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The L^2–order of magnitude of Vilenkin–Fourier coefficients

MOHAMMED S. YOUNIS
Yarmouk University, Irbid, JORDAN

ABSTRACT. Let G be a compact, metrizable, zero-dimensional, abelian gruop, i.e., a Vilenkin group. It is well known ([2], [6] for example) that if f belongs to the Lipschitz class $\mathrm{Lip}(\alpha, p, G)$, $0 < \alpha \leq 1$, $1 < p \leq 2$, then its Fourier transform $\hat{f}$ belongs to $\ell^{\beta}(\widehat{G})$ for $p/(p+\alpha p-1) < \beta \leq p' = p/(p-1)$, where $\widehat{G}$ is the dual of G. For Lipschitz functions on the real line $\mathbb{R}$ and on the circle group $\mathbb{T}$ ([3], Theorem 85, p. 117; [5], Theorem (1.3) c, p. 108), the special case $p = 2$, $0 < \alpha < 1$, reveals some reversibility between the conditions on f and $\hat{f}$. In the present work we extend, among other things, this reversibility to the L^2-Lipschitz functions on Vilenkin groups.

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1. Introduction

Titchmarsh ([3], p. 117) proved that if f belongs to $L^2(\mathbb{R})$, then the conditions

$$\left(\int_{\mathbb{R}} |f(x+h) - f(x)|^2 dx\right)^{1/2} = O(h^{\alpha}), \quad 0 < \alpha < 1, \quad h \to 0, \tag{1.1}$$

and

$$\left[\int_{-\infty}^{-X} + \int_{X}^{\infty}\right] |\hat{f}(u)|^2 du = O(X^{-2\alpha}), \quad X \to \infty, \tag{1.2}$$

are equivalent.

This theorem was respectively extended to functions in $L^2(\mathbb{R}^2)$ and in $L^2(\mathbb{T}^2)$ in [5], Theorem (1.2) c, p. 108, and in [6], Theorem 2.19, p. 44. For Vilenkin-Fourier coefficients, a partial result was proved by the present author in [6], Theorem 3.33, p. 69, where it was shown that (1.1) implies (1.2). A similar result was also proved by Vilenkin and Rubinstein in [4], Theorem 1. The purpose of this paper is to prove that in this context, (1.2) also implies (1.1).

2. Definitions and notation

We shall be brief here, since nearly all the definitions and notations needed are the usual in the literature. In particular, the reader is refered to [2] and [4] for the basic facts about G and $\widehat{G}$ as well as for the definitions and notations needed for our purpose. Thus, if G denotes a Vilenkin group, d will denote its metric and $\widehat{G}$ will be its dual. It is well known that there exists a countable basis $\{G_n\}$ of neighbourhoods of the identity element e in G such that

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots$$

and that

$$\bigcap_{n=0}^{\infty} G_n = \{e\}.$$

Also, if V_n is the anihilator in $\widehat{G}$ of G_n, then

$$\{e\} = V_0 \subset V_1 \subset V_2 \subset \cdots$$

and

$$\bigcup_{n=0}^{\infty} V_n = \widehat{G}.$$

There are numbers $m_0, m_1, m_2, \ldots$ such that $m_0 = 1$, $m_{k+1} = P_k m_k$, P_k being a prime number. The gropus V_n and V_n/V_{n+1} have Haar measures m_n and P_n, respectively.

Definition 2.1. Let $f \in L^p(G)$. The p-th integral modulus of continuity of f is defined by

$$\omega_p(f,k) = \sup_{h \in G_k} \left(\int_G |f(x+h) - f(x)|^p \, dx \right)^{1/p}.$$

If $\omega_p(f,k) = O(|h|^\alpha)$, $0 < \alpha \leq 1$, where $|h| = d(h,e)$, then f is said to belong to the Lipschitz class $\mathrm{Lip}(\alpha, p, G)$.

Definition 2.2. Let $x \in G$. Then the complex continuous characters in $\widehat{G}$ will be denoted by (n, x), $n \in \mathbb{Z}_+$. For $f \in L^p(G)$, the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(n) = \int_G f(x)\overline{(n,x)}dx, \quad n \in \mathbb{Z}_+,$$

where $\overline{(n,x)}$ stands for the complex conjugate of (n, x).

Definition 2.3. If P_k in $\widehat{G}$ is finite as $k \to \infty$, we say that G has the boundedness property (P).

3. Main results

We here prove the following

Theorem 3.1. *Let $f \in Lip\,(\alpha, 2, G)$, $0 < \alpha < 1$, where G is a Vilenkin group with property (P). Then the conditions*

$$\omega_2(f,k) = O(|h|^\alpha), \quad h \in G_k, \tag{3.1}$$

and

$$\sum_{n=m_k}^{\infty} |\hat{f}(n)|^2 = O(m_k^{-2\alpha}), \quad k \to \infty, \tag{3.2}$$

are equivalent.

Proof. That (3.1) implies (3.2) is easily deduced from the general estimate (see [2], (1))

$$\sum_{n=m_k}^{m_{k+1}-1} |\hat{f}(n)|^{p'} = O(\omega_p(f,k))^{p'}$$

which holds for $1 < p \le 2$. Here $p' = p/(p-1)$. For $p = p' = 2$ this yields

$$\sum_{n=m_k}^{\infty} |\hat{f}(n)|^2 = \sum_{n=m_k}^{m_{k+1}-1} + \sum_{m_{k+1}}^{m_{k+2}-1} + \cdots = O(m_k^{-2\alpha} + m_{k+1}^{-2\alpha} + \cdots). \tag{3.3}$$

Now $m_{k+1} \ge 2m_k$ for all $k \in \mathbb{N}$, and hence estimate (3.3) is equivalent to

$$\sum_{n=m_k}^{\infty} |\hat{f}(n)|^2 = O(m_k^{-2\alpha})(1 + 2^{-2\alpha} + 2^{-4\alpha} + \cdots).$$

Since the geometric series within brackets is convergent, this shows that (3.2) is valid.

To prove the converse, let (3.2) hold. Then by begining the summations at $n = m_k$ and $n = m_{k-1}$, respectively, and using (3.2) and the boundedness property of G, one sees that

$$\sum_{n=m_{k-1}}^{m_k} |\hat{f}(n)|^2 = O(m_{k-1}^{-2\alpha} - m_k^{-2\alpha}) = O(P_k^{2\alpha} m_k^{-2\alpha}) = O(m_k^{-2\alpha}).$$

Now, applying Parseval's identity we also get

$$\begin{aligned}\int_G |f(x+h) - f(x)|^2 dx &= \sum_{n=0}^{\infty} |\hat{f}(n)|^2 |(n,h) - 1|^2 \\ &\le \sum_{n=0}^{m_{k-1}} |\hat{f}(n)|^2 |(n,h) - 1|^2 + 4 \sum_{m_k}^{\infty} |\hat{f}(n)|^2 \\ &= O(m_k^{-2\alpha}), \quad h \in G_k,\end{aligned}$$

and the proof is thus complete. ☑

Remark 3.2. It should be noted that boundedness property (P) for G is needed only in the second part of the proof of Theorem 3.1. In fact, the first assertion was proved in [6], Theorem 3.33, without property (P) being imposed on G. It is worthwhile to mention here that in [4] (Theorem 1) an equivalence result is stated without G having property (P).

Remark 3.3. In [4] (Theorem 4) Vilenkin and Rubinstein stated an analogue of a theorem of G. G. Lorentz (see [1] Theorem 1) in the following form.

Theorem 3.4. *Let $f \in Lip\,(\alpha, G)$, and assume G has property (P). Then, for $\alpha > 1/p - 1/2$, $0 < p \le 2$,*

$$\sum_{n=m_k}^{\infty} |\hat{f}(n)|^p = O(m_k^{1-\alpha p - p/2}). \tag{3.4}$$

The authors suggest a proof of this theorem along the same lines of the original proof of Lorentz. Here we prove a more general theorem without imposing property (P) on G. Thus, we have:

Theorem 3.5. *Let $f \in Lip\,(\alpha, 2, G)$. Then (3.4) is valid for $0 < p \le 2$, $\alpha > 1/p - 1/2$.*

Proof. It is easily seen from Parseval's identity that

$$\sum_{n=m_k}^{\infty} |\hat{f}(n)|^2 \le A \sum_{n=1}^{\infty} |\hat{f}(n)|^2 |(n,h) - 1|^2 = O(m_k^{-2\alpha}).$$

Holder's inequality for $0 < p \le 2$ then yields

$$\sum_{n=m_k}^{\infty} |\hat{f}(n)|^p = O\left((m_k^{-2\alpha})^{p/2}(m_k^{1-p/2})\right) = O(m_k^{1-\alpha p-p/2}),$$

and this is equivalent to Theorem 4 in [4]. The proof is complete. ☑

We end with a few remarks in connection with the present work. First: Theorem 3.1 can be proved for higher differences of f with the same conclusion. Second: we hold it as a conjecture that Theorem 3.1 is valid on Vilenkin groups without property (P). Last: the result can possibly be extended to more general groups, such as the finite dimensional groups and some locally compact groups, abelian or non-abelian. This we will try to settle in subsequent papers.

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Mohammed S. Younis
Department of Mathematics
Yarmouk University
Irbid - Jordan