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Asymptotic formulae of generalized Chebyshev functions*

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ABSTRACT. In this paper we study the behavior of certain generalized Chebyshev functions over the square r-free integers and will prove Selberg's inequality for such functions

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1. Introduction and preliminaries

In the last three decades a number of elementary proofs of the prime number theorem have appeared (see [3] for a survey). Most of these proofs are based, at least in part, on ideas from the original proof of Erdös [4] and Selberg [9]. One of the main ingredients of the Erdös -Selberg proof is Selberg's formula

$$\sum_{p\leq x}\log^2 p+\sum_{pq\leq x}\log p\log q=2x\log x+O(x) \tag{1.1}$$

which appears, in some form, in almost all these proofs.

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Several proofs of Selberg's formula appeared soon (see [7], [11] ,[12]). Bombieri [1] used a class of analogues of Selberg's formulae of greater weight to improve the error term in the prime number theorem. Wirsing [13] introduced a recursion whereby each time that an error estimate was found for $\pi(x) - lix$, it was used again to obtain an improved form of Selberg's formula.

H.N. Shapiro [11] obtained a generalization of Selberg's formula and its equivalences. Also, A. Selberg [10] gave an elementary proof of the prime number theorem for arithmetic progressions. Its starting point is the following formula

$$\sum_{\substack{p\le x\\ p\equiv \ell(\mathrm{mod}\, k)}} \log^2 p + \sum_{\substack{pq\le x\\ pq\equiv \ell(\mathrm{mod}\, k)}} \log p \log q \tag{1.2}$$

$$= \frac{2}{\phi(k)} x\log x + O(x), \quad (k,\ell)=1.$$

In 1958 G.J. Rieger obtained formula (1.2) in an algebraic field [8].

Applying the inversion formula, K. Iseki and T. Tatuzawa [12] established the formula

$$\psi(x)\log x + \sum_{n\le x}\psi(x/n)\Lambda(n) = 2x\log x + O(x), \quad \psi(x)=\sum_{n\le x}\Lambda(n). \tag{1.3}$$

This result may be used in place of Selberg's formula to prove the prime number theorem.

In this paper we will prove Selberg's inequality for the generalized Chebyshev functions $\Psi^*_{r,k}(x)$, $\theta^*_{r,k}(x)$ which will be defined in (1.4). Let G_2 be the set of square integers and let Q_r be the set of r-free integers ($r > 2$). If $r = 1$ we take $G_2 \cap Q_1 = \{1\}$. If $r \ge 2$ is even, $G_2 \cap Q_r = G_2 \cap Q_{r-1}$, so that it is sufficient to consider the case when r is odd.

We denote by $C_{r,k}$ the set of natural numbers n such that $n = N$ or $n = pmN$, where $N \in G_2 \cap Q_r$ and m is an arbitrary integer with $\omega(m) \le k-1$, $(pm, N) = 1$, $\omega(m)$ being the number of distinct prime factors of m.

For positive integers r and k, let $\psi^*_{r,k}(x)$, $\theta^*_{r,k}(x)$ be the summatory functions

$$\Psi^*_{r,k}(x) = \sum_{n\le x}\Lambda^*_{r,k}(n), \quad \theta^*_{r,k}(x) = \sum_{\substack{n\le x\\ n\in C_{r,k}}}\Lambda^*_{r,k}(n) \tag{1.4}$$

where $\Lambda^*_{r,k}(n)$ is the function of Mangoldt type

$$\Lambda^*_{r,k}(n) = \sum_{d\delta=n}\mu^*_r(d)\log^k\delta, \tag{1.5}$$

$\mu_r^*(n)$ being given by $\mu_r^*(1) = 1$, $\mu_r^*(n) = 0$ if $p^{r+1}|n$ for some prime p, and $\mu_r^*(n) = (-1)^{\Omega(n)}$ if $n = \prod p_i^{\alpha_i}$, $0 \le \alpha_i \le r$, with $\Omega(n) = \sum \alpha_i$ (observe that $\mu_1^*(n)$ is the Moebius function $\mu(n)$).

The functions $\Lambda_{r,k}^*(n)$ generalize the well-known von Mangoldt function $\Lambda(n)$ and also Ivić functions $\Lambda_k(n)$ (cf. [5]). Since

$$\sum_{n=1}^{\infty} \mu_r^*(n) n^{-s} = (\zeta(2s)/\zeta(s))\gamma_r((r+1)s)$$

for $\mathrm{Re}(s) > 1$, where $\gamma_r(s) = 1/\zeta(s)$ if $r \ge 1$ is odd and $\gamma_r(s) = \zeta(s)/\zeta(2s)$ if $r \ge 2$ is even, the Dirichlet series for $\Lambda_{r,k}^*(n)$ is

$$\sum_{n=1}^{\infty} \Lambda_{r,k}^*(n) n^{-s} = (-1)^k (\zeta^{(k)}(s)/\zeta(s))\zeta(2s)\gamma_r((r+1)s),$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$. $\zeta(s)$ is the Riemann zeta function and $\zeta^{(k)}(s)$ its k-th order derivative. The function $\Lambda_{r,k}^*(n)$ has the property $\Lambda_{r,k}^*(n) = \sum_{d\delta=n} \Lambda_{1,k}^*(d) h_r(\delta)$, $h_r(n) = \sum_{d\delta=n} \mu_r^*(d)$. For odd integer r, $h_r(n)$ is the characteristic function of the square r-free integers.

Moreover, from [2, Theorem 2] we know that for fixed positive integers r, k, there exists a constant $C = C(k) > 0$ such that

$$\Psi_{r,k}^*(x) = x P_{k-1}(\log x) + O(x \exp(-C\delta(x))),$$
$$\delta(x) = \log^{3/5} x (\log\log x)^{-1/5} \tag{1.6}$$

where $P_{k-1}(t)$ is a polynomial of degree $k-1$ in t (the case $r = k = 1$ is formula (12.26) of [6]). An extension of (1.3) for $\Psi_{r,k}^*(x)$ is given in [2, Theorem 3]. We will give here other formulae of type (1.3) and (1.1) for $\Psi_{r,k}^*(x)$ and $\theta_{r,k}^*(x)$, the sums being extended over a certain class of integers (mentioned above).

2. The theorems

Theorem 1. *For integers $r, k \ge 1$, with r odd, we have*

$$0 \le \Psi_{r,k}^*(x) - \theta_{r,k}^*(x) \ll x^{1/2} \log^{2k} x. \tag{2.1}$$

Proof. From (1.4) we obtain that

$$\Psi_{r,k}^*(x) - \theta_{r,k}^*(x) = \sum_{\substack{n \le x \\ n \notin C_{r,k}}} \Lambda_{r,k}^*(n). \tag{2.2}$$

Moreover let $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ be the factorial descomposition of the positive integer n and let $\delta_i = \min\{\alpha_i, r\}$. Using (1.5) and the definition of $\mu_r^*(n)$ we get have that

$$\wedge_{r,k}^*(n) = \sum_{\beta_1=0}^{\delta_1} \cdots \sum_{\beta s=0}^{\delta_s} (-1)^{\sum_{i=1}^{s}\beta_i} (\sum_{i=1}^{s} (\alpha_i - \beta_i)\log p_i)^k =$$

$$= \sum_{n_1+\cdots+n_s=k} \frac{k!}{n_1!\dots n_s!} \log^{n_1} p_1 \dots \log^{n_s} p_s S(n_1)\dots S(n_s), \tag{2.3}$$

where $S(n_i) = \sum_{\beta=0}^{\delta_i} (-1)^\beta (\alpha_i - \beta)^{n_i}$. Since $0 \le S(n_i) < \alpha_i^{n_i}$, $i = 1, \dots, s$ from (2.3) we get

$$\wedge_{r,k}^*(n) \le \log^k n. \tag{2.4}$$

If at least $k+1$ exponents α_i are such that $\alpha_i \ge r$ or $\alpha_i < r$ is odd then $\wedge_{r,k}^*(n) = 0$ and, from the definition of $C_{r,k}$,

$$\{n : n \notin C_{r,k},\quad \wedge_{r,k}^*(n) \ne 0\} \subset M_{r,k} = \cup_{j=1}^{k} M_{r,k}^j$$

where

$$M_{r,k}^j = \{n = N \prod_{s=1}^{j} p_s^{\alpha_s} : 1 < \alpha_s(\text{odd }) < r \text{ or } \alpha_s \ge r, N \in G_2 \cap Q_r, (p_s, N) = 1\}.$$

We consider the sums

$$S_j = \sum_{\substack{n \le x \\ n \in M_{r,k}^j}} \log^k n \quad , \quad (1 \le j \le k). \tag{2.5}$$

Let $j = 1$, $N = 1$ and let $\pi(x)$ be the number of prime numbers which do not exceed x. Since the order of magnitude of $\pi(x)$ is $x/\log x$, then

$$\sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \log^k p^\alpha \ll \sum_{2 \le \alpha \le \log_2 x} \alpha^k \sum_{p \le x^{1/\alpha}} \log^k p$$

$$\ll \sum_{2 \le \alpha \le \log_2 x} \alpha^k \pi(x^{1/\alpha}) \log^k (x^{1/\alpha}) \ll x^{1/2} \log^{k+1} x.$$

Therefore

$$
\begin{aligned}
S_1 &= \sum_{\substack{n\le x\\ n\in M^1_{r,k}}} \log^k(n) \ll \max_{0\le\beta\le k} \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}} \log^{k-\beta} p^\alpha \sum_{\substack{N\le x/p^\alpha\\ N \text{ square}}} \log^\beta N \\
&\ll \max_{0\le\beta\le k} x^{1/2}\log^\beta x \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}} \frac{\log^{k-\beta} p^\alpha}{\sqrt{p^\alpha}} \ll x^{1/2}\log^{k+1} x,
\end{aligned}
\tag{2.6}
$$

because for a non-negative integer m

$$
\sum_{p^\alpha\le x,\alpha\ge 2} \frac{\log^m p^\alpha}{\sqrt{p^\alpha}} \ll \log^{m+1} x. \tag{2.7}
$$

From (2.6) and (2.7) we obtain the following estimate for S_2 (p, q are prime numbers):

$$
\begin{aligned}
S_2 &= \sum_{\substack{n\le x\\ n\in M^2_{r,k}}} \log^k(n) \\
&\ll \max_{0\le\beta\le k} \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}} \log^{k-\beta}(p^\alpha)\cdot \sum_{\substack{q^\gamma N\le x/p^\alpha\\ \gamma\ge 2, N \text{ square}}} \log^\beta(q^\gamma N) \\
&\ll \max_{0\le\beta\le k} x^{1/2}\log^{\beta+1} x \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}} \frac{\log^{k-\beta} p^\alpha}{\sqrt{p^\alpha}} \ll x^{1/2}\log^{k+2} x.
\end{aligned}
\tag{2.8}
$$

By repeating the above argument for every $j = 1, 2, \ldots, k$ we get

$$
S_j \ll x^{1/2}\log^{k+j} x. \tag{2.9}
$$

Therefore, we have

$$
\Psi^*_{r,k}(x) - \theta^*_{r,k}(x) \ll \max_{1\le j\le k} S_j \ll x^{1/2}\log^{2k} x. \qquad ☑
$$

It is well-known that

$$
\psi(x)\log x + \sum_{p\le x} \psi(x/p)\log p = 2x\log x + O(x).
$$

In the following theorem we will get the corresponding expression for the general case.

Theorem 2. *Let r be an odd positive integer and let k be a positive integer. Then*

$$\sum_{\substack{n\leq x\\ n\in G_2\cap Q_r}} \Psi^*_{r,k}\left(\frac{x}{n}\right)\log^k\left(\frac{x}{n}\right)+\sum_{i=1}^{k}\binom{k}{i}\sum_{\substack{n\leq x\\ n\in C_{r,i}}}\Psi^*_{r,k}\left(\frac{x}{n}\right)\log^{k-i}\left(\frac{x}{n}\right)\wedge^*_{r,i}(n)=$$

$$(2.10)\qquad =k\left[\frac{\zeta(2)}{\zeta(r+1)}\right]^2\left(\frac{k!}{(2k-1)!}\sum_{i=0}^{k}\frac{(2k-i-1)!}{(k-i)!}\right)x\log^{2k-1}x+O(x\log^{2k-2}x).$$

Proof. By [2, Theorem 3], we know that

$$\sum_{n\leq x}\Psi^*_{r,k}(x/n)\log^k(x/n)h_r(n)+\sum_{i=1}^{k}\binom{k}{i}\sum_{n\leq x}\Psi^*_{r,k}(x/n)\log^{k-i}(x/n)\wedge^*_{r,i}(n)=$$

$$=k\left[\frac{\zeta(2)}{\zeta(r+1)}\right]^2\left(\frac{k!}{(2k-1)!}\sum_{i=0}^{k}\frac{(2k-i-1)!}{(k-i)!}\right)x\log^{2k-1}x+O(x\log^{2k-2}x).$$

Since $h_r(n)$ is the characteristic function of $G_2\cap Q_r$, to deduce formula (2.10) it will be sufficient to prove that

$$(2.11)\qquad \max_{1\leq i\leq k}\left\{\sum_{\substack{n\leq x\\ n\notin C_{r,i}}}\Psi^*_{r,k}\left(\frac{x}{n}\right)\log^{k-i}\left(\frac{x}{n}\right)\wedge^*_{r,i}(n)\right\}\ll x\log^{2k-2}x$$

The natural numbers $n\notin C_{r,i}$ such that $\wedge^*_{r,i}(n)\neq 0$ are contained in $M_{r,i}$, therefore we can write the sum T_i within braces as

$$(2.12)\qquad T_i=\sum_{j=1}^{i}\sum_{\substack{n\leq x\\ n\in M^j_{r,i}}}\Psi^*_{r,k}\left(\frac{x}{n}\right)\log^{k-i}\left(\frac{x}{n}\right)\wedge^*_{r,i}(n).$$

As a consecuence of [2, Theorem 2] we know that $\Psi^*_{r,k}(x)\ll x\log^{k-1}x$, so, due to (2.4), we have that

$$(2.13)\qquad T_i\ll\max_{1\leq j\leq i}\sum_{\substack{n\leq x\\ n\in M^j_{r,i}}}\left(\frac{x}{n}\right)\log^{2k-i-1}\left(\frac{x}{n}\right)\log^i(n).$$

For $j = 1$, we have

$$\sum_{\substack{n\le x\\ n\in M^1_{r,i}}} \frac{x}{n}\log^{2k-i-1}\left(\frac{x}{n}\right)\log^i(n) \ll$$

$$\ll x \sum_{\substack{p^\alpha N\le x\\ \alpha\ge 2\ N \text{ square}}} \frac{\log^{2k-i-1}(x/p^\alpha N)\log^i(p^\alpha N)}{p^\alpha N}$$

$$\ll x \max_{0\le\gamma\le 2k-i-1} \log^{2k-i-1-\gamma} x \sum_{\substack{p^\alpha N\le x\\ \alpha\ge 2\ N \text{ square}}} \frac{\log^{i+\gamma}(p^\alpha N)}{p^\alpha N}$$

$$\ll x\log^{2k-i-1} x$$

because for integer numbers $m \ge 1$

$$\sum_{\substack{p^\alpha N\le x\\ \alpha\ge 2\ N \text{ square}}} \frac{\log^m(p^\alpha N)}{p^\alpha N} \ll \max_{0\le\delta\le m} \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}} \frac{\log^{m-\delta}(p^\alpha)}{p^\alpha} \sum_{\substack{N\le x\\ N \text{ square}}} \frac{\log^\delta N}{N}$$

$$\ll \max_{0\le\delta\le m} \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}} \frac{\log^{m-\delta}(p^\alpha)}{p^\alpha}$$

$$\ll \max_{0\le\delta\le m} \sum_{p\le\sqrt{x}} \sum_{\alpha\ge 2} \frac{\log^{m-\delta}(p^\alpha)}{p^\alpha}$$

$$\ll \max_{0\le\delta\le m} \sum_{p\le\sqrt{x}} \frac{\log^{m-\delta} p}{p^2} \ll 1.$$

In a similar way for each $j = 2, 3 \dots i$, the sum of (2.13) is $\ll x\log^{2k-i-j} x$. Hence, $T_i \ll x\log^{2k-i-1} x$, $(1 \le i \le k)$ and formula (2.11) is proved. ☑

It is well-known that

$$\theta(x)\log x + \sum_{p\le x}\theta(x/p)\log p = 2x\log x + O(x).$$

In the following theorem we will generalize the above formula for $\theta^*_{r,k}$.

Theorem 3. *For positive integers r, k with r odd we have,*
(2.14)

$$\sum_{\substack{n\le x\\ n\in G_2\cap Q_r}} \theta^*_{r,k}\left(\frac{x}{n}\right)\log^k\left(\frac{x}{n}\right) + \sum_{i=1}^{k}\binom{k}{i} \sum_{\substack{n\le x\\ n\in C_{r,i}}} \theta^*_{r,k}\left(\frac{x}{n}\right)\log^{k-i}\left(\frac{x}{n}\right)\Lambda^*_{r,i}(n) =$$

$$= k\left[\frac{\zeta(2)}{\zeta(r+1)}\right]^2\left(\frac{k!}{(2k-1)!}\sum_{i=0}^{k}\frac{(2k-i-1)!}{(k-i)!}\right)x\log^{2k-1}x+O(x\log^{2k-2}x).$$

Proof. By Theorem 2 it is sufficient to prove that

$$\sum_{\substack{n\le x\\ n\in G_2\cap Q_r}}[\Psi^*_{r,k}(x/n)-\theta^*_{r,k}(x/n)]\log^k(x/n)+$$

(2.15)

$$+\sum_{i=1}^{k}\binom{k}{i}\sum_{\substack{n\le x\\ n\in C_{r,i}}}[\Psi^*_{r,k}(x/n)-\theta^*_{r,k}(x/n)]\log^{k-i}(x/n)\wedge^*_{r,i}(n)$$

$$=\sum_1+\sum_2\ll x\log^{2k-2}x$$

By Theorem 1 ,

$$\sum_1\ll\sum_{m^2\le x}[\Psi^*_{r,k}(x/m^2)-\theta^*_{r,k}(x/m^2)]\log^k(x/m^2)$$

(2.16)

$$\ll x^{1/2}\sum_{m\le\sqrt{x}}\frac{\log^{3k}(x/m^2)}{m}\ll x$$

On the other hand, we can deduce

$$\sum_2\ll\max_{1\le i\le k}\log^{k-i}x\sum_{\substack{mn\le x\\ n\in C_{r,i},m\in M_{r,k}}}\wedge^*_{r,k}(m)\wedge^*_{r,i}(n)$$

$$\ll\max_{1\le i\le k}\log^{k-i}x\sum_{\substack{m\le x\\ m\in M_{r,k}}}\wedge^*_{r,k}(m)\Psi^*_{r,i}(x/m).$$

Applying the argument which has been used to estimate T_i in the proof of Theorem 2 we can write :

(2.17) $$\sum_2\ll x\max_{1\le i\le k}\log^{k-i}x\sum_{\substack{m\le x\\ m\in M_{r,k}}}\frac{\wedge^*_{r,k}(m)\log^{i-1}(x/m)}{m}\ll$$

$$\ll x\max_{1\le i\le k}\max_{0\le\gamma\le i-1}\log^{k-1-\gamma}x\left\{\sum_{\substack{p^\alpha N\le x\\ \alpha\ge2,N\text{ square}}}\frac{\log^{k+\gamma}p^\alpha N}{p^\alpha N}+\ldots+\right.$$

$$\left. + \sum_{\substack{p_1^{\alpha_1}\cdots p_k^{\alpha_k}N\le x\\ \alpha_i\ge 2, N \text{ square}}} \frac{\log^{k+\gamma} p_1^{\alpha_1}\dots p_k^{\alpha_k}N}{p_1^{\alpha_1}\dots p_k^{\alpha_k}N} \right\} \ll x\log^{k-1} x. \qquad ☑$$

Corollary. *When r is an odd positive integer and k is a positive integer the following estimate holds*

$$\sum_{\substack{nm\le x\\ m\in C_{r,k}\\ n\in G_2\cap Q_r}} \Lambda^*_{r,k}(m)\log^k(m) + \sum_{i=1}^{k}\binom{k}{i} \sum_{\substack{nm\le x\\ m\in C_{r,k}, n\in C_{r,i}}} \Lambda^*_{r,k}(m)\log^{k-i}(\frac{x}{n})\,\Lambda^*_{r,i}(n) =$$

(2.18)

$$= k\left[\frac{\zeta(2)}{\zeta(r+1)}\right]^2 \left(\frac{k!}{(2k-1)!}\sum_{i=0}^{k}\frac{(2k-i-1)!}{(k-i)!}\right) x\log^{2k-1} x + O(x\log^{2k-2} x).$$

Proof. By Abel's identity,

$$\sum_{\substack{m\le y\\ m\in C_{r,k}}} \Lambda^*_{r,k}(m)\log^k m = \theta^*_{r,k}(y)\log^k y - k\int_{3/2}^{y} \frac{\theta^*_{r,k}(z)\log^{k-1} z}{z}dz$$

Since $\theta^*_{r,k}(z) = O(z\log^{k-1} z)$, the last integral is $O(y\log^{2k-2} y)$ .Taking $y = x/N$ and adding over $N \le x$, N square r-free, we have

$$\sum_{\substack{N\le x\\ N\in G_2\cap Q_r}} \theta^*_{r,k}(x/N)\log^k(x/N) =$$

(2.19)

$$\sum_{\substack{N\le x\\ N\in G_2\cap Q_r}} \sum_{\substack{m\le x/N\\ m\in C_{r,k}}} \Lambda^*_{r,k}(m)\log^k m + O(x\log^{2k-2} x).$$

Replacing (2.19) in (2.14) the Corollary is deduced. ☑

3. Applications and special cases

1. When $r = 1$ we have $\Lambda^*_{1,k} = \Lambda_k$, $\Psi^*_{1,k} = \Psi_k$ and by Theorem 2 we deduce

that

$$\Psi_k(x)\log^k(x) + \sum_{i=1}^{k}\binom{k}{i}\sum_{\substack{pm\le x\\ \omega(m)<i}}\Psi_k\Big(\frac{x}{pm}\Big)\log^{k-i}\Big(\frac{x}{pm}\Big)\wedge_i(pm) =$$

(3.1)

$$= k\left(\frac{k!}{(2k-1)!}\sum_{i=0}^{k}\frac{(2k-i-1)!}{(k-i)!}\right)x\log^{2k-1}x + O(x\log^{2k-2}x).$$

2. When $r = 1$ we have $\theta^*_{1,k} = \theta_k$ and by Theorem 3 we deduce that

$$\theta_k(x)\log^k(x) + \sum_{i=1}^{k}\binom{k}{i}\sum_{\substack{pm\le x\\ \omega(m)<i}}\theta_k\Big(\frac{x}{pm}\Big)\log^{k-i}\Big(\frac{x}{pm}\Big)\wedge_i(pm) = \tag{3.2}$$

$$= k\left(\frac{k!}{(2k-1)!}\sum_{i=0}^{k}\frac{(2k-i-1)!}{(k-i)!}\right)x\log^{2k-1}x + O(x\log^{2k-2}x).$$

3. For $r = 3$, $k = 1$ we have

$$\sum_{m^2\le x}|\mu(m)|\Psi^*_{3,1}\Big(\frac{x}{m^2}\Big)\Big\{\log\Big(\frac{x}{m^2}\Big) + \wedge^*_{3,1}(m^2)\Big\}$$

$$+ \sum_{\substack{pm^2\le x\\ (p,m)=1}}|\mu(m)|\Psi^*_{3,1}\Big(\frac{x}{pm^2}\Big)\wedge^*_{3,1}(pm^2)$$

(3.3)

$$= k\left[\frac{\zeta(2)}{\zeta(4)}\right]^2 x\log x + O(x).$$

4. For $r = 3, k = 1$ we get

$$\sum_{m^2\le x}|\mu(m)|\theta^*_{3,1}\Big(\frac{x}{m^2}\Big)\Big\{\log\Big(\frac{x}{m^2}\Big) + \wedge^*_{3,1}(m^2)\Big\}$$

$$+ \sum_{\substack{pm^2\le x\\ (p,m)=1}}|\mu(m)|\theta^*_{3,1}\Big(\frac{x}{pm^2}\Big)\wedge^*_{3,1}(pm^2)$$

(3.4)

$$= 2\left[\frac{\zeta(2)}{\zeta(4)}\right]^2 x\log x + O(x).$$

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