Asymptotic formulae of generalized Chebyshev functions*

CATALINA CALDERÓN
MARÍA JOSÉ ZÁRATE
Universidad del País Vasco, ESPAÑA

ABSTRACT. In this paper we study the behavior of certain generalized Chebyshev functions over the square r-free integers and will prove Selberg's inequality for such functions.

Key words and phrases. Selberg's formula, Chebyshev functions, prime number theorem.

1991 Mathematics Subject Classification. Primary 11N05. Secondary 11N69.

1. Introduction and preliminaries

In the last three decades a number of elementary proofs of the prime number theorem have appeared (see [3] for a survey). Most of these proofs are based, at least in part, on ideas from the original proof of Erdös [4] and Selberg [9]. One of the main ingredients of the Erdös-Selberg proof is Selberg's formula

$$\sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x)$$

which appears, in some form, in almost all these proofs.

* Work supported by the University of the Basque Country and DGICYT PB91-0449.
Several proofs of Selberg's formula appeared soon (see [7], [11], [12]). Bombieri [1] used a class of analogues of Selberg's formulae of greater weight to improve the error term in the prime number theorem. Wirsing [13] introduced a recursion whereby each time that an error estimate was found for \( \pi(x) - li(x) \), it was used again to obtain an improved form of Selberg's formula.

H.N. Shapiro [11] obtained a generalization of Selberg's formula and its equivalences. Also, A. Selberg [10] gave an elementary proof of the prime number theorem for arithmetic progressions. Its starting point is the following formula

\[
\sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q
\]

\[
= \frac{2}{\phi(k)} x \log x + O(x), \quad (k, \ell) = 1.
\]

(1.2)

In 1958 G.J. Rieger obtained formula (1.2) in an algebraic field [8].

Applying the inversion formula, K. Iseki and T. Tatuzawa [12] established the formula

\[
\psi(x) \log x + \sum_{n \leq x} \psi(x/n) \Lambda(n) = 2x \log x + O(x), \quad \psi(x) = \sum_{n \leq x} \Lambda(n).
\]

(1.3)

This result may be used in place of Selberg's formula to prove the prime number theorem.

In this paper we will prove Selberg's inequality for the generalized Chebyshev functions \( \Psi_{r,k}(x) \), \( \theta_{r,k}(x) \) which will be defined in (1.4). Let \( G_2 \) be the set of square integers and let \( Q_r \) be the set of \( r \)-free integers \( (r > 2) \). If \( r = 1 \) we take \( G_2 \cap Q_1 = \{1\} \). If \( r \geq 2 \) is even, \( G_2 \cap Q_r = G_2 \cap Q_{r-1} \), so that it is sufficient to consider the case when \( r \) is odd.

We denote by \( C_{r,k} \) the set of natural numbers \( n \) such that \( n = N \) or \( n = pmN \), where \( N \in G_2 \cap Q_r \) and \( m \) is an arbitrary integer with \( \omega(m) \leq k - 1 \), \( (pm, N) = 1 \), \( \omega(m) \) being the number of distinct prime factors of \( m \).

For positive integers \( r \) and \( k \), let \( \Psi_{r,k}(x) \), \( \theta_{r,k}(x) \) be the summatory functions

\[
\Psi_{r,k}(x) = \sum_{n \leq x} \Lambda_{r,k}^{*}(n), \quad \theta_{r,k}(x) = \sum_{n \leq x} \Lambda_{r,k}^{*}(n)
\]

(1.4)

where \( \Lambda_{r,k}^{*}(n) \) is the function of Mangoldt type

\[
\Lambda_{r,k}^{*}(n) = \sum_{d \delta = n} \mu_r^{*}(d) \log^k \delta,
\]

(1.5)
\(\mu_r^*(n)\) being given by \(\mu_r^*(1) = 1, \mu_r^*(n) = 0\) if \(p^{r+1} | n\) for some prime \(p\), and 
\(\mu_r^*(n) = (-1)^{\Omega(n)}\) if \(n = \prod p_i^{\alpha_i}, 0 \leq \alpha_i \leq r\), with \(\Omega(n) = \sum \alpha_i\) (observe that \(\mu_1^*(n)\) is the Möbius function \(\mu(n)\)).

The functions \(\Lambda_{r,k}^*(n)\) generalize the well-known von Mangoldt function \(\Lambda(n)\) and also Ivić functions \(\Lambda_k(n)\) (cf. [5]). Since

\[
\sum_{n=1}^{\infty} \mu_r^*(n)n^{-s} = (\zeta(2s)/\zeta(s))\gamma_r((r + 1)s)
\]

for \(\text{Re}(s) > 1\), where \(\gamma_r(s) = 1/\zeta(s)\) if \(r \geq 1\) is odd and \(\gamma_r(s) = \zeta(s)/\zeta(2s)\) if \(r \geq 2\) is even, the Dirichlet series for \(\Lambda_{r,k}^*(n)\) is

\[
\sum_{n=1}^{\infty} \Lambda_{r,k}^*(n)n^{-s} = (-1)^k(\zeta(k)(s)/\zeta(s))\zeta(2s)\gamma_r((r + 1)s),
\]

which is absolutely convergent for \(\text{Re}(s) > 1\). \(\zeta(s)\) is the Riemann zeta function and \(\zeta(k)(s)\) its \(k\)-th order derivative. The function \(\Lambda_{r,k}^*(n)\) has the property \(\Lambda_{r,k}^*(n) = \sum_{d \delta = n} \Lambda_{r,k}^*(d)h_r(\delta), h_r(n) = \sum_{d \delta = n} \mu_r^*(d)\). For odd integer \(r\), \(h_r(n)\) is the characteristic function of the square \(r\)-free integers.

Moreover, from [2, Theorem 2] we know that for fixed positive integers \(r, k\), there exists a constant \(C = C(k) > 0\) such that

\[
\Psi_{r,k}^*(x) = xP_{k-1}(\log x) + O(x \exp(-C\delta(x))),
\]

(1.6) \(\delta(x) = \log^{3/5} x(\log \log x)^{-1/5}\)

where \(P_{k-1}(t)\) is a polynomial of degree \(k - 1\) in \(t\) (the case \(r = k = 1\) is formula (12.26) of [6]). An extension of (1.3) for \(\Psi_{r,k}^*(x)\) is given in [2, Theorem 3]. We will give here other formulae of type (1.3) and (1.1) for \(\Psi_{r,k}^*(x)\) and \(\theta_{r,k}^*(x)\), the sums being extended over a certain class of integers (mentioned above).

2. The theorems

Theorem 1. For integers \(r, k \geq 1\), with \(r\) odd, we have

\(0 \leq \Psi_{r,k}^*(x) - \theta_{r,k}^*(x) \ll x^{1/2}\log^{2k} x\).

Proof. From (1.4) we obtain that

\[
\Psi_{r,k}^*(x) - \theta_{r,k}^*(x) = \sum_{\substack{n \leq x \\text{n} \notin C_{r,k}}} \Lambda_{r,k}^*(n).
\]

\(\sum_{n \leq x} \Lambda_{r,k}^*(n)\)
Moreover let \( n = \prod_{i=1}^{s} p_i^{\alpha_i} \) be the factorial descomposition of the positive integer \( n \) and let \( \delta_i = \min\{\alpha_i, r\} \). Using (1.5) and the definition of \( \mu^*_r(n) \) we get have that

\[
\Lambda^*_r(k,n) = \sum_{\beta_1=0}^{\delta_1} \sum_{\beta_2=0}^{\delta_2} \cdots \sum_{\beta_s=0}^{\delta_s} (-1)^{\sum_{i=1}^{s} \beta_i} (\sum_{i=1}^{s} (\alpha_i - \beta_i) \log p_i)^k =
\]

\[
= \sum_{n_1 + \cdots + n_s = k} \frac{k!}{n_1! \cdots n_s!} \log^{n_1} p_1 \log^{n_2} p_2 \cdots \log^{n_s} p_s S(n_1) \cdots S(n_s),
\]

where \( S(n_i) = \sum_{\beta=0}^{\delta_i} (-1)^{\beta} (\alpha_i - \beta)^{n_i} \). Since \( 0 \leq S(n_i) < \alpha_i^{n_i}, i = 1, \ldots, s \) from (2.3) we get

\[
\Lambda^*_r(k,n) \leq \log^k n.
\]

If at least \( k + 1 \) exponents \( \alpha_i \) are such that \( \alpha_i \geq r \) or \( \alpha_i < r \) is odd then \( \Lambda^*_r(k,n) = 0 \) and, from the definition of \( C_{r,k} \),

\[
\{ n : n \notin C_{r,k}, \quad \Lambda^*_r(k,n) \neq 0 \} \subset M_{r,k} = \bigcup_{j=1}^{k} M_{r,k}^j,
\]

where

\[
M_{r,k}^j = \{ n = N \prod_{s=1}^{j} p_s^{\alpha_s} : 1 < \alpha_s (\text{odd}) < r \text{ or } \alpha_s \geq r, N \in G_2 \cap Q_r, (p_s, N) = 1 \}.
\]

We consider the sums

\[
S_j = \sum_{\substack{n \leq x \\ n \in M_{r,k}^j}} \log^k n \quad (1 \leq j \leq k).
\]

Let \( j = 1 \), \( N = 1 \) and let \( \pi(x) \) be the number of prime numbers which do not exceed \( x \). Since the order of magnitude of \( \pi(x) \) is \( x / \log x \), then

\[
\sum_{p^\alpha \leq x \atop \alpha \geq 2} \log^k p^\alpha \ll \sum_{2 \leq \alpha \leq \log_2 x} \alpha^k \sum_{p \leq x^{1/\alpha}} \log^k p \\
\ll \sum_{2 \leq \alpha \leq \log_2 x} \alpha^k \pi(x^{1/\alpha}) \log^k (x^{1/\alpha}) \ll x^{1/2} \log^{k+1} x.
\]
Therefore

\[ S_1 = \sum_{n \leq x, n \in M_{r,k}^1} \log^k(n) \ll \max_{0 \leq \beta \leq k} \sum_{p^\alpha \leq x, \alpha \geq 2} \log^{k-\beta} p^\alpha \sum_{N \leq x/p^\alpha} \log^\beta N \]

(2.6)

\[ \ll \max_{0 \leq \beta \leq k} x^{1/2} \log^\beta x \sum_{p^\alpha \leq x, \alpha \geq 2} \frac{\log^{k-\beta} p^\alpha}{\sqrt{p^\alpha}} \ll x^{1/2} \log^{k+1} x, \]

because for a non-negative integer \( m \)

\[ \sum_{p^\alpha \leq x, \alpha \geq 2} \frac{\log^m p^\alpha}{\sqrt{p^\alpha}} \ll \log^{m+1} x. \]

From (2.6) and (2.7) we obtain the following estimate for \( S_2 \) (\( p, q \) are prime numbers):

\[ S_2 = \sum_{n \leq x, n \in M_{r,k}^2} \log^k(n) \]

\[ \ll \max_{0 \leq \beta \leq k} \sum_{p^\alpha \leq x, \alpha \geq 2} \log^{k-\beta} (p^\alpha) \cdot \sum_{q^\gamma N \leq x/p^\alpha, \gamma \geq 2, N \text{ square}} \log^\beta (q^\gamma N) \]

(2.8)

\[ \ll \max_{0 \leq \beta \leq k} x^{1/2} \log^{\beta+1} x \sum_{p^\alpha \leq x, \alpha \geq 2} \frac{\log^{k-\beta} p^\alpha}{\sqrt{p^\alpha}} \ll x^{1/2} \log^{k+2} x. \]

By repeating the above argument for every \( j = 1, 2, \ldots, k \) we get

(2.9) \[ S_j \ll x^{1/2} \log^{k+j} x. \]

Therefore, we have

\[ \Psi^*_r(x) - \theta^*_r(x) \ll \max_{1 \leq j \leq k} S_j \ll x^{1/2} \log^{2k} x. \]

\( \checkmark \)

It is well-known that

\[ \psi(x) \log x + \sum_{p \leq x} \psi(x/p) \log p = 2x \log x + O(x). \]

In the following theorem we will get the corresponding expression for the general case.
Theorem 2. Let $r$ be an odd positive integer and let $k$ be a positive integer. Then

$$
\sum_{n \leq x} \Psi_{r,k}^* \left( \frac{x}{n} \right) \log^k \left( \frac{x}{n} \right) + \sum_{i=1}^{k} \binom{k}{i} \sum_{n \leq x} \sum_{n \in C_{r,i}} \Psi_{r,k}^* \left( \frac{x}{n} \right) \log^{k-i} \left( \frac{x}{n} \right) \Lambda_{r,i}^* (n) = \\
\left( \sum_{n \in G_2 \cap Q_r} \right) (2.10)
$$

$$
= k \left[ \frac{\zeta(2)}{\zeta(r+1)} \right]^2 \left( \frac{k!}{(2k-1)!} \sum_{i=0}^{k} \frac{(2k-i-1)!}{(k-i)!} \right) x \log^{2k-1} x + O(x \log^{2k-2} x).
$$

Proof. By [2, Theorem 3], we know that

$$
\sum_{n \leq x} \Psi_{r,k}^* (x/n) \log^k (x/n) h_r(n) + \sum_{i=1}^{k} \binom{k}{i} \sum_{n \leq x} \Psi_{r,k}^* (x/n) \log^{k-i} (x/n) \Lambda_{r,i}^* (n) = \\
= k \left[ \frac{\zeta(2)}{\zeta(r+1)} \right]^2 \left( \frac{k!}{(2k-1)!} \sum_{i=0}^{k} \frac{(2k-i-1)!}{(k-i)!} \right) x \log^{2k-1} x + O(x \log^{2k-2} x).
$$

Since $h_r(n)$ is the characteristic function of $G_2 \cap Q_r$, to deduce formula (2.10) it will be sufficient to prove that

$$
\max_{1 \leq i \leq k} \left\{ \sum_{n \leq x} \sum_{n \in C_{r,i}} \Psi_{r,k}^* \left( \frac{x}{n} \right) \log^{k-i} \left( \frac{x}{n} \right) \Lambda_{r,i}^* (n) \right\} \ll x \log^{2k-2} x
$$

(2.11)

The natural numbers $n \notin C_{r,i}$ such that $\Lambda_{r,i}^* (n) \neq 0$ are contained in $M_{r,i}$, therefore we can write the sum $T_i$ within braces as

$$
(2.12) \quad T_i = \sum_{j=1}^{i} \sum_{n \leq x} \sum_{n \in M_{r,i}} \Psi_{r,k}^* \left( \frac{x}{n} \right) \log^{k-i} \left( \frac{x}{n} \right) \Lambda_{r,i}^* (n).
$$

As a consequence of [2, Theorem 2] we know that $\Psi_{r,k}^* (x) \ll x \log^{k-1} x$, so, due to (2.4), we have that

$$
(2.13) \quad T_i \ll \max_{1 \leq i \leq k} \sum_{n \leq x} \sum_{n \in M_{r,i}} \left( \frac{x}{n} \right) \log^{2k-i-1} \left( \frac{x}{n} \right) \log^i (n).
$$
For $j = 1$, we have

$$\sum_{\substack{n \leq x \\ n \in M_{r,i}^1}} \frac{x}{n} \log^{2k-i-1} \left( \frac{x}{n} \right) \log^i (n) \ll \ll x \sum_{\substack{p^\alpha N \leq x \\ \alpha \geq 2 \ N \text{ square}}} \frac{\log^{2k-i-1} \left( x/p^\alpha N \right) \log^i (p^\alpha N)}{p^\alpha N}$$

$$\ll x \max_{0 \leq \gamma \leq 2k-i-1} \log^{2k-i-1-\gamma} x \sum_{\substack{p^\alpha N \leq x \\ \alpha \geq 2 \ N \text{ square}}} \frac{\log^{i+\gamma} (p^\alpha N)}{p^\alpha N}$$

$$\ll x \log^{2k-i-1} x$$

because for integer numbers $m \geq 1$

$$\sum_{\substack{p^\alpha N \leq x \\ \alpha \geq 2 \ N \text{ square}}} \frac{\log^m (p^\alpha N)}{p^\alpha N} \ll \max_{0 \leq \delta \leq m} \sum_{\substack{p^\alpha N \leq x \\ \alpha \geq 2 \ N \text{ square}}} \frac{\log^{m-\delta} (p^\alpha)}{p^\alpha} \sum_{N \leq x} \frac{\log^\delta N}{N}$$

$$\ll \max_{0 \leq \delta \leq m} \sum_{p \leq \sqrt{x}} \sum_{\alpha \geq 2} \frac{\log^{m-\delta} (p^\alpha)}{p^\alpha}$$

$$\ll \max_{0 \leq \delta \leq m} \sum_{p \leq \sqrt{x}} \frac{\log^{m-\delta} p}{p^2} \ll 1.$$  

In a similar way for each $j = 2, 3 \ldots i$, the sum of (2.13) is $\ll x \log^{2k-i-j} x$. Hence, $T_i \ll x \log^{2k-i-1} x$, $(1 \leq i \leq k)$ and formula (2.11) is proved.

It is well-known that

$$\theta(x) \log x + \sum_{p \leq x} \theta(x/p) \log p = 2x \log x + O(x).$$

In the following theorem we will generalize the above formula for $\theta^*_r,k$.

**Theorem 3.** For positive integers $r$, $k$ with $r$ odd we have,

(2.14)

$$\sum_{\substack{n \leq x \\ n \in G_2 \cap Q_r}} \theta^*_r,k \left( \frac{x}{n} \right) \log^k \left( \frac{x}{n} \right) + \sum_{i=1}^{k} \binom{k}{i} \sum_{\substack{n \leq x \\ n \in G_{r,i}}} \theta^*_r,k \left( \frac{x}{n} \right) \log^{k-i} \left( \frac{x}{n} \right) \wedge^*_i (n) =$$
\[ k \left( \frac{\zeta(2)}{\zeta(r + 1)} \right)^2 \left( \frac{k!}{(2k - 1)!} \sum_{i=0}^{k} \frac{(2k - 1)!}{(k - i)!} \right) x \log^{2k-1} x + O(x \log^{2k-2} x). \]

**Proof.** By Theorem 2 it is sufficient to prove that

\[
\sum_{n \leq x} \frac{[\Psi_{r,k}(x/n) - \theta_{r,k}^*(x/n)] \log^k (x/n)}{n \in \mathbb{G}_2 \cap Q_r}
\]

\[= \sum_{1} + \sum_{2} \ll x \log^{2k-2} x. \]

By Theorem 1,

\[
\sum_{m^2 \leq x} \ll \sum_{n \in C_{r,i}} \frac{[\Psi_{r,k}(x/m^2) - \theta_{r,k}^*(x/m^2)] \log^k (x/m^2)}{m \leq \sqrt{x}} \ll x.
\]

On the other hand, we can deduce

\[
\sum_{2} \ll \max_{1 \leq i \leq k} \log^{k-i} x \sum_{m \leq x} \Lambda_{r,k}^*(m) \Lambda_{r,i}^*(n) \ll \max_{1 \leq i \leq k} \log^{k-i} x \sum_{m \leq x} \Lambda_{r,k}^*(m) \Psi_{r,i}^*(x/m).
\]

Applying the argument which has been used to estimate \( T_i \) in the proof of Theorem 2 we can write:

\[
\sum_{2} \ll x \max_{1 \leq i \leq k} \log^{k-i} x \sum_{m \leq x} \frac{\Lambda_{r,k}^*(m) \log^{i-1}(x/m)}{m} \ll
\]

\[\ll x \max_{1 \leq i \leq k} \log^{k-1-\gamma} x \left\{ \sum_{p^\alpha N \leq x} \frac{x \log^{k+\gamma} p^\alpha N}{p^\alpha N} + \ldots + \right\} \]
Corollary. When \( r \) is an odd positive integer and \( k \) is a positive integer the following estimate holds

\[
\sum_{nm \leq x} \Lambda^*_{r,k}(m) \log^k(m) + \sum_{i=1}^{k} \binom{k}{i} \sum_{nm \leq x} \Lambda^*_{r,k}(m) \log^{k-i} \left( \frac{x}{n} \right) \Lambda^*_{r,i}(n) =
\]

\[
= k \left[ \frac{\zeta(2)}{\zeta(r+1)} \right]^2 \left( \frac{k!}{(2k-1)!} \sum_{i=0}^{k} \frac{(2k-i-1)!}{(k-i)!} \right) x \log^{2k-1} x + O(x \log^{2k-2} x).
\]

Proof. By Abel's identity,

\[
\sum_{m \leq y \atop m \in \mathcal{C}_{r,k}} \Lambda^*_{r,k}(m) \log^k m = \theta^*_{r,k}(y) \log^k y - k \int_{3/2}^{y} \frac{\theta^*_{r,k}(z) \log^{k-1} z}{z} \, dz
\]

Since \( \theta^*_{r,k}(z) = O(z \log^{k-1} z) \), the last integral is \( O(y \log^{2k-2} y) \). Taking \( y = x/N \) and adding over \( N \leq x \), \( N \) square \( r \)-free, we have

\[
\sum_{N \leq x \atop N \in \mathcal{G}_2 \cap \mathbb{Q}_r} \theta^*_{r,k}(x/N) \log^k (x/N) =
\]

\[
= \sum_{N \leq x \atop N \in \mathcal{G}_2 \cap \mathbb{Q}_r} \sum_{m \leq x/N \atop m \in \mathcal{C}_{r,k}} \Lambda^*_{r,k}(m) \log^k m + O(x \log^{2k-2} x).
\]

Replacing (2.19) in (2.14) the Corollary is deduced.

3. Applications and special cases

1. When \( r = 1 \) we have \( \Lambda^*_{1,k} = \Lambda_k \), \( \Psi^*_{1,k} = \Psi_k \) and by Theorem 2 we deduce
\[ \Psi_k(x) \log^k(x) + \sum_{i=1}^{k} \binom{k}{i} \sum_{\substack{pm \leq x \omega(m) < i}} \Psi_k \left( \frac{x}{pm} \right) \log^{k-i} \left( \frac{x}{pm} \right) \Lambda_i(pm) = \]

\[
= k \left( \frac{k!}{(2k-1)!} \sum_{i=0}^{k} \frac{(2k-i-1)!}{(k-i)!} \right) x \log^{2k-1} x + O(x \log^{2k-2} x). \tag{3.1}
\]

2. When \( r = 1 \) we have \( \theta_{1,k}^* = \theta_k \) and by Theorem 3 we deduce that

\[
\theta_k(x) \log^k(x) + \sum_{i=1}^{k} \binom{k}{i} \sum_{\substack{pm \leq x \omega(m) < i}} \theta_k \left( \frac{x}{pm} \right) \log^{k-i} \left( \frac{x}{pm} \right) \Lambda_i(pm) = \]

\[
= k \left( \frac{k!}{(2k-1)!} \sum_{i=0}^{k} \frac{(2k-i-1)!}{(k-i)!} \right) x \log^{2k-1} x + O(x \log^{2k-2} x). \tag{3.2}
\]

3. For \( r = 3, k = 1 \) we have

\[
\sum_{m^2 \leq x} |\mu(m)| \Psi_{3,1}^* \left( \frac{x}{m^2} \right) \left\{ \log \left( \frac{x}{m^2} \right) + \Lambda_{3,1}^* (m^2) \right\} + \sum_{\substack{pm^2 \leq x \omega(m) = 1}} |\mu(m)| \Psi_{3,1}^* \left( \frac{x}{pm^2} \right) \Lambda_{3,1}^* (pm^2) \]

\[
= k \left[ \frac{\zeta(2)}{\zeta(4)} \right]^2 x \log x + O(x). \tag{3.3}
\]

4. For \( r = 3, k = 1 \) we get

\[
\sum_{m^2 \leq x} |\mu(m)| \theta_{3,1}^* \left( \frac{x}{m^2} \right) \left\{ \log \left( \frac{x}{m^2} \right) + \Lambda_{3,1}^* (m^2) \right\} + \sum_{\substack{pm^2 \leq x \omega(m) = 1}} |\mu(m)| \theta_{3,1}^* \left( \frac{x}{pm^2} \right) \Lambda_{3,1}^* (pm^2) \]

\[
= 2 \left[ \frac{\zeta(2)}{\zeta(4)} \right]^2 x \log x + O(x). \tag{3.4}
\]
References


(Recibido en marzo de 1995; revisado en febrero de 1996)

Catalina Calderón
Departamento de Matemáticas
Universidad del País Vasco
E-48080 Bilbao
España

Maria José Zárate
Departamento de Matemáticas
Universidad del País Vasco
E-48080 Bilbao
España