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# Elementary remarks on Anderson's hyperfinite random walk

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ABSTRACT. We give a combinatorial proof that the standard part of Anderson's hyperfinite random walk has the Gaussian distribution. We also show the connection between moments of Gaussian variables and some combinatorial properties.

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Following the notation and the definition of Anderson's nonstandard construction of Brownian motion given in [1, p. 78-84], we let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, $\Delta t = N^{-1}$, $T = \{\Delta t, 2\Delta t, \ldots, 1\}$, and let $\mu$ be the counting measure on $\Omega = \{-1, 1\}^T$. We define the hyperfinite random walk as $B : \Omega \times T \to {}^*\mathbb{R}$ such that $B(\omega, t) = B_t(\omega) = \sum_{s<t} \omega(s)\sqrt{\Delta t}$. Then under the Loeb measure of $\mu$, the standard part of $B$ forms a Brownian motion.

The first remark here is a combinatorial proof that ${}^\circ B_t$ has the Gaussian distribution $\mathcal{N}(0, t)$ (mean 0 and variance $t$). Although it is well-known that this can be done, the author was unable to find a reference; instead the result is usually proved by using either the central limit theorem or the Fourier transform. The second remark gives a combinatorial explanation of the significance of the moments of Gaussian random variables.

## 1. Gaussian distribution

We verify that for $t \in T$, ${}^\circ t > 0$, $x \in {}^*\mathbb{R}$, $\mu\{B_t \leq x\} \approx \int_{-\infty}^{{}^\circ x} \gamma(z,t)dz$, where $\gamma(z,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$, the Gaussian density of $\mathcal{N}(0,t)$.

By writing $t = H\Delta t$, $H$ infinite, and let $\Delta u = H^{-1}$, $\tilde{B}_r(\omega) = \sum_{s<r} \omega(s)\sqrt{\Delta u}$, we note that $B_r(\omega) = y$ iff $\tilde{B}_r(\omega) = \frac{y}{\sqrt{t}}$. Notice that $\tilde{B}$ can be thought of as a Brownian motion on another Anderson's model. (Using $\Delta u$ as the step size of the time line). So by changing the variable in the integral, it suffices to check

$$\mu\{B_1 \leq x\} \approx \int_{-\infty}^{{}^\circ x} \gamma(z,1)dz. \tag{1}$$

Write $y = M\sqrt{\Delta t}$. If $B_1(\omega) = y$, then $\sum_{t<1} \omega(t) = M$. So in the coordinates of $\omega$, $\frac{1}{2}(N+M)$ entries are 1 and the rest are $-1$. Therefore the proportion of such $\omega$ in $\Omega$ is $\frac{1}{2^N}\binom{N}{\frac{1}{2}(N+M)}$.

Suppose that $x = K\sqrt{\Delta t}$ is a position attained by some path at time 1. Then at time 1, the set of possible positions not exceeding $x$ which are attained by some paths is $S_x = \{K\sqrt{\Delta t}, (K-2)\sqrt{\Delta t}, \cdots, -N\sqrt{\Delta t}\}$. So

$$\mu\{B_1 \leq x\} = \sum_{M\sqrt{\Delta t} \in S_x} \frac{1}{2^N}\binom{N}{\frac{1}{2}(N+M)} = \sum_{z \in S_x} \Gamma(z)\Delta s,$$

where $\Gamma(M\sqrt{\Delta t}) = \frac{\sqrt{N}}{2^{N+1}}\binom{N}{\frac{1}{2}(N+M)}$, $\Delta s = 2\sqrt{\Delta t}$. Hence we only need to show the following

$$\text{If } z = M\sqrt{\Delta t} \text{ is finite, then } \Gamma(z) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{{}^\circ z^2}{2}}. \tag{2}$$

As a consequence of this, $\Gamma$ is $S-$integrable and (1) holds.

First recall the Stirling's formula:

$$m! = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m e^{\frac{\varepsilon}{m}}, \text{ for some } 0 < \varepsilon < \tfrac{1}{12}. \tag{3}$$

To prove (2), we first write

$$\Gamma(z) = \frac{\sqrt{N}}{2^{N+1}} \frac{\sqrt{2\pi N}\left(\frac{N}{e}\right)^N \varepsilon'}{\sqrt{\pi(N+M)}\left(\frac{N+M}{2e}\right)^{\frac{N+M}{2}} \sqrt{\pi(N-M)}\left(\frac{N-M}{2e}\right)^{\frac{N-M}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{N^2}{N^2 - M^2}} \left(\frac{N}{N+M}\right)^{\frac{N+M}{2}} \left(\frac{N}{N-M}\right)^{\frac{N-M}{2}} \varepsilon',$$

for some $\varepsilon' \approx 1$. The second term $\sqrt{\frac{N^2}{N^2-M^2}} = \sqrt{\frac{1}{1-\frac{z^2}{N}}} \approx 1$. So it suffices to show

$$\left(\frac{N}{N+M}\right)^{N+M}\left(\frac{N}{N-M}\right)^{N-M} \approx e^{-z^2}. \tag{4}$$

**Claim.** *If $\frac{M^2}{H}$ is finite, $H > 0$ is infinite, then* $\frac{e^M}{(1+\frac{M}{H})^H} \approx e^{\frac{M^2}{2H}}$.

*Proof.* By expanding $e^{\frac{M}{H}}$ into a power series, the left side has the form $(1+\frac{A}{M+H})^H$, where

$$A = H\sum_{k\geq 2}\frac{1}{k!}(\frac{M}{H})^k = \frac{M^2}{H}\sum_{k\geq 0}\frac{1}{(k+2)!}(\frac{M}{H})^k \approx \frac{M^2}{2H}.$$

(Use $\frac{M}{H} \approx 0$). Since $(1+\frac{A}{M+H})^M = (1+\frac{A\varepsilon''}{M})^M$ for some $\varepsilon'' \approx 0$, we have $(1+\frac{A}{M+H})^H \approx (1+\frac{A}{M+H})^{M+H} \approx e^{\frac{M^2}{2H}}$. The claim is now proved.

Now rewrite the left side of (4)

$$(1-\frac{M}{N+M})^{N+M}\left(1+\frac{M}{N-M}\right)^{N-M}$$

$$= (1-\frac{M^2}{(N+M)^2})^{N+M}\frac{\left(1+\frac{M}{N-M}\right)^{N-M}}{\left(1+\frac{M}{N+M}\right)^{N+M}}\frac{e^M}{e^M}$$

$$\approx (1-\frac{M^2}{(N+M)^2})^{N+M} \text{ (by applying the claim to } H = N \pm M)$$

$$= \left(1-\frac{1}{N+M}\left(\frac{z^2}{1+\frac{z}{\sqrt{N}}}\right)\right)^{N+M} \approx e^{-z^2}.$$

So (4) holds as required.

## 2. Moments of Gaussian variables

Suppose that $\xi$ is a Gaussian random variable of mean 0 and variance $t$. Then by a simple argument using a moment generating function, one obtains

$$E[\xi^{2n}] = \frac{(2n)!}{2^n n!}t^n. \tag{5}$$

On the other hand, there are exactly $\binom{2n}{n} n! \frac{1}{2^n} = \frac{(2n)!}{2^n n!}$ ways of partitioning a set of size $2n$ into $n$ pairs. We now explain their connections by using Anderson's model.

Write $\omega_s = \omega(s)\sqrt{\Delta t}$ for $\omega \in \Omega$. So $B_t(\omega) = \sum_{s<t} \omega_s$ and $E[B_t^{2n}] = E[(\sum_{s<t} \omega_s)^{2n}] = E[\sum_{s_1,\dots,s_{2n}<t} \omega_{s_1} \cdots \omega_{s_{2n}}]$. By the independence, each product $\omega_{s_1} \cdots \omega_{s_{2n}}$ has zero expectation unless it can be put into the form

$$\omega_{u_1}^2 \cdots \omega_{u_n}^2 = (\Delta t)^n = \frac{1}{N^n}.$$

Let $A_{t,n}$ (respectively $C_{t,n}$) be the $2n-$tuples $(s_1, \dots s_{2n})$ from $\{u \in T : u < t\}$, where $(s_1, \cdots s_{2n})$ can be divided into $n$ pairs (respectively $n$ distinct pairs) such that each pair consists of indentical elements. So

$$E[B_t^{2n}] = \sum_{(s_1,\dots,s_{2n})\in A_{t,n}} E[\omega_{s_1} \cdots \omega_{s_{2n}}] = \sum_{(s_1,\dots,s_{2n})\in A_{t,n}} N^{-n} = |A_{t,n}| N^{-n}.$$

From the definition, we have the following inequality for $A_{t,n}$ and $C_{t,n}$ :

$$\frac{(2n)!}{2^n n!} Nt(Nt-1)\cdots(Nt-n+1) = |C_{t,n}| \le |A_{t,n}| \le \frac{(2n)!}{2^n n!}(Nt)^n. \tag{6}$$

Therefore $E[B_t^{2n}] \approx \frac{(2n)!}{2^n n!} t^n$. From the previous section and nonstandard integration theory, $E[B_t^{2n}]$ lifts $E[\xi^{2n}]$, so (5) holds.

## References

1. S. Albeverio, J. E. Fenstad, R. Høegh-Krohn, T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York, 1986.

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