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Differential equations associated to algebraic functions and their solutions

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ABSTRACT. A method to find the general solution of algebraic equations, including the quintic, is presented. The procedure is essentially based on the analytical properties of the algebraic functions and leads to the solution of certain linear differential equations that these functions must satisfy.

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I. Introduction

It is well known that the algebraic equation (A.E.) of degree n

$$Y^n + a_1 Y^{n-1} + \cdots + a_{n-1} Y + a_n = 0, \tag{1}$$

with $a_m \in \mathbb{C}$, has in general n "roots", which can be explicitly given in terms of simple functions of the a_m's only for $n \leq 4$.

A non traditional approach to the general solution of (1) is proposed in the present work. On the basis of rather elementary arguments, the existence of certain linear differential equations (D.E.) to be satisfied by the roots of the A.E.'s will be established for the cases $n = 2, 3, 4$ and 5. For each case, an appropriate transformation allows to remove all but one parameter of (1), so that the solutions will be complex functions of one independent variable, and the D.E.'s ought to be of the ordinary type. It will be found that the basic solutions to these D.E.'s are essentially generalized hypergeometric functions. The reduced A.E., the associated linear D.E. and one of its basic solutions are illustrated in the ensuing scheme:

Algebraic Equation in reduced form.	Associated Differential Equation ($c_o, \ldots,$ are rational numbers, characteristic for each D.E.)	Typical member of the set of $(n-1)$ basic solutions to the (homogeneous) differential equation. ($\|z\| \leq 1$)
$y^2 - 2y + z = 0$	$(z-1)y' + c_o y = -\frac{1}{2}$	${}_1F_o(-\frac{1}{2}; z)$
$y^3 - 3y + 2z = 0$	$(z^2-1)y'' + c_1 z y' + c_o y = 0$	${}_2F_1(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; z^2)$
$y^4 - 4y + 3z = 0$	$(z^3-1)y''' + c_2 z^2 y'' + c_1 z y' + c_o y = 0$	${}_3F_2(-\frac{1}{12}, \frac{2}{12}, \frac{5}{12}; \frac{1}{3}, \frac{2}{3}; z^3)$
$y^5 - 5y + 4z = 0$	$(z^4-1)y^{iv} + c_3 z^3 y''' + c_2 z^2 y'' + c_1 z y' + c_o y = 0$	${}_4F_3(-\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{11}{20}; \frac{1}{4}, \frac{2}{4}, \frac{3}{4}; z^4)$

For the cases $n > 5$, not considered here, the same method of reduction leads to a minimum of $(n-4)$ parameters, and the D.E.'s would be linear partial differential equations.

The paper is organized as follows. Section II contains a brief summary of some known results from the classical theories of both A.E.'s and algebraic functions, but only to the extent that they are useful in Section III, devoted to the construction of the D.E.'s, and which constitutes the core of this work. In Section IV the solutions to the D.E.'s which will be simultaneously solutions of the corresponding A.E.'s are found.

II. Basic concepts and results

II.1. The left hand side of (1) can be considered as an entire function $\phi(Y)$ with complex independent parameters a_m which, by the fundamental theorem of algebra, can be expressed as

$$\phi(Y) = Y^n + a_1 Y^{n-1} + \cdots + a_{n-1} Y + a_n = \prod_{j=1}^{n} (Y - Y_j), \tag{2}$$

where the Y_j's are the roots satisfying $\phi(Y_j) = 0$.

II.2. The product of pairs of differences of roots (for fixed j) is

$$\prod_{k \neq j} (Y_j - Y_k) = \left(\frac{\partial \phi}{\partial Y}\right)_{Y=Y_j} = nY_j^{n-1} + (n-1)a_1 Y_j^{n-2} + \cdots + a_{n-1}, \tag{3}$$

the existence of repeated roots being characterized by

$$\phi(Y_j) = \left(\frac{\partial \phi}{\partial Y}\right)_{Y=Y_j} = 0. \tag{4}$$

II.3.1. The elementary symmetric functions of the Y_j's are

$$\begin{aligned} s_1 &= \sum_1^n Y_j = -a_1, \quad s_2 = \sum_{j<k} Y_j Y_k = a_2, \dots, \\ s_n &= \prod_1^n Y_j = \pm a_n \qquad \text{(depending on whether } n \text{ is even or odd).} \end{aligned} \tag{5}$$

II.3.2. The (Newton) sums of powers of the Y_j's are given by

$$S_1 = s_1, \qquad S_2 = \sum_1^n Y_j^2 = s_1^2 - 2s_2, \dots, S_m = \sum_1^m Y_j^m, \dots. \tag{6}$$

For a method fo find the S_m's for higher m, see [1], [2].

II.3.3. The discriminant function associated with $\phi(Y)$ is defined by

$$D_n(a_1, a_2, \dots, a_n) = \prod_{j<k} (Y_j - Y_k)^2, \qquad j = 1, \dots, n-1. \tag{7}$$

D_n can be expressed as a sum of products of powers of the elementary symmetric functions, and vanishes if there exist two or more repeated roots.

II.4. The Tschirnhaus transformation. If Y_k is some root of (1) and the expression

$$Y_k' = b_o + b_1 Y_k + \cdots + b_{n-1} Y_k^{n-1}, \quad b_m \in \mathbb{C} \tag{8}$$

is formed, then the Y_k''s are the roots of a new A.E. of degree n

$$Y'^n + a_1' Y'^{n-1} + \cdots + a_{n-1}' Y' + a_n' = 0, \tag{9}$$

which is known as the "Tschirnhaus-transform" of the original A. E. Its usefulness lies in the fact that the determination of the Y_k''s leads in general to the original Y_k with no need of solving a higher degree A.E. [1]. It is natural to look for a transformation which makes equation (9) easier to solve as compared to (1), through the vanishing of some of the a_m''s. This possibility —and its limitations— are established in a proposition (sometimes called Jerrard's theorem):

"By a Tschirnhaus transformation —involving only square and cube roots— the second, third and fourth terms of (9) can be removed" [3].

That the elimination of more parameters is impossible is a consequence of the following: in the process of determining the b_m's from (8) in terms of the original a_m's, the necessity of solving an A.E. of degree higher than n itself would be unavoidable.

II.5. Analytical properties of the roots Y_k considered as functions of the n complex independent parameters a_m. Each root Y_k of equation (1) defines an element of an algebraic function, whose (well known) basic properties are summarized (c.f. [4], [5]) by:

(i) Each Y_k is an analytical function of the a_m's.
(ii) The only singularities of the Y_k's are the points where the discriminat function (7) vanishes – the so called "critical points" - and the point at infinity.
(iii) The Y_k's are continuous at the critical points.
(iv) The singularities of the Y_k's can only be algebraic branch points or algebraic poles.

III. The differential equations

III.1. The following proposition is fundamental to the purpose of establishing the existence and specific form of the D.E.'s:

Theorem. *Let Y_i be some root of $\phi(Y)$ as defined in (2). If the discriminant function $D_n(a_1,\ldots,a_n)$ does not vanish, then:*

(i) *Every partial derivative $\frac{\partial Y_i}{\partial a_j}$ exists and can be reduced to the functional form*

$$(10) \qquad \frac{\partial Y_i}{\partial a_j} = A_{j1}Y_i^{n-1} + \cdots + A_{j(n-1)}Y_i + A_{jn},$$

where the A_{jm}'s are rational functions of the a_k's, the same for all roots.

(ii) *All higher order partial derivatives, including mixed ones, exist and can be reduced to a form similar to (10).*

To show part (i), the analytical properties II.5(i) and (ii) of the roots are recalled. Next, the derivative from the A.E. satisfied by each Y_i,

$$(11) \qquad Y_i^n + a_1Y_i^{n-1} + \cdots + a_{n-1}Y_i + a_n = 0,$$

with respect to a_j, leads to

$$(12) \qquad (nY_i^{n-1} + (n-1)Y_i^{n-2} + \ldots a_{n-1})\frac{\partial Y_i}{\partial a_j} + Y_i^{n-j} = 0.$$

But, according to equation (3), (12) can be recasted as

$$\left(\frac{\partial\phi}{\partial Y}\right)_{Y_i} \cdot \frac{\partial Y_i}{\partial a_j} + Y_i^{n-j} = 0. \tag{13}$$

If repeated roots are excluded, $(\partial\phi/\partial Y)_{Y_i} \neq 0$, and (13) can be formally solved in the form

$$\frac{\partial Y_i}{\partial a_j} = -\frac{Y_i^{n-j}}{(\partial\phi/\partial Y)_{Y_i}}, \tag{14}$$

which can be reformulated as

$$\frac{\partial Y_i}{\partial a_j} = Y_i^{n-j} \frac{(-1)}{(\partial\phi/\partial Y)_{Y_i}} = Y_i^{n-j} \left(\frac{\partial Y_i}{\partial a_n}\right). \tag{15}$$

By using the A.E. (11) itself, any power of Y_i equal to or higher than Y_i^n, can be reduced to a polynomial in Y_i, of degree not greater than $(n-1)$, so that it is enough to obtain the asserted functional form of $\frac{\partial Y_i}{\partial a_j}$ for $a_j = a_n$ only.

To simplify the notation, only the case $i = 1$ will be considered. Using relation (3) for $\left(\frac{\partial\phi}{\partial Y}\right)_{Y_i}$,

$$\begin{aligned}\frac{\partial Y_1}{\partial a_n} &= \frac{(-1)}{(Y_1 - Y_2)(Y_1 - Y_3)\cdots(Y_1 - Y_n)} \\ &= \frac{(-1)(Y_1 - Y_2)\cdots(Y_1 - Y_n)\cdot(Y_2 - Y_3)^2\cdots(Y_2 - Y_n)^2\cdots(Y_{n-1} - Y_n)^2}{\prod\limits_{k<\ell}(Y_k - Y_\ell)^2},\end{aligned}$$

which, by means of expressions (3) and (7), can be written

$$\frac{\partial Y_1}{\partial a_n} = -\frac{(nY_1^{n-1} + \cdots + a_{n-1})\prod'_{\ell' \neq \ell''}(Y_{\ell'} - Y_{\ell''})^2}{D_n(a_1, \ldots, a_n)} \quad (\ell', \ell'' \neq 1). \tag{16}$$

The primed product in (16), where Y_1 does not appear, can be recognized as the product of two identical $(n-1)\times(n-1)$ Vandermonde determinants:

$$\prod\nolimits'_{\ell'\neq\ell''}(Y_{\ell'}-Y_{\ell''})^2$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 \\ Y_2 & Y_3 & \dots & Y_n \\ Y_2^2 & Y_3^2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ Y_2^{n-1} & Y_3^{n-1} & \dots & Y_n^{n-1} \end{vmatrix} \begin{vmatrix} 1 & Y_2 & Y_2^2 & \dots & Y_2^{n-1} \\ 1 & Y_3 & Y_3^2 & \dots & Y_3^{n-1} \\ 1 & Y_4 & Y_4^2 & \dots & Y_4^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & Y_n & Y_n^2 & \dots & Y_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} n-1 & S'_1 & \dots & S'_{n-1} \\ S'_1 & S'_2 & \dots & S'_n \\ S'_2 & S'_3 & \dots & S'_{n+1} \\ \dots & \dots & \dots & \dots \\ S'_{n-1} & S'_s & \dots & S'_{2n-1} \end{vmatrix},$$

where $S'_m = S_m - Y_1^m$ is the sum of the m^{th} - powers of the roots different from Y_1. S_m can be expressed as an entire function of the a_j's, and Y_1^m, for $m > (n-1)$, can be reduced to a polynomial whose degree does not exceed $n-1$. Thus, the primed product can be finally set in the form

$$\prod{}'(Y_{\ell'}-Y_{\ell''})^2 = B_1Y_1^{n-1}+\dots+B_{n-1}Y_1+B_n.$$

The B_m's are entire functions of the a_m's and are the same for all Y_i.

After the product in (16) has been carried out, the functional form (10) is obtained for $\frac{\partial Y_1}{\partial a_n}$ and, consequently, for all $\frac{\partial Y_i}{\partial a_n}$. Then, through relation (15), for all $\frac{\partial Y_i}{\partial a_j}$.

To close the proof observe that the rules of differentiation and multiplication of polynomials lead to the conclusion that any higher partial derivative, including mixed ones, with respect to each a_j, will exist if $D_n(a_1,\dots,a_n)\neq 0$, and they will have a functional form similar that given in (10). ☑

The fundamental theorem allows to pose the question of the existence of linear partial D.E.'s for the Y_i in simplest terms: the possibility of eliminating the $(n-2)$ non-linear terms $Y_i^{n-1}, Y_i^{n-2},\dots,Y_i^2$ in some set of $(n-1)$ different partial derivatives of Y_i. Although it does not seem to be an easy matter to find the general conditions additional to the basic one $D_n(a_1,\dots,a_n)\neq 0$ which would ensure the feasibility of this elimination, it can be stated that in the specific cases considered here no new requirements emerge.

III.2. D.E. for $n=2$. The A.E.

$$(17) \qquad Y_j^2 - 2aY_j + b = 0$$

can be transformed, through the replacements $Y_j = ay_j$, $z = a^{-2}b$, $a \neq 0$, into

$$y_j^2 - 2y_j + z = 0 \tag{18}$$

The elementary symmetric functions and discriminat function associated with the roots of (18) are given by:

$$s_1 = 2, \quad s_2 = z; \quad D_2(z) = -2^2(z-1). \tag{19}$$

The D.E. to be satisfied by the roots can be obtained directly from the fundamental theorem:

$$y_j' = \frac{dy_j}{dz} = -\frac{(y_j - y_k)}{D_2(z)} = -\frac{y_j - (s_1 - y_j)}{D_2(z)} = -\frac{2\,(y_j - 1)}{D_2(z)}, j = 1, 2,$$

so that both y_1 and y_2 satisfy the linear D.E.

$$(z-1)y' - \frac{1}{2}y = -\frac{1}{2}, \tag{20}$$

which is exceptionally — due to the fact that $s_1 \neq 0$ — non-homogeneous.

III.3. D.E. for $n = 3$. The transformation $Y_j' = \left(\frac{1}{3}\right) a_1 + Y_j$, followed by an obvious redefinition of the parameters, leads to the reduced form of the general cubic

$$Y_j'^3 - 3aY_j' + 2b = 0. \tag{21}$$

Next, the substitutions

$$Y_j' = a^{\frac{1}{2}} y_j, \quad z = a^{-\frac{3}{2}} b, \quad a \neq 0, \tag{22}$$

where $\arg\left(a^{m/p}\right) = (m/p)\arg(a)$ is adopted from now on, lead to the form

$$y_j^3 - 3y_j + 2z = 0 \tag{23}$$

of the A. E. The elementary functions, sums of powers of the roots and the discriminat function associated with (23) are:

$$s_1 = 0, \qquad s_2 = -3, \qquad s_3 = -2z$$

$$S_1 = 0, \qquad S_2 = 6, \qquad S_3 = -6z, \qquad S_4 = 18, \dots, \tag{24}$$

$$D_3(z) = -2^2 \cdot 3^3 \cdot \quad (z^2 - 1).$$

To obtain the D.E. for the roots, the following recurrence procedure is developed:

According to the basic theorem, both the k-th and $(k+1)$-th order derivatives of a particular root can be written as

$$
(25)\qquad \begin{aligned} y_j^{(k)} &= A_k y_j^2 + B_k y_j + C_k \\ y_j^{(k+1)} &= A_{k+1} y_j^2 + B_{k+1} y_j + C_{k+1} \end{aligned} \qquad k = 0, 1, \ldots
$$

If the first of equations (25) is summed over all three possible values of j, it is easy to see, due to relations (24) for the sums of powers of the roots, that $C_k = -2A_k$ for all k.

Next, the combination of equations (25) and $y^{(k+1)} = dy^{(k)}/dz$, and the foregoing connection between C_k and A_k, lead to

$$
(A_{k+1} - A'_k)(y_j^2 - 2) + (B_{k+1} - B'_k)\, y_j = (2A_k y_j + B_k) y'_j \ ,
$$

where the "prime" symbol now stands for "d/dz". If this relation is multiplied, first by y_j and then by y_j^2, and if each of the new expressions is summed up over all possible values of index j, with due account for the sums of powers from (24), the following system turns out

$$
\begin{aligned} z\,(A_{k+1} - A'_k) - (B_{k+1} - B'_k) &= 2A_k/3 \\ (A_{k+1} - A'_k) - z\,(B_{k+1} - B'_k) &= -B_k/3. \end{aligned}
$$

When this system is solved, a recurrence relation between the "$k+1$" and the "k" coefficients is established. Exposed in matrix form, it is

$$
(26)\qquad \begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{dz} + \frac{1}{3(z^2-1)} \begin{pmatrix} 2z & 1 \\ 2 & z \end{pmatrix} \right] \begin{pmatrix} A_k \\ B_k \end{pmatrix}, C_k = -2A_k.
$$

When applied first to $k = 0$, to which $A_0 = C_o = 0, B_o = 1$ correspond, and then to $k = 1$, recurrence relation (26) leads to

$$
(27)\qquad \begin{aligned} y'(z) &= \tfrac{1}{3(z^2-1)} \left[y^2 + zy - 2 \right] \\ y''(z) &= \tfrac{(-1)}{3^2(z^2-1)^2} \left[3zy^2 + (2z^2+1)y - 6z \right]. \end{aligned}
$$

The elimination of the non-linear term y^2 in equations (27) is possible under the assumption $z^2 - 1 \neq 0$, and the linear second order homogeneous D.E. for each of the three roots of (23)

$$
(28)\qquad (z^2 - 1)y'' + z \cdot y' - \frac{1}{9} y = 0,
$$

is obtained, which can be easily solved by the simple change of variable $dz = 3(z^2-1)^{\frac{1}{2}} dt$, i.e., $z = \cosh 3t$, which leads to the D.E., $\frac{d^2 y(t)}{dt^2} - y(t) = 0$. If "initial" conditions for $y(t)$ and $y'(t)$ are taken at $t_o = i\frac{\pi}{6}$ $(z_o = 0)$, from the A.E. (23) and the first of equations (27) the following set of 3 solutions is obtained:

$$(29)\qquad y(t) = \begin{cases} \cosh t \pm i\sqrt{3}\mathrm{senh}\, t, \\ -2\cosh t, \end{cases} \qquad \text{with} \quad e^t = \sqrt[3]{z + \sqrt{z^2-1}}.$$

For higher degrees, there seems not to exist such a fortunate change of variable, and the D.E. (28) will be solved by a method common to all four cases $n = 2, 3, 4, 5$.

III.4. D.E. for $n = 4$. For the general $n = 4$ A.E.

$$Y_j^4 + a_1 Y_j^3 + a_2 Y_j^2 + a_3 Y_j + a_4 = 0,$$

a Tschirnhaus transformation of the following form is proposed:

$$(30)\qquad Y_j' = b_o + b_1 Y_j + Y_j^2,$$

where b_o and b_1 should be determined by the condition that Y_j' satisfies the new A.E.

$$(31)\qquad Y_j'^4 + a_3' Y_j' + a_4' = 0.$$

The details can be found in [1] and [2]. A last simplification of equation (31) can be achieved through the replacements

$$a_3' = -4a,\ a_4' = 3b, \quad Y_j' = a^{\frac{1}{3}} y_j, \quad z = a^{-\frac{4}{3}} b, \quad a \neq 0,$$

so that the A.E. for each $y_j(z)$ becomes

$$(32)\qquad y_j^4 - 4y_j + 3z = 0.$$

The elementary symmetric functions, the sums of powers of the roots, and the discriminant function associated to (32) are

$$s_1 = 0, \qquad s_2 = 0, \qquad s_3 = 4, \qquad s_4 = 3z;$$

$$(33)\qquad S_1 = 0, \qquad S_2 = 0, \qquad S_3 = 12, \qquad S_4 = -12z, \dots,$$

$$D_4(z) = 3^3 \cdot 4^4 \cdot \ (z^3 - 1).$$

An iterative procedure for the determination of the derivatives of any order can be again developed. If the k-th order derivative is written as

$$y^{(k)} = A_k y^3 + B_k y^2 + C_k y + D_k,$$

it can be shown that the coefficients of the $k+1$–th order derivative are related to the preceding ones through

$$
(34) \qquad \begin{pmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d}{dz} + \frac{1}{4(z^3-1)} \begin{pmatrix} 3z^2 & 2z & 1 \\ 3 & 2z^2 & z \\ 3z & 2 & z^2 \end{pmatrix} \right] \begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix},
$$

and $D_k = -3A_k$ for all k.

Starting with $k = 0$, for which only $C_o = 1$ is not zero, the following set of derivatives turns out:

$$
(35) \qquad \begin{aligned}
y' &= \tfrac{1}{4(z^3-1)} \left[y^3 + zy^2 + z^2y - 3 \right] \\
y'' &= \tfrac{(-1)}{4^2\cdot(z^3-1)^2} \left[6z^2y^3 + (5z^3+1)y^2 + 3(z^4+z)y - 18z^2 \right] \\
y''' &= \tfrac{1}{4^3\cdot(z^3-1)^3} \left[(65z^4+43z)y^3 + (47z^5+61z^2)y^2 + \right. \\
&\qquad \left. + (21z^6+77z^3+10)y - 3(65z^4+43z) \right]
\end{aligned}
$$

The elimination of the terms y^3 and y^2 in the set of equations (35) leads to the linear homogeneous D.E.

$$
(36) \qquad (z^3-1)y''' + \frac{9}{2}z^2y'' + \frac{43}{16}zy' - \frac{10}{64}y = 0,
$$

which must be satisfied by the four roots of the A.E. (32).

III.5. D.E. for $n = 5$. The corresponding general A. E. can be reduced to

$$
(37) \qquad Y_j'^5 + a_4'Y_j' + a_5' = 0
$$

by means of a Tschirnhaus transformation of the type

$$
Y_j' = b_o + b_1Y_j + b_2Y_j^2 + Y_j^3,
$$

where b_1, b_2 and b_3 are obtained as the solutions to a system composed of a linear, a second-degree and a cubic equation. See [1] and [2] for details. A further modification of equation (37) is achieved through the replacements

$$
a_4' = -5a, \quad a_5' = 4b, \quad Y_j' = a^{\frac{1}{4}}y, \quad z = a^{-\frac{5}{4}}b \quad (a \neq 0),
$$

leading to the final form

$$y_j^5 - 5y_j + 4z = 0. \tag{38}$$

The elementary symmetric functions, the sums of powers of the roots, and the discriminat function associated to the roots of (38) are

$$\begin{aligned} &s_1 = s_2 = s_3 = 0, \qquad s_4 = -5, \qquad s_5 = -4z; \\ &S_1 = S_2 = S_3 = 0, \qquad S_4 = 20, \qquad S_5 = -20z, \ldots, \\ &D_5(z) = 4^4 5^5 (z^4 - 1). \end{aligned} \tag{39}$$

The process of finding the explicit form of the derivatives is once more simplified by a recurrence procedure: if A_k, B_k, C_k, D_k and E_k are the coefficients in the k-th order derivative, it is found that the coefficients of the next derivative are given by

(40)
$$\begin{pmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \\ D_{k+1} \end{pmatrix} =$$
$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{d}{dz} + \frac{1}{5(z^4-1)} \begin{pmatrix} 4z^3 & 3z^2 & 2z & 1 \\ 4 & 3z^3 & 2z^2 & z \\ 4z & 3 & 2z^3 & z^2 \\ 4z^2 & 3z & 2 & z^3 \end{pmatrix}\right] \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix},$$

and $E_k = -4A_k$ for all k.

Starting with $k = 0$, for which only $D_o = 1$ does not vanish, the following set of four derivatives is obtained:

$$\begin{aligned} y' &= \tfrac{1}{5(z^4-1)} \left[y^4 + zy^3 + z^2y^2 + z^3y - 4 \right], \\ y'' &= \tfrac{(-1)}{5^2(z^4-1)^2} \left[10z^3y^4 + (9z^4+1)y^3 + (7z^5+3z)y^2 \right. \\ &\quad \left. + (4z^6 + 6z^2)y - 40z^3 \right], \end{aligned} \tag{41}$$

$$y''' = \frac{1}{5^3(z^4-1)^3}\left[(165z^6+135z^2)y^4+(135z^7+165z^3)y^3 \right.$$
$$+(87z^8+201z^4+12)y^2+(36z^9+213z^5+51z)y$$
$$\left. -4(165z^6+135z^2)\right],$$

$$y^{iv} = \frac{(-1)}{5^4(z^4-1)^4}\left[\left(3675z^9+10050z^5+1275z\right)y^4\right.$$
$$+\left(2760z^{10}+10380z^6+1860z^2\right)y^3$$
$$+\left(1530z^{11}+9840z^7+3630z^3\right)y^2$$
$$+\left(504z^{12}+7623z^8+6642z^4+231\right)y$$
$$\left.-4\left(3675z^9+10050z^5+1275z\right)\right].$$

The elimination of y^4, y^3, y^2 in the set of equations (41), which is possible under the condition that $D_5(z) \neq 0$, leads to the D.E.

$$(42) \qquad \left(z^4-1\right)y^{iv}+10z^3y'''+\frac{117}{5}z^2y''+\frac{51}{5}zy'-\frac{231}{625}y=0.$$

to be satisfied by the five roots of (37).

IV. The solutions to the D.E.'s

In view of the common features of the D.E.'s (20), (28), (36) and (42), it is not surprising that the basic set of solutions for each of them can be expressed in terms of the same type of transcendental functions: generalized hypergeometric functions. The adopted notation for these functions and some of their properties are reviewed in what follows (see [6] for more details).

(i) If $\alpha_r (r=1,\ldots,p)$, the numerator parameters, and $\beta_s (s=1,\ldots,q)$ the denominator-parameters, are given, the generalized hypergeometric functions of the complex variable t with these parameters will be

$$(43) \qquad {}_pF_q(\alpha_r;\beta_s;t)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\cdot\frac{t^k}{k!},$$

where $(\gamma)_k$ is the Pochhammer symbol defined by

$$(44) \qquad (\gamma)_o=1, \quad (\gamma)_k=\gamma(\gamma+1)\ldots(\gamma+k-1)=\Gamma(\gamma+k)/\Gamma(\gamma).$$

(ii) If $p=q+1$, $\beta_s \neq 0,-1,-2,\ldots$, and

$$(45) \qquad \sigma = \operatorname{Re}(\beta_1+\cdots+\beta_q-\alpha_1-\cdots-\alpha_p)\in[0,1],$$

the series (43) converges for all $|t| \leq 1, \quad t \neq 1$.

Some common features of the procedure to be followed are mentioned in order to avoid obvious repetitions. For each D.E., the (regular) singular points are the solutions of $z^{n-1} = 1$, and $z = \infty$; $z = 0$ is a regular point. Accordingly, the z-plane will be divided in the regions $|z| < 1$ and $|z| > 1$, and the usual Ansätze (Frobenius method) will be adopted for the solutions:

$$w(z) = \sum_{0}^{\infty} a_k z^k, \quad |z| < 1; \tag{46}$$

$$w(z) = z^{-\rho} \sum_{0}^{\infty} a_k z^{-k}, \quad |z| > 1. \tag{47}$$

The trial series (46) and (47) will lead in all cases to generalized hypergeometric functions depending on the arguments z^{n-1}, $z^{-(n-1)}$, respectively, with $p = q + 1 = n - 1$ and $\sigma = \frac{1}{2}$, so that the convergence of all series for $|z| = 1$, $z^{n-1} \neq 1$, is ensured. The existence of (algebraic) branch points at $z^{n-1} = 1$ will force to introduce $(n-1)$ cuts on each of the corresponding sub-regions.

IV.1. The solutions for $n = 2$. If the Ansatz (46) is used in the homogeneous part of the D. E. (20), the corresponding recurrence relation is found to be

$$a_{k+1} = \frac{(-\frac{1}{2} + k)}{(k+1)} a_k, \tag{48}$$

which leads to

$$w_h(z) = {}_1F_o(\alpha_r; z) = {}_1F_o(-\frac{1}{2}; z), \ |z| \leq 1. \tag{49}$$

The general solutions, satisfying both the D.E. (20) and the A.E. (18), are then given by $y_j(z) = 1 + A_j w_h(z)$, $j = 1, 2$. The constants A_j can be determined from the initial values $y_j(0)$, which are readily found from the A.E. In a conventional order, $y_1(0) = 0$ and $y_2(0) = 2$. Since $w_h(0) = 1$, the set of solutions is

$$\begin{aligned} y_1(z) &= 1 - w_h(z) \\ y_2(z) &= 1 + w_h(z) \end{aligned} \qquad |z| \leq 1. \tag{50}$$

It is interesting to notice that the value of ${}_1F_o(-\frac{1}{2}; z)$ at "unity", i.e., at $z = 1$, where $D_2(z)$ vanishes, can be obtained from (50) and the A.E. (18), which gives $y_1(1) = y_2(1) = 1$. It must then be ${}_1F_o(-\frac{1}{2}; 1) = 0$. A quite similar situation will be found in the cases $n = 3, 4$ and 5.

The $n = 2$ degree will not be prosecuted any longer. It suffices to state that the generalized hypergeometric function (49) can be recognized as the elementary function

$$ {}_1F_o(-\frac{1}{2}; z) = (1-z)^{\frac{1}{2}}, $$

as it ought to be.

IV.2. The solutions for $n = 3$. The replacement of the Ansatz (46) for $|z| \leq 1$ in the D.E. (28) leads to the following set of basic solutions *:

$$ \begin{aligned} w_1^{(1)}(z) &= {}_2F_1(-\tfrac{1}{6}, \tfrac{1}{6}; \tfrac{1}{2}; z^2) \\ w_2^{(1)}(z) &= z\, {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; z^2) \end{aligned} \qquad |z| \leq 1. \tag{51} $$

If the independent variable in the D.E. is changed according to $z \to 1/z$ and the Ansatz (4) is now used in the transformed differential equation, the new set of basic solutions

$$ \begin{aligned} w_1^{(2)}(z) &= \left(z^{\frac{1}{3}}\right) {}_2F_1(-\tfrac{1}{6}, \tfrac{2}{6}; \tfrac{2}{3}; z^{-2}) \\ w_2^{(2)}(z) &= \left(z^{-\frac{1}{3}}\right) {}_2F_1\left(\tfrac{1}{6}, \tfrac{4}{6}; \tfrac{4}{3}; z^{-2}\right) \end{aligned} \qquad |z| \geq 1, \tag{52} $$

is obtained. On the region $|z| \leq 1$, the solutions to the A.E. (23) will be given by

$$ y_j(z) = A_{j1} w_1^{(1)}(z) + A_{j2} w_2^{(1)}(z), \quad j = 1, 2, 3, \tag{53} $$

where the constants are to be obtained from the initial values $y_j(0)$ and $y_j'(0)$ (equations (23) and (27)). Adopting a conventional order, and using ${}_2F_1(\alpha_r; \beta_s; 0) = 1$, the set of solutions

$$ \begin{pmatrix} y_1(z) \\ y_2(z) \\ y_3(z) \end{pmatrix} = \begin{pmatrix} \sqrt{3} & -\frac{1}{3} \\ -\sqrt{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} w_1^{(1)}(z) \\ w_2^{(1)}(z) \end{pmatrix} \tag{54} $$

$$ y_1(0) = \sqrt{3}, \quad y_2(0) = -\sqrt{3}, \quad y_3(0) = 0. $$

is obtained. In order to establish the continuations to the region $|z| > 1$, the values of each $y_j(z)$ at some point on the common boundary $|z| = 1$ are needed. A natural set of values is given precisely at the critical points z_c, $z_c^2 - 1 = 0$, for which the A.E. (23) can be solved. For $z_c \in \{-1, 1\}$, $y_j(z_c) \in \{z_c, -2z_c\}$, and the value $y_j(z_c)$ is doubly repeated. The fact that at the points z_c both hypergeometric functions appearing in (51) are real and positive (the first one

*Here, and in the cases $n = 4, 5$ as well, the identification of the solutions is made somewhat simpler through the change of variable $z \to z^{n-1}$, which leads to the usual form of the D.E. to be satisfied by ${}_pF_q(\alpha_r; \beta_s; z)$, as quoted in [6].

being smaller, the second, greater, than 1) gives more than enough information to determine their values at "unity", as follows: suppose that $y_3(1) = -2$; according to (54), it should then be ${}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; 1) = -3$, which is obviously false; it must then be $y_3(1) = 1$, leading to ${}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}\, 1) = \frac{3}{2}$. A similar trial process with the other roots leads to the results summarized in (55):

(55)

	y_1	y_2	y_3
$z = 1$	1	-2	1
$z = -1$	2	-1	-1

$$ {}_2F_1(-\tfrac{1}{6}, \tfrac{1}{6}; \tfrac{1}{2}; 1) = \tfrac{\sqrt{3}}{2} $$
$$ {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; 1) = \tfrac{3}{2}. $$

Of course, both values of ${}_2F_1(1)$ could have been obtained by means of the well known formula

$$ (56) \qquad {}_2F_1(\alpha_1, \alpha_2; \beta; 1) = \frac{\Gamma(\beta)\Gamma(\beta - \alpha_1 - \alpha_2)}{\Gamma(\beta - \alpha_1)\Gamma(\beta - \alpha_2)}. $$

The essential point is that no such general formula is known for the values ${}_3F_2(1)$ and ${}_4F_3(1)$, which will appear when solving the cases $n = 4$ and 5.

On the region $|z| \geq 1$, the solutions will be constructed as the superposition

$$ (57) \qquad y_j(z) = B_{j1} w_1^{(2)}(z) + B_{j2} w_2^{(2)}(z). $$

To find the constants, it is convenient to replace (57) in the A.E. (23) and take $z \gg 1$, noticing that each hypergeometric function tends to 1 when $(1/z) \to 0$. Collecting equal powers of z, it is then obtained that

$$ (58) \qquad B_{j1} = 2^{\frac{1}{3}} \theta_j, \quad B_{j2} = -2^{-\frac{1}{3}} \theta_j^2, $$

where $\theta_j^3 = -1$, i.e., $\theta_j \in \{e^{i\frac{\pi}{3}}, -1, e^{-i\frac{\pi}{3}}\}$.
Equation (57) then becomes

$$ (59) \qquad y_j(z) = 2^{\frac{1}{3}} \theta_j w_1^{(2)}(z) - 2^{-\frac{1}{3}} \theta_j^2 w_2^{(2)}(z), $$

but the exact assignation of θ_j is still to be found. The first step is the determination of the values at unity for both ${}_2F_1$ appearing in the set (52). By a logical process wholly analogous to the one exposed for $|z| \leq 1$, except for the fact that the values $y_j(z_c)$ are known from (55), the values

$$ (60) \qquad {}_2F_1(-\frac{1}{6}, \frac{2}{6}; \frac{2}{3}; 1) = 2^{-\frac{1}{3}}, \quad {}_2F_1(\frac{1}{6}, \frac{4}{6}; \frac{4}{3}; 1) = 2^{\frac{1}{3}}, $$

are obtained. To fix the correct θ_j, it must be recalled that, due to the existence of branch points at $z_c^2 = 1$, the region should be cut twice. Both cuts will be taken along the real axis, from $-\infty$ to -1, and from $+1$ to $+\infty$, so that the argument of z will be constrained to the intervals $0 \leq \arg z < \pi$ and $\pi \leq \arg z < 2\pi$.

If the values of ${}_2F_1(1)$, from (60), and of the $y_j(z_c)'s$, from (55), are used, the following set of solutions is found on the upper half of the $|z| \geq 1$ region:

$$
(61) \qquad \begin{pmatrix} y_1(z) \\ y_2(z) \\ y_3(z) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{3}} & e^{i\frac{\pi}{3}} \\ -1 & -1 \\ e^{i\frac{\pi}{3}} & e^{-i\frac{\pi}{3}} \end{pmatrix} \begin{pmatrix} 2^{\frac{1}{3}} w_1^{(2)}(z) \\ 2^{-\frac{1}{3}} w_2^{(2)}(z) \end{pmatrix}
$$

$$
(|z| \geq 1, \qquad 0 \quad \leq \text{ argz } < \pi).
$$

For the lower half of the region, $\pi \leq$ argz $< 2\pi$, it is found that $y_1(z)$ preserves the functional form given in (61), buth $y_2(z)$ and $y_3(z)$ permute their former forms, a fact indeed predicted by the elementary theory of algebraic functions.

Closing the case $n = 3$, it can be mentioned that all four ${}_2F_1$ which appear in (51) and (52) can be expressed in terms of elementary functions. For instance, both functions in (52) can be casted in the form

$$
{}_2F_1(\alpha; \alpha + \frac{1}{2}; 1 + 2\alpha; t) = 2^{2\alpha}[1 + (1-t)^{\frac{1}{2}}]^{-2\alpha}.
$$

When the values $\alpha = -\frac{1}{6}$ and $\alpha = \frac{1}{6}$ are replaced, the solutions given in (61) become

$$
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{3}} & e^{i\frac{\pi}{3}} \\ -1 & -1 \\ e^{i\frac{\pi}{3}} & e^{-i\frac{\pi}{3}} \end{pmatrix} \begin{pmatrix} \left[z + (z^2-1)^{\frac{1}{2}}\right]^{\frac{1}{3}} \\ \left[z - (z^2-1)^{\frac{1}{2}}\right]^{\frac{1}{3}} \end{pmatrix},
$$

which can be shown to be the same set of solutions (29) found earlier.

IV.3. The solutions for $n = 4$. The method to be followed is analogous to the one applied to the foregoing $n = 3$ case, so that a succint report is justified.

By means of the Ansätze (46) and (47), the ensuing sets of basic solutions to the D.E. (36) are obtained:

$$
\begin{aligned}
w_1^{(1)} &= {}_3F_2(-\tfrac{1}{12}, \tfrac{2}{12}, \tfrac{5}{12}; \tfrac{1}{3}, \tfrac{2}{3}; z^3) \\
(62) \qquad w_2^{(1)} &= (z)\,{}_3F_2(\tfrac{3}{12}, \tfrac{6}{12}, \tfrac{9}{12}; \tfrac{2}{3}, \tfrac{4}{3}; z^3) \qquad |z| \leq 1. \\
w_3^{(1)} &= (z^2)\,{}_3F_2(\tfrac{7}{12}, \tfrac{10}{12}, \tfrac{13}{12}; \tfrac{4}{3}, \tfrac{5}{3}; z^3)
\end{aligned}
$$

$$
\begin{aligned}
w_1^{(2)} &= (z^{\frac{1}{4}})\,{}_3F_2(-\tfrac{1}{12}, \tfrac{3}{12}, \tfrac{7}{12}; \tfrac{2}{4}, \tfrac{3}{4}; z^{-3}) \\
(63) \qquad w_2^{(2)} &= (z^{-\frac{2}{4}})\,{}_3F_2(\tfrac{2}{12}, \tfrac{6}{12}, \tfrac{10}{12}; \tfrac{3}{4}, \tfrac{5}{4}; z^{-3}) \qquad |z| \geq 1. \\
w_3^{(2)} &= (z^{-\frac{5}{4}})\,{}_3F_2(\tfrac{5}{12}, \tfrac{9}{12}, \tfrac{13}{12}; \tfrac{5}{4}, \tfrac{6}{4}; z^{-3})
\end{aligned}
$$

On the region $|z| \leq 1$, the solutions which must satisfy both the A.E. (32) and the D.E. (36) are written as

$$y_j(z) = \sum_1^3 A_{jk} w_k^{(1)} \qquad j = 1, \ldots, 4, \tag{64}$$

where the matrix elements A_{jk} can be found by means of the initial values $y_j(0)$, $y_j'(0)$ and $y_j''(0)$, which in turn are obtained from the A.E. (32) and the first two equations (35). For the conventional order

$$y_1(0) = 4^{\frac{1}{3}}, \qquad y_2(0) = 4^{\frac{1}{3}} e^{i2\frac{\pi}{3}},$$

$$y_3(0) = 4^{\frac{1}{3}} e^{-i2\frac{\pi}{3}}, \qquad y_4(0) = 0,$$

the matrix A is

$$(A_{jk}) = \begin{pmatrix} 4^{\frac{1}{3}} & -4^{-1} & -4^{-\frac{5}{3}} \\ 4^{\frac{1}{3}} e^{i2\frac{\pi}{3}} & -4^{-1} & -4^{-\frac{5}{3}} e^{-i2\frac{\pi}{3}} \\ 4^{\frac{1}{3}} e^{-i2\frac{\pi}{3}} & -4^{-1} & -4^{-\frac{5}{3}} e^{i2\frac{\pi}{3}} \\ 0 & 3 \cdot 4^{-1} & 0 \end{pmatrix}. \tag{65}$$

To establish the continuations onto the region $|z| > 1$, the values of the y_j's at some point on the boundary must be known. The set of possible values can be obtained from the A.E. (32) at the natural set determined by $z^3 - 1 = 0$: for each $z_c \in \left\{1, e^{i2\frac{\pi}{3}}, e^{-i2\frac{\pi}{3}}\right\}$, we have the values

$$y_j(z_c) \in \left\{z_c, z_c(-1, i\sqrt{2}), -z_c(1 + i\sqrt{2})\right\}. \tag{66}$$

of the solutions. The values at $z_c^3 = 1$ for each ${}_3F_2$ appearing in equation (62) can be obtained by a systematic trial process, leading to:

$$\begin{aligned} {}_3F_2(-\tfrac{1}{12}, \tfrac{2}{12}, \tfrac{5}{12}; \tfrac{1}{3}, \tfrac{2}{3}; 1) &= 2^{-\frac{2}{3}} \left[\sqrt{\tfrac{2}{3}} + \tfrac{2}{3}\right] \\ {}_3F_2(\tfrac{3}{12}, \tfrac{6}{12}, \tfrac{9}{12}; \tfrac{2}{3}, \tfrac{4}{3}; 1) &= \tfrac{4}{3} \\ {}_3F_2(\tfrac{7}{12}, \tfrac{10}{12}, \tfrac{13}{12}; \tfrac{4}{3}, \tfrac{5}{3}; 1) &= 2^{\frac{11}{3}} \left[\sqrt{\tfrac{2}{3}} - \tfrac{2}{3}\right]. \end{aligned} \tag{67}$$

By means of the set of ${}_3F_2(1)$'s from (67), the $y_j(z_c)$'s are readily found to be:

(68)

z_c	$y_1(z_c)$	$y_2(z_c)$	$y_3(z_c)$	$y_4(z_c)$
1	1	$(-1+i\sqrt{2})$	$-(1+i\sqrt{2})$	1
$e^{i2\frac{\pi}{3}}$	$-z_c(1+i\sqrt{2})$	z_c	$z_c(-1+i\sqrt{2})$	z_c
$e^{-i2\frac{\pi}{3}}$	$z_c(-1+i\sqrt{2})$	$-z_c(1+i\sqrt{2})$	z_c	z_c

On the region $|z| \geq 1$, the solutions will be given by:

$$y_j(z) = \sum_{1}^{3} B_{jk}\, w_k^{(2)}(z), \quad j = 1, \ldots, 4, \tag{69}$$

and the matrix elements B_{jk} can be determined, up to a phase factor, by replacing y_j from (63) in the A.E. (32), taking again $z \gg 1$, and considering that each ${}_3F_2(z^3)$ in (63) behaves as $1 + O(z^{-3})$ for $(1/z) \to 0$. It is so obtained that

$$B_{j1} = 3^{\frac{1}{4}}\theta_j, \quad B_{j2} = -3^{-\frac{2}{4}}\theta_j^2, \quad B_{j3} = -2^{-1} \cdot 3^{-\frac{5}{4}}\,\theta_j^3, \tag{70}$$

where $\theta_j^4 = -1$, i.e., $\theta_j \in \{u, iu, -u, -iu\}$, $u = e^{i\frac{\pi}{4}}$.

The θ_j corresponding to a certain y_j can be determined by means of the values (68). For that purpose, the ${}_3F_2$ from (63) at unity must be found. Once again, a trial process which makes use of the $y_j(z_c)$'s from (68) leads to the values

$$\begin{aligned} {}_3F_2(-\tfrac{1}{12}, \tfrac{3}{12}, \tfrac{7}{12}; \tfrac{2}{4}, \tfrac{3}{4}; 1) &= 2^{-1} \cdot 3^{-\frac{1}{4}}(\sqrt{2}+1) \\ {}_3F_2(\tfrac{2}{12}, \tfrac{6}{12}, \tfrac{10}{12}; \tfrac{3}{4}, \tfrac{5}{4}; 1) &= \sqrt{\tfrac{3}{2}} \\ {}_3F_2(\tfrac{5}{12}, \tfrac{9}{12}, \tfrac{13}{12}; \tfrac{5}{4}, \tfrac{6}{4}; 1) &= 3^{\frac{5}{4}}(\sqrt{2}-1). \end{aligned} \tag{71}$$

Due to the existence of three branch points at $z_c^3 = 1$, the $|z| \geq 1$ region must be cut three times, and the θ_j's are to be found separately on each of the subregions. We only report the final result for the matrix (B_{jk}) on each region:

(i) On the "fundamental" region $0 \leq \arg z < 2\frac{\pi}{3}$,

$$(72) \qquad (B_{jk}) = \begin{pmatrix} 3^{\frac{1}{4}}u^* & i3^{-\frac{2}{4}} & 2^{-1}\cdot 3^{-\frac{5}{4}}u \\ -3^{\frac{1}{4}}u^* & i3^{-\frac{2}{4}} & -2^{-1}\cdot 3^{-\frac{5}{4}}u \\ -3^{\frac{1}{4}}u & -i3^{-\frac{2}{4}} & -2^{-1}\cdot 3^{-\frac{5}{4}}u^* \\ 3^{\frac{1}{4}}u & -i3^{-\frac{2}{4}} & 2^{-1}\cdot 3^{-\frac{5}{4}}u^* \end{pmatrix}, \qquad u = e^{i\frac{\pi}{4}}.$$

(ii) On the region $\frac{2\pi}{3} \leq \arg z < 4\frac{\pi}{3}$, only one permutation of the y_j's is needed: in the matrix (B_{jk}) from (i), the second and third row are interchanged.

(iii) On the region $4\frac{\pi}{3} \leq \arg z < 2\pi$, only one permutation is again needed: in the new matrix from (ii), the second and fourth row must be now interchanged.

Note that $y_1(z)$ preserves its form on the whole $|z| > 1$ region. The permutations take place only on the subset $\{y_2, y_3, y_4\}$.

IV.4. The solutions for $n = 5$. There is nothing essentially new in the case $n = 5$, and only a short account of results will be given. Through the replacement of the Ansätze (46) and (47) in the D.E. (42), the two sets of basic solutions are obtained:

$$(73) \qquad |z| \leq 1: \quad \begin{aligned} w_1^{(1)} &= {}_4F_3(-\tfrac{1}{20}, \tfrac{3}{20}, \tfrac{7}{20}, \tfrac{11}{20}; \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}; z^4) \\ w_2^{(1)} &= (z)\,{}_4F_3(\tfrac{4}{20}, \tfrac{8}{20}, \tfrac{12}{20}, \tfrac{16}{20}; \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{5}{4}; z^4) \\ w_3^{(1)} &= (z^2)\,{}_4F_3(\tfrac{9}{20}, \tfrac{13}{20}, \tfrac{17}{20}, \tfrac{21}{20}; \tfrac{3}{4}, \tfrac{5}{4}, \tfrac{6}{4}; z^4) \\ w_4^{(1)} &= (z^3)\,{}_4F_3(\tfrac{14}{20}, \tfrac{18}{20}, \tfrac{22}{20}, \tfrac{26}{20}; \tfrac{5}{4}, \tfrac{6}{4}, \tfrac{7}{4}; z^4) \end{aligned}$$

$$(74) \qquad |z| \geq 1: \quad \begin{aligned} w_1^{(2)} &= (z^{\frac{1}{5}})\,{}_4F_3\left(-\tfrac{1}{20}, \tfrac{4}{20}, \tfrac{9}{20}, \tfrac{14}{20}; \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; z^{-4}\right) \\ w_2^{(2)} &= (z^{-\frac{3}{5}})\,{}_4F_3\left(\tfrac{3}{20}, \tfrac{8}{20}, \tfrac{13}{20}, \tfrac{18}{20}; \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{6}{5}; z^{-4}\right) \\ w_3^{(2)} &= (z^{-\frac{7}{5}})\,{}_4F_3\left(\tfrac{7}{20}, \tfrac{12}{20}, \tfrac{17}{20}, \tfrac{22}{20}; \tfrac{4}{5}, \tfrac{6}{5}, \tfrac{7}{5}; z^{-4}\right) \\ w_4^{(2)} &= (z^{-\frac{11}{5}})\,{}_4F_3\left(\tfrac{11}{20}, \tfrac{16}{20}, \tfrac{21}{20}, \tfrac{26}{20}; \tfrac{6}{5}, \tfrac{7}{5}, \tfrac{8}{5}; z^{-4}\right) \end{aligned}$$

For $|z| \leq 1$, the solutions are written

$$(75) \qquad y_j(z) = \sum_{1}^{4} A_{jk}\, w_k^{(1)}(z) \qquad j = 1, \ldots, 5$$

and the A_{jk} are found by the initial values $y_j(0), y_j'(0), y_j''(0)$ and $y_j'''(0)$, obtained from the A.E. (38) and the first three equations in (41). The resulting

(A_{jk}) matrix, whose first column contains the conventional assignments for the $y_j(0)$'s, is

$$(A_{jk}) = \begin{pmatrix} 5^{\frac{1}{4}} & -5^{-1} & -2^{-1}\cdot 5^{-\frac{5}{4}} & -2\cdot 5^{-\frac{5}{2}} \\ i5^{\frac{1}{4}} & -5^{-1} & i2^{-1}\cdot 5^{-\frac{5}{4}} & 2\cdot 5^{-\frac{5}{2}} \\ -5^{\frac{1}{4}} & -5^{-1} & 2^{-1}\cdot 5^{-\frac{5}{4}} & -2\cdot 5^{-\frac{5}{2}} \\ -i5^{\frac{1}{4}} & -5^{-1} & -i2^{-1}\cdot 5^{-\frac{5}{4}} & 2\cdot 5^{-\frac{5}{2}} \\ 0 & 4\cdot 5^{-1} & 0 & 0 \end{pmatrix}. \tag{76}$$

To establish the continuations of the solutions to $|z| > 1$, their values at points on the common boundary are needed. In the process of finding those values, the ${}_4F_3$ from (73) at unity are also obtained. Once more, the set of points given by $z_c^4 - 1 = 0$ is convenient. It is found that the A.E. (38) has $y(z_c) = z_c$ as a double root, and the remaining ones must be solutions of the cubic

$$y^3 + 2z_c y^2 + 3z_c^2 y + 4z_c^3 = 0. \tag{77}$$

Equation (77) can be reduced by the procedure developed in Section III.3, and explicitly solved using equation (29). Summarizing, for $z_c \in \{1, i, -1, -i\}$,

$$y_j(z_c) \in \{z_c, z_c Y_1, z_c Y_2, z_c Y_3\}, \tag{78}$$

where

$$\begin{aligned} Y_1 &= -\tfrac{\sqrt{5}}{3}(r_1 - r_2) - \tfrac{2}{3} \\ Y_2 &= i\tfrac{\sqrt{5}}{3}\left(e^{-\frac{i\pi}{6}} r_1 + e^{\frac{i\pi}{6}} r_2\right) - \tfrac{2}{3} \\ Y_3 &= Y_2^* \\ r_{1,2} &= \sqrt[3]{\tfrac{\sqrt{54}\pm 7}{5}}. \end{aligned} \tag{79}$$

Using the set (78) of possible values for the solutions, the ${}_5F_3(1)$ and the exact assignments $y_j(z_c)$ are obtained:

(80)

$$\begin{aligned} {}_4F_3\left(-\tfrac{1}{20}, \tfrac{3}{20}, \tfrac{7}{20}, \tfrac{11}{20}; \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}; 1\right) &= \tfrac{5^{\frac{1}{4}}}{12}\left[\sqrt{3}(r_1 + r_2) + (r_1 - r_2) + \sqrt{5}\right] \\ {}_4F_3\left(\tfrac{4}{20}, \tfrac{8}{20}, \tfrac{12}{20}, \tfrac{16}{20}; \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{5}{4}; 1\right) &= \tfrac{5}{4} \\ {}_4F_3\left(\tfrac{9}{20}, \tfrac{13}{20}, \tfrac{17}{20}, \tfrac{21}{20}; \tfrac{3}{4}, \tfrac{5}{4}, \tfrac{6}{4}; 1\right) &= \tfrac{5^{\frac{7}{4}}}{6}\left[\sqrt{3}(r_1 + r_2) - (r_1 - r_2) - \sqrt{5}\right] \\ {}_4F_3\left(\tfrac{14}{20}, \tfrac{18}{20}, \tfrac{22}{20}, \tfrac{26}{20}; \tfrac{5}{4}, \tfrac{6}{4}, \tfrac{7}{4}; 1\right) &= \tfrac{5^3}{24}\left[2(r_1 + r_2) - \sqrt{5}\right] \end{aligned}$$

(81)

z_c	$y_1(z_c)$	$y_2(z_c)$	$y_3(z_c)$	$y_4(z_c)$	$y_5(z_c)$
1	1	Y_2	Y_1	Y_3	1
i	iY_3	i	iY_2	iY_1	i
-1	$-Y_1$	$-Y_3$	-1	$-Y_2$	-1
$-i$	$-iY_2$	$-iY_1$	$-iY_3$	$-i$	$-i$

For the region $|z| \geq 1$, the solutions will be given by

$$y_j(z) = \sum_{1}^{4} B_{jk}\, w_k^{(2)} \qquad j = 1, \ldots, 5, \tag{82}$$

and the B_{jk} are determined, up to a phase factor, through the replacement of (82) in the A.E. (38) and taking $z \gg 1$. It is then obtained that

$$\begin{aligned} B_{j1} &= 4^{\frac{1}{5}}\theta_j, & B_{j2} &= -4^{-\frac{3}{5}}\theta_j^2, \\ B_{j4} &= -4^{-\frac{11}{5}}\theta_j^4; & B_{j3} &= -4^{-\frac{7}{5}}\theta_j^3, \end{aligned} \tag{83}$$

where $\theta_j^5 = -1$, i.e., $\theta_j \in \left\{ e^{\frac{i\pi}{5}}, -e^{\frac{i2\pi}{5}}, -1, e^{-\frac{i2\pi}{5}}, -e^{-\frac{i\pi}{5}} \right\}$. To find the θ_j's, the correspondence between the $y_j(z_c)$'s from(82) and the values fixed in (81) must be determined. Once more, the finding of the ${}_4F_3$ in (74) at unity is necessary. They are given by

$$\begin{aligned} {}_4F_3(-\tfrac{1}{20}, \tfrac{4}{20}, \tfrac{9}{20}, \tfrac{14}{20}; \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; 1) &= 4^{-\frac{1}{5}}a \\ {}_4F_3(\tfrac{3}{20}, \tfrac{8}{20}, \tfrac{13}{20}, \tfrac{18}{20}; \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{6}{5}; 1) &= 4^{\frac{3}{5}}b \\ {}_4F_3(\tfrac{7}{20}, \tfrac{12}{20}, \tfrac{17}{20}, \tfrac{22}{20}; \tfrac{4}{5}, \tfrac{6}{5}, \tfrac{7}{5}; 1) &= 4^{\frac{7}{5}}c \\ {}_4F_3(\tfrac{11}{20}, \tfrac{16}{20}, \tfrac{21}{20}, \tfrac{26}{20}; \tfrac{6}{5}, \tfrac{7}{5}, \tfrac{8}{5}; 1) &= 4^{\frac{11}{5}}d, \end{aligned} \tag{84}$$

where a, d, b, and c are solutions to the system

$$
\text{(85)} \quad
\begin{aligned}
a+d &= \tfrac{1}{6}\left[2\left(\sqrt{5}+1\right)+\left(\sqrt{5}-1\right)\left(r_1-r_2\right)\right] \\
a-d &= \tfrac{1}{10}\sqrt{\tfrac{5}{3}}\left(r_1+r_2\right)\left[2\sqrt{5+2\sqrt{5}}-\sqrt{10-2\sqrt{5}}\right] \\
b+c &= \tfrac{1}{10}\sqrt{\tfrac{5}{3}}\left(r_1+r_2\right)\left[\sqrt{10+2\sqrt{5}}-2\sqrt{5-2\sqrt{5}}\right] \\
b-c &= \tfrac{1}{6}\left[\left(\sqrt{5}+1\right)\left(r_1-r_2\right)-2\left(\sqrt{5}-1\right)\right],
\end{aligned}
$$

and r_1, r_2 are defined in (79).

Finally, due to the existence of branch points at $z^4 = 1$, the region $|z| > 1$ has to be cut four times, and the assignation of the θ_j's must be done separately on each of the sub-regions. We only report the corresponding matrix elements B_{jk} for (82):

(i) For $0 \leq \arg z < \frac{\pi}{2}$, the fundamental region,

$$
\text{(86)} \quad (B_{jk}) = \begin{pmatrix}
4^{\frac{1}{5}}u^* & -4^{-\frac{3}{5}}u^{*2} & 4^{-\frac{7}{5}}u^2 & 4^{-\frac{11}{5}}u \\
-4^{\frac{1}{5}}u^{*2} & 4^{-\frac{3}{5}}u & -4^{-\frac{7}{5}}u^* & -4^{-\frac{11}{5}}u^2 \\
-4^{\frac{1}{5}} & -4^{-\frac{3}{5}} & 4^{-\frac{7}{5}} & -4^{-\frac{11}{5}} \\
-4^{\frac{1}{5}}u^2 & 4^{-\frac{3}{5}}u^* & -4^{-\frac{7}{5}}u & -4^{-\frac{11}{5}}u^{*2} \\
4^{\frac{1}{5}}u & -4^{-\frac{3}{5}}u^2 & 4^{-\frac{7}{5}}u^{*2} & 4^{-\frac{11}{5}}u^*
\end{pmatrix}
$$

$$
(u = e^{\frac{i\pi}{5}}).
$$

(ii) For $\frac{\pi}{2} \leq \arg z < \pi$, the second and fifth rows are permuted in (86).

(iii) For $\pi \leq \arg z < 3\frac{\pi}{2}$, the third and fifth rows are permuted in the resulting matrix (ii).

(iv) For $3\frac{\pi}{2} \leq \arg z < 2\pi$, the fourth and the fifth rows of matrix (iii) are permuted.

Once more, the root $y_1(z)$ preserves its form on the whole $|z| > 1$ region, as the permutations only occur within the subset $\{y_2, y_3, y_4, y_5\}$.

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