maximality of T_0 in S. Thus T_0 is a tree of representatives for the action of G on X.

Now we need to prove the existence of a fundamental domain for the action of G on X. Let Λ be the set of all Y, where Y is a subgraph of X containing the chosen tree of representatives T_0 such that each edge in Y has at least one end in T_0 and contains at most one edge from $G(y)$, unless $G(\bar{y}, y) \neq \emptyset$, in which case it contains at most one pair y, $\bar{y}$ from $G(y)$, for all $y \in E(X)$. Since T_0 contains at most one edge from each G-orbit, $T_0 \in \Lambda$. For Y_1 and Y_2 in Λ, we define $Y_1 \leq Y_2$ if Y_1 is a subgraph of Y_2, so Λ becomes a partially ordered set. As in the proof of the existence of a tree of representatives, we can show that Λ contains a maximal element Y_0, say. Let $y \in E(X)$. We need to show that Y_0 contains exactly one edge from $G(y)$, if $G(\bar{y}, y) = \emptyset$ and exactly one pair y, $\bar{y}$ from $G(y)$, if $G\bar{y}, y) \neq \emptyset$. Suppose there exists $y \in E(X)$ such that $E(Y_0) \cap G(y) = \emptyset$. Since X is connected, ther is a shortest path $y_1, y_2, \cdots, y_n = y$ joining a vertex in Y_0 to $t(y)$. Let y_i be the first edge of this path such that $E(y_0) \cap G(Y_i) = \emptyset$. Since $V(T_0) \cap G(o(y_i)) \neq \emptyset$ and T_0 is the tree of representatives in Y_0, there exists $g \in G$ such that $o(g(Y_i)) \in V(T_0)$. Let Y be a subgraph with $V(Y) = V(Y_0) \cup \{t(g(y_i))\}$ and $E(Y) = E(Y_0) \cup \{g(y_i), g(\bar{y}_i)\}$. It is clear that each edge of Y has at least one end in T_0. Moreover, it is clear that $E(Y)$ satisfies the conditions on edges for elements of Λ and properly contains Y_0. This contradicts the maximality of Y_0 in Λ. Thus Y_0 is the fundamental domain for the action of G on X. This completes the proof.

For the rest of this paper G will be the group acting on a tree X and Y a fundamental domain for the action of G on X containing a tree of representatives of T.

Properties of T and Y.

(1) *If u, $v \in V(T)$ such that $G(u, v) \neq \emptyset$, the $u = v$.*
(2) *If $u \in V(X)$, then $G(v) \cap T$ consists of exactly one vertex.*
(3) *$G(\bar{y}, y) = \emptyset$ for all $y \in E(T)$.*
(4) *$V(T)$ is in one-to-one correspondence with $V(X/G)$ under the map $v \to G(V)$.*
(5) *If $y_1, y_2 \in E(Y)$ such that $G(y_1, y_2) \neq \emptyset$, then $y_1 \in \{y_2, \bar{y}_2\}$.*
(6) *If G acts without inversions on X, then Y is in one-to-one correspondence with X/G under the map $y \to G(y)$.*
(7) *If $u \in V(X)$, then there exists an element $g \in G$ and a unique vertex v of T such that $u = g(v)$.*
(8) *If $x \in E(X)$, then there exists $g \in G$ and $y \in E(Y)$ such that $x = g(y)$. If G acts on X without inversions, then y is unique.*
(9) *The set $G(Y) = \{g(y) : g \in G, y \in Y\} = X$. Also $G(E(Y)) = \{g(y) : g \in G, y \in E(Y)\} = E(X)$.*
(10) *The set $G(V(T)) = \{g(v) : g \in G, v \in V(T)\} = V(X)$.*

Definition 2.2. Let G, X, T and Y as above. For each $v \in V(X)$ let v^* be the unique vertex of T such that $G(v, v^*) \neq \emptyset$. It is clear that $v^* = v$ if $v \in V(T)$, and in general $(v^*)^* = v^*$. Also if $G(u, v) \neq \emptyset$, then $U^* = v^*$ for $u, v \in V(X)$.

Note that $G(v) \cap T = \{v^*$, for all $v \in V(X)\}$.

Definition 2.3. For each $y \in E(y)$, define $[y]$ to be an element of $G(t(y), t(y^*))$, that is, $[y](t(y)^*) = t(y)$, to be chosen as follows:

If $(o(y) \in V(T)$, then
(i) $[y] = 1$ if $y \in E(T)$,
(ii) $[y](y) = \bar{y}$ if $G(\bar{y}, y) \neq \emptyset$.

If $o(y) \notin V(T)$, then $[\bar{y}] = [y]^{-1}$ if $G(\bar{y}, y) = \emptyset$, otherwise $[\bar{y}] = [y]$.

It is clear that $[y][\bar{y}] = 1$ if $G(\bar{y}, y) = \emptyset$, otherwise $[y][\bar{y}] = [y]^2$. Let $-y = [y]^{-1}(y)$ if $o(y) \in V(T)$, otherwise let $-y = y$. It is clear that $t(-y) = t(y)^*$, and $G_{-y} = g_y$ if $G(\bar{y}, y) \neq \emptyset$.

Lemma 2.4. *G is generated by the set* $\{[y] : y \in E(Y)\} \cup \{g : g \in G_v,\ v \in V(T)\}$.

Proof. . See Mahmud [4].

Definition 2.5. By a *word* of G we mean an expression w of the form $w = g_0 \cdot y_1 \cdot g_1 \cdot \cdots \cdot y_n \cdot g_n$, $n \geq 0$, where $y_i \in E(Y)$, $i = 11, \cdots, n$, such that

(1) $g_0 \in G_{(o(y_1))^*}$;
(2) $G_i \in G_{(t(y_i))^*}$, for $i = 1, \cdots, n$;
(3) $(t(y_i))^* = (o(y_{i+1}))^*$, for $i = 1, \cdots, n-1$,

w is called *trivial* if $w = 1$. We define n to be the *length* of w and denote it by $|w|$. The inverse w^{-1} of w is defined by the word

$$w^{-1} = g_n^{-1} \cdot \bar{y}_n \cdot g_{n-1}^{-1} \cdot \cdots \cdot g_1^{-1} \cdot \bar{y}_1 \cdot g_0^{-1} .$$

w is called a *reduced word* of G if w contains no expression of the forms

(1) $1 \cdot y_i \cdot g_i \cdot \bar{y}_i \cdot 1$ with $g_i \in G_{-y_i}$, for $i = 1, \cdots, n$, or
(2) $1 \cdot y_i \cdot g_i \cdot y_i \cdot 1$ with $g_i \in G_{y_i}$ such that $G(\bar{y}_i, y_i) \neq \emptyset$, for $i = 1, \cdots, n$.

If $o(y_1))^* = (t(y_n))^*$, then w is called a *closed word* of G of type $(o(y_1))^*$. The *value* $[w]$ of w is the element

$$[w] = g_0[y_1]g_1 \cdots [y_n]g_n$$

of G. If $w_1 = h_n \cdot y_{n+1} \cdot h_{n+1} \cdot \cdots \cdot y_m \cdot h_m$ is a word of G such that $(t(y_n))^* = (o(y_{n+1}))^*$ then $w \cdot w_1$ is defined to be the word

$$w \cdot w_1 = g_0 \cdot y_1 \cdot g_1 \cdot \cdots \cdot y_n \cdot g_n \cdot h_n \cdot y_{n+1} \cdot h_{n+1} \cdot \cdots \cdot y_m \cdot h_m .$$

Lemma 2.6. *Every element of G is the value of a closed and reduced word of G, and if w is a non-trivial closed and reduced word of G, then* $[w]$ *is not the identity element of G. Moreover, if* $w_1 = g_0 \cdot y_1 \cdot \cdots \cdot y_n \cdot g_n$ *and* $w_2 = h_0 \cdot x_1 \cdot \cdots \cdot x_n \cdot h_n$ *are two reduced and closed words of G of the same value and type, then* $n = m$ *and* $y_i = x_i$ *(or* $y_i = \bar{x}_i$ *if* $G(\bar{x}_i, x_i) \neq \emptyset$*) for* $i = 1, \cdots, n$.

Proof. Let $v_0 \in V(T)$. By Lemma 2.4, the set $\{[y] : y \in E(Y)\} \cup \{g : g \in G_v, v \in V(T)\}$ generates G. Let g be an element of G. Then g can be expressed as a product $g_0[y_1] \cdots [y_n]g_n$, where $g_i \in G_{u_i}$, for some vertices $u_0, u_1, \cdots, u_n$ in T and edges $y_1, y_2 \cdots, y_n$ in Y. By taking the unique reduced paths in T between v_0 and v_i, between v_0 and $(o(y_i))^*$, and between $(t(y_i))^*$

and v_0 and the identities of $G(t(y_i))^*$, we may choose this product so that $w = g_0 \cdot y_1 \cdot \cdots \cdot y_n \cdot g_n$ is a word of type v_0. Thus g is the value of the word of G of type v_0 (not necessarily unique). The performance of the following operations called a y-reduction on w, where y is an edge of Y occurs in w

(1) replacing the form $y \cdot g' \cdot \bar{y}$ by $[y]g'[y]^{-1}$, if $g' \in G_{-y}$, or

(2) replacing the form $y \cdot g' \cdot y$ by $[y]g'[y]$ if $G(\bar{y}, y) \neq \emptyset$ and $g \in G_y$,

yields a reduced word w' of G such that $g = [w] = [w']$, $o(w) = o(w')$ and $t(w) = t(w')$. Thus every element of G is the value of a closed and reduced word of G. Now by Corollary 1 of [5], if w is not a trivial closed and reduced word of G, then $[w] \neq 1$. For a similar proof see [2] (Theorem 2,1, p. 82). Now we show that $|w_1 = |w_2|$. Since $[w_1] = [w_2]$, the word $w = w_1 w_2^{-1} = g_0 \cdot y_1 \cdot g_1 \cdot \cdots \cdot y_n \cdot g_n \cdot h_m^{-1} \cdot \bar{x}_m \cdot \cdots \cdot h_1^{-1} \cdot \bar{x}_1 \cdot h_0^{-1}$ has value 1, i. e. $[w] = 1$. Since w_1 and w_2 are reduced, the only way in which the word w can fail to be reduced is that $x_m = y_n$ (or $\bar{x}_m = y_n$ if $G(\bar{x}_m, x_m) \neq \emptyset$) and $g_n h_m^{-1} \in G_{-y_n}$. Making succesive y_i-reductions we see that $n = m$, i. e. $|w_1| = |w_2|$ and $y_i = x_i$ (or $y_i = \bar{x}_i$ if $G(\bar{x}_i, x_i) \neq \emptyset$). This completes the proof.

§3. The main Theorem

Theorem 3.1. *Let G be a group acting on a tree X, and $f, g \in G$ such that $fg = gf$. Then*

(i) *f or g may be in G_y for $y \in E(X)$;*

(ii) *If $g \in G_v$, for $v \in V(X)$ but $g \notin G_y$, for all $y \in E(X)$, $t(y) = v$, then $f \in G_v$;*

(iii) *If neither f or g are in G_v, for all $v \in V(X)$, then there xists an edge x of $E(X)$ and an element c of G such that $f = f^* c^j$ and $g = g^* c^k$, (j, k are integers), where f^*, $g^* \in G_x$, and f^*, g^* and c commute in pairs.*

Proof. If g is in G_y for $y \in E(X)$, then $g \in G_x$, where $x = f(y)$. If $f \in G_y$, or $f \in G_x$, there is nothing to prove. Let $g \in G_v$, for $v \in V(X)$, but not in G_y, for all $y \in E(X)$, $t(y) = v$. We need show that $f \in G_v$. Let T be a tree of representatives for the action of G on X, and let Y be a fundamental domain such that $T \subseteq Y$. Now $v = p(v^*)$, where $v^* \in V(T)$ and $p \in G$. Therefore, $g = pap^1$, where $a \in G_{v^*}$, $a \neq G_{-y}$, for all $y \in E(Y)$, $(t(y))^* = v^*$. Then $p^{-1}fp$ commutes with a. Let $w = y_0 \cdot y_1 \cdot y_1 \cdot \cdots \cdot y_n \cdot g_n$ be a reduced word of G of value $p^{-1}fp$ and type v^*. Since $a \notin G_{-y_n}$, $w \cdot a \cdot w^{-1}$ is a reduced word of G of type v^* and value a. By Lemma 2.6, $|w \cdot a \cdot w^{-1}| = |a|$. Since $|w \cdot a \cdot w^{-1}| = 2n$ and $|a| = 0$, we get $n = 0$. Hence, $p^{-1}fp = g_o$, which implies that $f = pg_0p^{-1}$. Since $g_0 \in G_{v^*}$, we have $f \in pG_{v^*}p^{-1} = G_{p(v^*)} = G_v$.

Now we prove (iii) by contradiction. Let f be an element of G of value a closed and reduced word of G of smallest length for which there exists some element g of G falsifying the assertion. Let g be an element of G such that g is the value of a closed and reduced word of G of smallest length and g

falsifies the assertion with f. Clearly f and g are not in G_v for all $x \in V(X)$. Suppose then that $w_1 = f_0 \cdot x_1 \cdot f_1 \cdot \cdots \cdot x_m \cdot f_m$ and $w_2 = g_0 \cdot y_1 \cdot g_1 \cdot \cdots \cdot y_m \cdot g_m$ are reduced and closed words of G such that w_1 and w_2 are of the same type, and $[w_1] = f$ and $[w_2] = g$. By simmetry, we may suppose that $m \leq n$. Now w_1^2 is reduced, or equivalently, if $f_m f_0 \in G_{-x_m}$, then $x_1 \neq \bar{x}_m$ or viceversa, for otherwise $[x_m] f_m f f_m^{-1} [x_m]^{-1}$ is the value of the closed and reduced word $L = [x_m] f_m f_0 [x_m]^{-1} f_1 \cdot x_2 \cdot f_2 \cdot \cdots \cdot x_{m-1} f_{m-1}$ of lenght $m-2$ which leads to a contradiction, because the elements $[x_m] f_m f f_m^{-1} [x_m]^{-1}$ and $[x_m] f_m g f_m^{-1} [x_m]^{-1}$ falsify the assertion.

Now we consider the following cases:

Case 1. $w_1 \cdot w_2$ is reduced. Since $fg = gf$, or equivalently $w_1 \cdot w_2$ and $w_2 \cdot w_1$ have the same value, then by Lemma 2.6, $w_1 \cdot w_2$ is reduced. But $m \leq n$ and so $y_{n-m+i} = x_i$ (or $\bar{x}_i$ if $G(\bar{x}_i, x_i) \neq \emptyset$), for $i = 1, \cdots, m$. Hence gf^{-1} is the value of the closed and reduced word $M = g_0 \cdot y_1 \cdot g_1 \cdot \cdot \cdot y_{n-m} \cdot a$, where a comes from the cancelation in the word $w_2 \cdot w_1^{-1}$. Since gf^{-1} commutes with f and $|M| < |w_2|$, then gf^{-1} and f satisfy the assertion of the theorem. If gf^{-1} is in G_x, where $x \in E(X)$, then $gf^{-1} = hf'h^{-1}$, where $x = h(y)$, $y \in E(Y)$, and $f' \in G_y$. Hence $f = h^{--1}f$ and $g = gf^{-1} = hf'h^{-1}$, and the assertion of the theorem would hold for f and g ,contrary to ssumption. If f^{-1} is in G_v, where $v \in V(X)$, then f is also, and so $g = gf^{-1}f$. This contradicts our assumption that f and g are not in G_v, for all $v \in V(X)$. Thus it must be that $f = hf'h^{-1}c^j$, $gf^{-1} = hg'h^{-1}c^k$, $g' \in G_y$, where $hf'h^{-1}$, $hg'h^{-1}$ and c commute in pairs. We take $f^* = hf'h^{-1}$ and $g^* = hg'h^{-1}$. Hence $g = gf^{-1}f = hg'h^{-1}c^k hf'h^{-1}c^j = hg'f'h^{-1}c^{k+j}$, and again the assertion of the theorem would hold for f and g. Thus, case 1 leads to a contradiction.

Case 2. $w_1 \cdot w_2$is not reduced. In this case $w_1 \cdot w_2^{-1}$ is reduced. Indeed, if $w_1 \cdot w_2^{-1}$ is not reduced, then we get that w_1 is not reduced. This contradicts that w_1^2 is reduced. But then f^{-1} can be used in place of f and w_1^{-1} in place of w_1 in the preceding paragraph, for $|w_1^{-1}| = |w_1|$, and f and g falsify the assertion of the theorem if and only if f^{-1} and g falsify the assertion. Thus case 2 leads to a contradiction.

This completes the proof of the main theorem.

Corollary 3.2. *Let $G = \prod^*(G_i; A_{jk} = A_{kj})$ be a free product of the groups G_i and $f, g \in G$ such that $fg = gf$. Then*

(i) *f and g may be in a conjugate of A_{jk};*

(ii) *If neither f nor g is in a conjugate of A_i, but f is in a conjugate of G_j, then g is in that conjugate of G_j;*

(iii) *If neither f nor g is in a conjugate of a factor G_i, then $f = hf'h^{-1}c^m$ and $g = hg'h^{-1}c^n$, where $h, c \in G$, $h, c \in G$, $f', g' \in A_{jk}$, $m, n \in Z$, and $hf'h^{-1}$, $hg'h^{-1}$, c commute in pairs.*

Proof. There is a tree on which G acts in such a way that the G-vertex stabilizers are the conjugates og G_i, and the G-edge stabilizers are the conjugates of A_{jk}. By the main theorem the proof of Corollary 3.2 follows.

Corollary 3.3.. *Let* $G =< H, t_i \mid rel\ H, t_i A_i t_1^{-1} = B_i >$ *be an* HNN *group and* $f, g \in G$ *such that* $fg = gf$. *Then*

(i) *f or g may be conjugates of A_i;*

(ii) *If neither f or g in a conjugate of A_i, but f is a conjugate of H, then g is in that conjugate of H;*

(iii) *If neither f nor g in a conjugate of H, then $f = hf'h^{-1}c^m$ and $g = hg'h^{-1}c^m$, where $h, c \in G$, $f', g' \in A_i$, $m, n \in Z$, and $hf'h^{-1}$, $hg'h^{-1}$, c commute in pairs.*

Proof. There is a tree on which G acts in such a way that G is transitive on the set of vertices, and the G-vertex stabilizers are the conjugates of H, and the G-edge stabilizers are the conjugates of A_i. By the main theorem the proof of Corollary 3.3 follows.

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REFERENCES

1. Khanfar, M. I. and Mahmud, R. M. S., *A note on groups acting on connected graphs*, J. Univ. Kuwait (Sci.) **16** (1989).
2. Lyndon, R. C. and Schupp, P. E., *Combinatorial Group Theory*, Springer-Verlag, New York, 1977.
3. Magnus, W., Karras, A. and Solitar, D., *Combinatorial Group Theory. Presentation of Groups in terms of Generators and Relations*, Interscience, New York, 1966.
4. Mahmud, R. M. S., *Presentation of groups acting on trees with inversions*, Proc. Royal Soc. of Edinburgh **113A** (1989), 235–241.
5. Mahmud, R. M. S., *The normal form theorem of groups acting on trees with inversions*, J. Univ. Kuwait (sci) **18** (1991).
6. Serre, J.-P., *Arbres, amalgames et $S\ell_2$*, Asterisque **46** (1977), Société Math. France.

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