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Eight lectures on quantum groups and q-special functions

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ABSTRACT. The lectures contain an introduction to quantum groups, q-special functions and their interplay. After generalities on Hopf algebras, orthogonal polynomials and basic hypergeometric series we work out the relation between the quantum $SU(2)$ group and the Askey-Wilson polynomials out in detail as the main example. As an application we derive an addition formula for a two-parameter subfamily of Askey-Wilson polynomials. A relation between the Al-Salam and Chihara polynomials and the quantised universal enveloping algebra for $su(1,1)$ is given. Finally, more examples and other approaches as well as some open problems are given.

Key words and phrases. Quantum groups, special functions, Hopf algebras, Askey-Wilson polynomials, addition formula, convolution

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Introduction

In the latter half of this century it has become clear that there is an intimate relation between special functions of hypergeometric type, such as Jacobi, Hahn and Krawtchouk polynomials (or other polynomials within the Askey scheme of hypergeometric orthogonal polynomials), Bessel functions and representation theory of groups, notably Lie groups. See the books by Vilenkin [101] and its succesor by Vilenkin and Klimyk [102] for a large number of relations between special functions and group representations, and how group properties can imply interesting formulae for the special functions involved.

Basic, or q-hypergeometric series and q-special functions are almost as ancient as their $q = 1$ counterparts. But the development for the q-special functions has been much slower than the development for special functions. In the 1970's the works of Andrews and Askey initiated a great number of papers on q-hypergeometric series and q-special functions. One of the highlights in this development is the introduction by Askey and Wilson of a very general four parameter set of orthogonal polynomials, nowadays called the Askey-Wilson polynomials. However, before the introduction of quantum groups there was not much known about where these q-special functions 'live' in an analogous way as special functions live on Lie groups. Of course, there were some isolated results such as the interpretation of certain q-analogues of special functions on finite groups of Lie type.

Since the introduction of quantum groups by Drinfeld, Jimbo and Woronowicz in the mid-eighties, a lot of research has been going on developing the relations between q-special functions and quantum groups. A large number of such relations are known by now. Since then a huge amount of papers on quantum groups have appeared, and also a number of books on quantum groups have been published. Since we are not dealing with all kinds of aspects of quantum groups, we refer to the books by Chari and Pressley [15], Jantzen [31], Joseph [32], Kassel [40], Lusztig [64], Majid [68] for more information. Chari and Pressley [15] and also Vilenkin and Klimyk [102] have a chapter on the relation between quantum groups and q-special functions. For a more physics point of view to quantum groups and related special functions Biedenharn and Lohe [12] can be consulted.

It is the purpose of these lectures to get some feeling for quantum groups and its relation with q-special functions. I have chosen to do so by studying various aspects of one particular simple well-known example, namely the quantum group analogue of the compact Lie group of 2×2-unitary matrices, $SU(2)$. At the end we briefly discuss some other examples which are related to the treatment of the quantum $SU(2)$ group of these lectures. After reading these course notes it must be clear that there is a very nice and important interplay between quantum groups and q-special functions.

What are quantum groups? Firstly, they are *not* groups, but they are somehow related to groups in the sense that they are deformations of certain structures reflecting the group properties. This is rather vague, but more precise information can be found in the books on quantum groups mentioned. We can think of a quantum group as a deformation of the algebra of functions on a group. For a Lie group we have the duality between functions on the group and the universal enveloping algebra of the corresponding Lie algebra. It turns out that for a large class of quantum groups, there exist deformations of universal enveloping algebras such that the duality survives. These deformations are known as quantised universal enveloping algebras, or quantum algebras, or q-algebras, and for each simple Lie algebra there is a 'canonical' deformation, due to Jimbo.

In these lectures we first discuss the fundamental concept in quantum group theory, namely Hopf algebras. Two important examples, namely the algebra of functions on a (finite) group and the universal enveloping algebra, are discussed. Duality is an important concept. In the second lecture we investigate in detail deformations of the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ and the algebra of polynomials on $SL(2,\mathbb{C})$. Orthogonal polynomials and q-hypergeometric series are discussed in lecture 3, and we combine the two in a discussion of the Askey-Wilson polynomials. In lecture 4 we describe how we can characterise quantum subgroups of the quantum $SU(2)$ group, and we derive an explicit expression for the analogue of the Haar measure for left and right invariant (with respect to such a subgroup) functions. In lecture 5 we then show how the full four-parameter family of Askey-Wilson polynomials can be interpreted on the quantum $SU(2)$ group as generalised matrix elements. Using this interpretation an addition formula for a two-parameter family of Askey-Wilson polynomials is derived in lecture 6. In lecture 7 we discuss how a general convolution formula for a subclass of Askey-Wilson polynomials can be derived from the quantised universal enveloping algebra for $\mathfrak{su}(1,1)$. An overview of related results is finally given in the last lecture.

The main line in the relation between quantum groups and q-special functions as presented in these lectures uses the duality between deformed function algebras and quantised universal enveloping algebras. It is shown that results in the theory of quantum groups can be proved using q-special functions, and that identitites for q-special functions can be derived using their interpretation on quantum groups. Another approach, due to Kalnins, Miller and coworkers, and Floreanini and Vinet, only uses the quantum algebra, and we discuss this alternative shortly in lecture 8. Lecture 7 contains a result which is a kind of mixture of these two approaches.

Each lecture ends with a number of references to the literature, which, due to the enormous amount of papers in this area, cannot be complete. For each lecture a number of exercises is given. There are more published lecture notes on this subject, notably Koornwinder [57], [61] and Noumi [72]. For the general lecture 1 I have used [61] a lot. Lectures 2, 4 and 5 follow my survey paper [50] with a different proof of the expression for the Haar functional in lecture 4, which is taken from joint work with Verding [54]. Lecture 6 is based on unpublished work [47]. It is contained a limiting case of the far more complex and computational result of [51]. Lecture 7 is a special case of joint work [53] with Van der Jeugt. Lecture 8 gives a biased look to other cases, and discusses some open ends.

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1. Hopf algebras

The concept of a *Hopf algebra* is fundamental for the theory of quantum groups. In the first lecture we study this concept in some detail and we treat some important examples.

The ground field is $\mathbb{C}$, although everything goes through when working over a commutative ring with unit. The tensor product $V \otimes W$ of two linear spaces is the *algebraic tensor product*, which means that elements of $V \otimes W$ consist of finite linear combinations of the form $v \otimes w$, $v \in V$, $w \in W$.

§1.1. Algebras, bi-algebras and Hopf algebras. Recall that an *algebra*, or better, an associative algebra with unit, is a linear space A with a bilinear mapping $A \times A \to A$, $(a,b) \mapsto ab$, called *multiplication*, and a distinguished non-zero element $1 \in A$, called the *unit*, such that $a(bc) = (ab)c$ and $1a = a = a1$ for all $a, b, c \in A$. This leads to two mappings, $m: A \otimes A \to A$, also called multiplication, and $\eta: \mathbb{C} \to A$, also called unit, defined by $m(a \otimes b) = ab$ and $\eta(z) = z1$. Then we can rephrase the associatitvity and unit in terms of the following commuting diagrams;

$$
(1.1.1) \qquad
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\
{\scriptstyle id \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{C} \otimes A & \xrightarrow{\eta \otimes id} & A \otimes A & \xleftarrow{id \otimes \eta} & A \otimes \mathbb{C} \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle m} & & \downarrow {\scriptstyle \cong} \\
A & \xrightarrow{id} & A & \xleftarrow{id} & A
\end{array}
$$

where we use $\mathbb{C}\otimes A \cong A \cong A\otimes\mathbb{C}$ by identifying $z\otimes a$ and $a\otimes z$ with za for $z \in \mathbb{C}$ and $a \in A$. An algebra homomorphism means a unital algebra homomorphism, i.e. mapping unit onto unit.

For an algebra A the tensor product $A \otimes A$ is again an algebra with multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and unit $1 \otimes 1$. Note that

$$m_{A\otimes A}: A^{\otimes 4} \to A^{\otimes 2}, \qquad m_{A\otimes A} = (m_A \otimes m_A) \circ (id \otimes \sigma \otimes id) \tag{1.1.2}$$

where $\sigma: A \otimes A \to A \otimes A$ is the flip automorphism, $\sigma(a \otimes b) = b \otimes a$. Finally, note that commutativity of A is equivalent to the condition $\sigma \circ m = m$.

Definition 1.1.1. *A coalgebra, or better, a coassociative coalgebra with counit, is a linear space A with a linear mapping $\Delta: A \to A \otimes A$, called the comultiplication, and a non-zero linear mapping $\varepsilon: A \to \mathbb{C}$, called the counit, such that the following diagram is commutative;*

$$\begin{array}{ccccccc} A\otimes A\otimes A & \xleftarrow{\Delta\otimes id} & A\otimes A & \quad & \mathbb{C}\otimes A & \xleftarrow{\varepsilon\otimes id} & A\otimes A & \xrightarrow{id\otimes\varepsilon} & A\otimes\mathbb{C} \\ \uparrow{\scriptstyle id\otimes\Delta} & & \uparrow{\scriptstyle\Delta} & & \cong\downarrow & & \uparrow{\scriptstyle\Delta} & & \cong\downarrow \\ A\otimes A & \xleftarrow{\Delta} & A & & A & \xleftarrow{id} & A & \xrightarrow{id} & A \end{array} \tag{1.1.3}$$

The commutative diagram of (1.1.3) is obtained from (1.1.1) by reversing arrows.

We say that the coalgebra is cocommutative if $\sigma \circ \Delta = \Delta$.

Definition 1.1.2. *A bialgebra is an algebra A, such that A is also a coalgebra and the comultiplication Δ and counit ε are algebra homomorphisms.*

Remark 1.1.3. An equivalent definition of a bialgebra can be obtained replacing the condition that Δ and ε are algebra homomorphism by the condition that m and η are coalgebra homomorphisms.

Definition 1.1.4. *A Hopf algebra is a bialgebra A with a linear mapping $S: A \to A$, the antipode, such that the following diagram is commutative;*

$$\begin{array}{ccccc} A\otimes A & \xrightarrow{S\otimes id} & A\otimes A & \xleftarrow{id\otimes S} & A\otimes A \\ \uparrow{\scriptstyle\Delta} & & {\scriptstyle m}\downarrow & & \uparrow{\scriptstyle\Delta} \\ A & \xrightarrow{\eta\circ\varepsilon} & A & \xleftarrow{\eta\circ\varepsilon} & A \end{array} \tag{1.1.4}$$

A Hopf algebra morphism $\phi: A \to B$ of two Hopf algebras A and B is an algebra morhism $\phi: A \to B$ such that $\varepsilon_B \circ \phi = \varepsilon_A$, $\Delta_B \circ \phi = \phi \otimes \phi \circ \Delta_A$ and $S_B \circ \phi = \phi \circ S_A$.

Proposition 1.1.5. (i) *If A is a bialgebra and S an antipode making A into a Hopf algebra, then S is unique.*

(ii) *Let A be a Hopf algebra, then the antipode S is unital, counital, antimultiplicative and anticomultiplicative. Or, with σ the flip automorphism,*

$$S(1)=1,\quad \varepsilon\circ S=\varepsilon,\quad S\circ m=m\circ\sigma\circ(S\otimes S),\quad \Delta\circ S=\sigma\circ(S\otimes S)\circ\Delta.$$

Proof. (i) Let F and G be linear mappings of A into itself, then we define the convolution product $F*G$ by $F*G=m\circ(F\otimes G)\circ\Delta$. This convolution product is associative, which follows from the associativity of m and the coassociativity of Δ. Moreover, $\eta\circ\varepsilon\colon A\to A$ is the unit for the convolution product. So the endomorphism algebra of A, $End(A)$, becomes algebra. Now assume that A is a Hopf algebra with antipode S, then $S*id=\eta\circ\varepsilon=id*S$. Or, S is a two-sided inverse of the identity mapping in $End(A)$ with respect to the convolution product and thus unique.

(ii) Use (1.1.4) to $1\in A$ to find $S(1)=1$. To ease notation we introduce

$$\text{(1.1.5)}\qquad \Delta(a)=\sum_{(a)}a_{(1)}\otimes a_{(2)},\qquad (id\otimes\Delta)\Delta(a)=\sum_{(a)}a_{(1)}\otimes a_{(2)}\otimes a_{(3)},$$

which is well-defined by (1.1.3). Then by (1.1.3) we have $\sum_{(a)}\varepsilon(a_{(1)})a_{(2)}=a$. Apply S and next ε to see that

$$\varepsilon\big(S(a)\big)=\varepsilon\otimes\varepsilon\big((id\otimes S)\Delta(a)\big)=\varepsilon\circ m\circ(id\otimes S)\circ\Delta(a)=\varepsilon\circ\eta\circ\varepsilon(a)=\varepsilon(a).$$

Next

$$\begin{aligned} S(b)S(a)&=\sum_{(a),(b)}S(b_{(1)})S(a_{(1)})\varepsilon(a_{(2)}b_{(2)})\\ &=\sum_{(a),(b)}S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}S(a_{(3)}b_{(3)})\\ &=\sum_{(a),(b)}\varepsilon(a_{(1)})\varepsilon(b_{(1)})S(a_{(2)}b_{(2)})=S(ab).\end{aligned}$$

The last statement is left as an exercise. ☑

Example 1.1.6. Let G be a finite group and $A=C(G)$, the space of (continuous) complex-valued functions on G. Then A is a commutative algebra under pointwise multiplication and its unit is the constant function equal to 1. Since G is finite we have $A\otimes A\cong C(G\times G)$, and then $m(F)(g)=F(g,g)$, $\eta(z)(g)=z$. Moreover, A is a bialgebra if we define the comultiplication and counit by

$$\begin{aligned}&\Delta(f)(g,h)=f(gh),\qquad f\in A,\ g,h\in G,\\ &\varepsilon(f)=f(e),\qquad f\in A, e\in G \text{ unit of the group.}\end{aligned}$$

We define $S(f)(g) = f(g^{-1})$ and then A is a Hopf algebra by a straightforward, but instructive, check. Note that A is a commutative Hopf algebra with $S^2 = 1$.

The important observation is that the group structure –multiplication, unit and inverse– is stored in the Hopf algebra structure –comultiplication, counit and antipode– and so instead of studying G we can study $C(G)$.

This works nicely for a finite group. Now suppose that G is an algebraic subgroup of $SL(n,\mathbb{C})$, the group of $n \times n$-matrices with complex entries and determinant one, so G is some complex matrix group. Then we can take $A = \mathrm{Pol}(G)$ consisting of complex-valued functions in g, such that considered as functions of its matrix entries g_{ij} they are polynomials. Then $\mathrm{Pol}(G \times G) \cong \mathrm{Pol}(G) \otimes \mathrm{Pol}(G)$. Let $t_{ij} \in A$ be defined by $t_{ij}(g) = g_{ij}$, then the t_{ij} generate A and $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$, $\varepsilon(t_{ij}) = \delta_{ij}$, $S(t_{ij}) = T_{ji}$, where T_{ji} is the cofactor of the (ji)-th entry in the matrix $(t_{ij})_{1 \leq i,j \leq n}$.

Example 1.1.7. Let $\mathfrak{g}$ be a complex Lie algebra, i.e. a vector space over $\mathbb{C}$ equipped with a Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is a bilinear mapping such that $[X,Y] = -[Y,X]$ and $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi identity). An example of a Lie algebra is $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, the space of $n \times n$-matrices with complex coefficients with zero trace. The Lie bracket is $[X,Y] = XY - YX$, where the product on the right hand side is matrix multiplication.

Let A be the universal enveloping algebra of $\mathfrak{g}$, $A = U(\mathfrak{g})$. This is a unital algebra generated by X, $X \in \mathfrak{g}$, with relations $XY - YX = [X,Y]$ for $X, Y, \in \mathfrak{g}$. Then we define

$$\Delta(X) = 1 \otimes X + X \otimes 1,\ \varepsilon(X) = 0,\ S(X) = -X, \qquad X \in \mathfrak{g}$$

and extend Δ and ε to $U(\mathfrak{g})$ as algebra homomorphisms and S as an antihomomorphism. Then $U(\mathfrak{g})$ is a Hopf algebra. Note that it is a cocommutative Hopf algebra, but that it is not commutative unless $\mathfrak{g}$ is abelian, i.e. $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.

§1.2. Duality and Hopf $*$-algebras. A pairing between two vector spaces A and U is a bilinear mapping $A \times U \to \mathbb{C}$, $(a,u) \mapsto \langle a,u\rangle$. We say that the pairing is non-degenerate, or better, doubly non-degenerate, if $\langle a,u\rangle = 0$ for all $u \in U$ implies $a = 0$ and $\langle a,u\rangle = 0$ for all $a \in A$ implies $u = 0$. Such a pairing can be extended to a pairing of $A \otimes A$ and $U \otimes U$ by $\langle a \otimes b, u \otimes v\rangle = \langle a,u\rangle\langle b,v\rangle$.

Definition 1.2.1. *Two Hopf algebras A and U are said to be in duality if there exists a non-degenerate pairing $\langle\cdot,\cdot\rangle\colon A \times U \to \mathbb{C}$ such that*

$$\langle a \otimes b, \Delta(u)\rangle = \langle ab, u\rangle, \qquad \langle \Delta(a), u \otimes v\rangle = \langle a, uv\rangle,$$
$$\langle 1, u\rangle = \varepsilon(u), \qquad \langle a, 1\rangle = \varepsilon(a), \qquad \langle S(a), u\rangle = \langle a, S(u)\rangle.$$

Example 1.2.2. Recall that if G is an (algebraic) subgroup of $SL(n,\mathbb{C})$ and its Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{sl}(n,\mathbb{C})$, then we have a natural pairing of

$\mathrm{Pol}(G)$ and $\mathfrak{g}$ by

$$\langle X, f\rangle = \frac{d}{dt}\Big|_{t=0} f(\exp tX), \qquad X\in\mathfrak{g},\ f\in \mathrm{Pol}(G).$$

From Definition 1.2.1 and Examples 1.1.6 and 1.1.7 we find the familiar expression

$$\langle X_1\ldots X_k, f\rangle = \frac{\partial^k}{\partial t_1\ldots\partial t_k}\Big|_{t_1=\ldots=t_k=0} f\big((\exp t_1X_1)\ldots(\exp t_kX_k)\big)$$

for $X_1,\ldots,X_k\in\mathfrak{g}$, $f\in\mathrm{Pol}(G)$.

Using this pairing we can define a left and right action of $\mathfrak{g}$ on $\mathrm{Pol}(G)$ by left and right invariant first order differential operators; for $g\in G$, $X\in\mathfrak{g}$, $f\in\mathrm{Pol}(G)$,

$$X.f(g) = \frac{d}{dt}\Big|_{t=0} f(g\exp tX), \qquad f.X(g) = \frac{d}{dt}\Big|_{t=0} f((\exp tX)g)$$

and we extend this to $U(\mathfrak{g})$ by $(XY).f = X.(Y.f)$ and $f.(XY) = (f.X).Y$. This gives the action of $U(\mathfrak{g})$ on smooth functions on G by left or right invariant differential operators. This can be done for arbitrary Hopf algebras in duality.

Proposition 1.2.3. *Let A and U be two Hopf algebras in duality, then for $u\in U$ and $a\in A$*

$$u.a = (id\otimes\langle\cdot,u\rangle)\circ\Delta(a), \qquad a.u = (\langle\cdot,u\rangle\otimes id)\circ\Delta(a)$$

define a left and right action of U on A.

Proof. Using (1.1.5) we see that

$$(vu).a = \sum_{(a)}\langle a_{(2)}, vu\rangle a_{(1)} = \sum_{(a)} a_{(1)}\langle a_{(2)},v\rangle\langle a_{(3)},u\rangle = v.(u.a),$$

and and similarly for the right action. ☑

Recall that an algebra A is $*$-algebra if there exists an antilinear antimultiplicative involution $a\mapsto a^*$. So $(\lambda a+\mu b)^* = \bar\lambda a^* + \bar\mu b^*$, $(ab)^* = b^*a^*$, $(a^*)^* = a$ for $a,b\in A$, $\lambda,\mu\in\mathbb{C}$. If $\phi\colon A\to B$ is an algebra homomorphism of two $*$-algebras A and B, then ϕ is a $*$-homomorphism if $\phi(a^*) = \phi(a)^*$.

Definition 1.2.4. *A Hopf $*$-algebra is a Hopf algebra A, such that A is a $*$-algebra and Δ and ε are $*$-homomorphisms. A Hopf $*$-algebra morphism $\phi\colon A\to B$ of two Hopf $*$-algebras A and B is a Hopf algebra morphism such that $\phi(a^*) = \big(\phi(a)\big)^*$.*

There is no requirement on the relation between the antipode S and $*$ in Definition 1.2.4.

Proposition 1.2.5. *Let A be a Hopf $*$-algebra, then $(S\circ *)^2 = id$. In particular, S is invertible.*

Proof. For an algebra A we define the opposite algebra A_{opp} as the algebra with the same vector space structure and multiplication and unit defined by

$$m_{\mathrm{opp}} = m\circ\sigma\colon A_{\mathrm{opp}}\otimes A_{\mathrm{opp}}\to A_{\mathrm{opp}}, \qquad \eta_{\mathrm{opp}} = \eta\colon A_{\mathrm{opp}}\to\mathbb{C}.$$

A_{opp} is a bialgebra with unchanged comultiplication and counit. If A_{opp} is also a Hopf algebra then S is invertible and the antipode for A_{opp} is S^{-1}.

Let A be a Hopf $*$-algebra. Use (1.1.4) on a^*, and apply again the $*$-operator to get, using the notation (1.1.5),

$$\sum_{(a)} a_{(2)}\big(S(a^*_{(1)})\big)^* = \varepsilon(a)1 = \sum_{(a)}\big(S(a^*_{(a)})\big)^* a_{(2)}$$

or $*\circ S\circ *$ is the antipode for A_{opp}. Using Proposition 1.1.5(i) and the previous paragraph it follows that $S^{-1} = *\circ S\circ *$. ☑

Example 1.2.6. Let $G\subset SL(n,\mathbb{C})$ be an algebraic group as in Example 1.1.6, and let G_0 be a real connected group which is a real form of G. E.g. if $G = SL(n,\mathbb{C})$ we can take G_0 equal to $SL(n,\mathbb{R})$ ($n\times n$ matrices with real entries and determinant one), which are the fixed points of the involution which is entry-wise complex conjugation, or to $SU(n)$ ($n\times n$ unitary matrices with determinant one), which are the fixed points of the involution which takes adjoints. A polynomial p on G is then completely determined by its restriction to the real form G_0. Suppose that for every $p\in \mathrm{Pol}(G)$ there exists a polynomial $p^*\in\mathrm{Pol}(G)$ such that $p^*(g_0) = \overline{p(g_0)}$ for all $g_0\in G_0$, then $\mathrm{Pol}(G)$ is a Hopf $*$-algebra. Conversely, if $\mathrm{Pol}(G)$ is a Hopf $*$-algebra, then

$$G_0 = \{g\in G \mid p^*(g) = \overline{p(g)}\,\forall p\in\mathrm{Pol}(G)\}$$

defines a real form of G.

So $\mathrm{Pol}(G)$ as a Hopf algebra carries the properties of a complex group, and $\mathrm{Pol}(G)$ considered as a Hopf $*$-algebra carries the properties of a real group.

Definition 1.2.7. *Two Hopf $*$-algebras A and U are in duality (as Hopf $*$-algebras), if they are in duality as Hopf algebras and $\langle a^*, u\rangle = \overline{\langle a, (S(u))^*\rangle}$.*

Example 1.2.8. From Example (1.1.7) we know that $U(\mathfrak{g})$ is a Hopf algebra, which is in duality with the Hopf algebra $\mathrm{Pol}(G)$ if the Lie algebra of G is $\mathfrak{g}$. Suppose $\mathrm{Pol}(G)$ is a Hopf $*$-algebra and let G_0 be the corresponding real form of G and $\mathfrak{g}_0$ the corresponding real form of $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$. Then for $X\in\mathfrak{g}_0$ we have

$$\langle X, p\rangle = \overline{\langle X, (S(p))^*\rangle} = \overline{\frac{d}{dt}\Big|_{t=0} p(\exp(-tX))} = -\langle X, p\rangle,$$

so $X^* = -X$ for $X\in\mathfrak{g}_0$. E.g. in case $G = SL(n,\mathbb{C})$ and $G_0 = SU(n)$, then the $*$-operator on the Lie algebra level is taking adjoints.

§1.3. Invariant functional. Let A be a Hopf algebra and assume that a linear functional $h: A \to \mathbb{C}$ satisfies

$$(id \otimes h) \circ \Delta = \eta \circ h = (h \otimes id) \circ \Delta \tag{1.3.1}$$

then we say that h is an *invariant functional*, or h is a *Haar functional*. If only the first, respectively last, equation of (1.3.1) holds, then we say that h is a left, respectively right, invariant functional. If it exists we normalise h by $h(1) = 1$.

Theorem 1.3.1. *If the Haar functional exists on a Hopf algebra A, then it is unique up to a scalar multiple.*

Example 1.3.2. Let G be an algebraic subgroup of $SL(n, \mathbb{C})$, then the left Haar measure $d\mu$ gives a left invariant functional on $\mathrm{Pol}(G)$ by $h(p) = \int_G p(g)\, d\mu(g)$, assuming that the polynomials are integrable on G with respect to the Haar measure. Here we have used that

$$(id \otimes h)\Delta(p)(g') = \int_G p(g'g)\, d\mu(g) = \int_G p(g)\, d\mu(g) = h(p).$$

If $d\mu$ is also right invariant, i.e. if G is unimodular, then h is also a right invariant functional.

For the special case of the quantised function algebra on the compact group $SU(2)$ we give a proof of the existence of the invariant integral.

Notes and references. The standard references for the theory of Hopf algebras before the introduction of quantum groups are Abe [1] and Sweedler [91]. Since its introduction a wealth of papers has appeared, and we have used Chari and Pressley [15, Ch. 4] and especially Koornwinder [61]. Important concepts within the theory of Hopf algebras related to quantum groups are quasitriangular Hopf algebras and the quantum double construction, both due to Drinfeld [21], see also [15], [68]. Also, we haven't mentioned corepresentations of a coalgebra, see Exercise 1.6.

For Hopf $*$-algebras in duality we refer to Van Daele [98]. The invariant functional is an important tool in the harmonic analysis on quantum groups and proofs of Theorem 1.3.1 can be found for the C^*-algebra setting in Woronowicz's influential fundamental paper [103], see also Van Daele [99] for an up-to-date version. A purely algebraic proof can be found in Dijkhuizen and Koornwinder [17], see als [61].

Exercises.

1. In (1.1.2) the multiplication $m_{A\otimes A}$ is defined using the flip automorphism σ. Now let $\Psi: A \otimes A \to A \otimes A$ be a linear map and define $m_{A\otimes A} = m \otimes m \circ id \otimes \Psi \otimes id$. Suppose that $\Psi(a \otimes 1) = 1 \otimes a$, $\Psi(1 \otimes a) = a \otimes 1$ for all $a \in A$, and that

$$\Psi \circ (m \otimes id) = (id \otimes m) \circ (\Psi \otimes id) \circ (id \otimes \Psi),$$
$$\Psi \circ (id \otimes m) = (m \otimes id) \circ (id \otimes \Psi) \circ (\Psi \otimes id).$$

Prove that $A \otimes A$ is an associative algebra with unit $1 \otimes 1$. The map Ψ is called the *braiding*.

2. Define the notion of coalgebra homomorphism and prove Remark 1.1.3.
3. Prove the last statement of Proposition 1.1.5(ii).
4. Check that the examples in Example 1.1.6 and 1.1.7 are Hopf algebras. (First prove that in Example 1.1.7 the mappings Δ, ε and S are well-defined as follows; $U(\mathfrak{g}) = T(\mathfrak{g})/I$ where $T(\mathfrak{g})$ is the tensor algebra for $\mathfrak{g}$ and I is the ideal in $T(\mathfrak{g})$ generated by $X \otimes Y - Y \otimes X - [X, Y]$ for all $X, Y \in \mathfrak{g}$. Then show that $\Delta(I) \subset T(\mathfrak{g}) \otimes I + I \otimes T(\mathfrak{g})$, $\varepsilon(I) = 0$, $S(I) \subset I$. This means that $I \subset T(\mathfrak{g})$ is a Hopf ideal.)
5. Prove that A_{opp} is a Hopf algebra, see proof of Proposition 1.2.5. Similarly, if A is a coalgebra we define A^{opp} as the coalgebra with comultiplication $\Delta^{\mathrm{opp}} = \sigma \circ \Delta$ and $\varepsilon^{\mathrm{opp}} = \varepsilon$. Suppose that A is a Hopf algebra with invertible antipode S, prove that A^{opp} is a Hopf algebra with unchanged multiplication and unit and antipode S^{-1}. Prove also that $A^{\mathrm{opp}}_{\mathrm{opp}} = (A_{\mathrm{opp}})^{\mathrm{opp}} = (A^{\mathrm{opp}})_{\mathrm{opp}}$ is a Hopf algebra with antipode S.
6. A (left) representation π of an algebra A in the linear space V is a bilinear mapping $\pi: A \otimes V \to V$, $\pi: a \otimes v \mapsto a \cdot v$, such that $(ab) \cdot v = a \cdot (b \cdot v)$ and $1 \cdot v = v$. Rephrase this in terms of commutative diagrams using the multiplication m and unit η. Define the notion of a corepresentation of a coalgebra by reversing arrows.

2. The quantum $SL(2, \mathbb{C})$ group

In this lecture we consider the fundamental example of a non-commutative, non-cocommutative Hopf algebra; the so-called quantised universal enveloping algebra for $\mathfrak{sl}(2, \mathbb{C})$.

§2.1. The Hopf algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$. Let $U_q(\mathfrak{sl}(2, \mathbb{C}))$ be the complex unital associative algebra generated by A, B, C, D subject to the relations

(2.1.1)
$$AD = 1 = DA, \quad AB = qBA, \quad AC = q^{-1}CA, \quad BC - CB = \frac{A^2 - D^2}{q - q^{-1}}.$$

On the level of generators we define the comultiplication, counit and antipode by

$$\begin{aligned} \Delta(A) &= A \otimes A, & \Delta(B) &= A \otimes B + B \otimes D, \\ \Delta(C) &= A \otimes C + C \otimes D, & \Delta(D) &= D \otimes D, \\ \varepsilon(A) &= \varepsilon(D) = 1, & \varepsilon(C) &= \varepsilon(B) = 0, \\ S(A) = D, \quad S(B) &= -q^{-1}B, & S(C) = -qC, \quad S(D) &= A. \end{aligned} \tag{2.1.2}$$

Here q is thought of as a deformation parameter, and at first we take $q \in \mathbb{C} \setminus \{-1, 0, 1\}$. (It is also possible to view $U_q(\mathfrak{sl}(2, \mathbb{C}))$ as an algebra over $\mathbb{C}(q)$.)

The case $q \to 1$ is considered in a moment. However, we will always assume that q is not a root of unity, i.e. $q^m \neq 1$ for all $m \in \mathbb{Z}_+$. Observe that S is invertible, but $S^2 \neq 1$ since $q^2 \neq 1$.

Proposition 2.1.1. *Define Δ and ε on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ by (2.1.2) as (unital) algebra homomorphisms and S by (2.1.2) as (unital) anti-algebra homomorphisms, then $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is a Hopf algebra.*

Proof. We have to check that Δ, ε and S are well-defined and that they satisfy the axioms of a Hopf algebra. These are straighforward computations. E.g.

$$\Delta(AB) = A^2 \otimes AB + AB \otimes 1 = q(A^2 \otimes BA + BA \otimes 1) = q\Delta(BA)$$

and

$$m \circ (id \otimes S) \circ \Delta(B) = AS(B) + BS(D) = -q^{-1}AB + BA = 0 = \varepsilon(B).$$

Continuing in this way proves the proposition. ☑

The element

$$\Omega = \frac{q^{-1}A^2 + qD^2 - 2}{(q^{-1}-q)^2} + BC = \frac{qA^2 + q^{-1}D^2 - 2}{(q^{-1}-q)^2} + CB \tag{2.1.3}$$

is the Casimir element of the quantised universal enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$. Ω belongs to the centre of $U_q(\mathfrak{sl}(2,\mathbb{C}))$.

To justify the name for this Hopf algebra, we replace A by $\exp((q-1)H/2)$, and hence D by $\exp((1-q)H/2)$ and we let $q \uparrow 1$. Then we can deduce from (2.1.1) that in the limit we get

$$[H,B] = 2B, \qquad [H,C] = -2C, \qquad [B,C] = H.$$

To see the first relation, use $A = \exp((q-1)H/2)$ in $AB = qBA$ to get

$$B + \frac{1}{2}(q-1)HB = qB + \frac{1}{2}q(q-1)BH + \mathcal{O}((q-1)^2) \Longrightarrow$$
$$\frac{1}{2}(q-1)(HB - qBH) = (q-1)B + \mathcal{O}((q-1)^2).$$

Divide both sides by $q-1$ and let $q \to 1$ to get the first relation. The other relations are obtained similarly. With

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

we see that $\{H, B, C\}$ forms a basis for the three-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Moreover, we see that (2.1.2) tends to the standard Hopf algebra structure on $U(\mathfrak{sl}(2,\mathbb{C})$ as in Example 1.1.7 as $q \to 1$.

For the universal enveloping algebra the Poincaré-Birkhoff-Witt theorem gives a basis for the underlying linear space. Here we have a similar result, but the proof is not a straightforward generalisation of the PBW-theorem.

Lemma 2.1.2. *A linear basis for $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is given by $D^l C^k B^m$ for $k, m \in \mathbb{Z}_+$, $l \in \mathbb{Z}$ with the convention $D^{-l} = A^l$ for $l \in \mathbb{Z}_+$.*

§2.2. Finite dimensional representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$. We first concentrate on the algebra structure of $U_q(\mathfrak{sl}(2,\mathbb{C}))$. Let H be a finite dimensional vector space, then $\mathcal{B}(H)$ is the (unital associative) algebra of linear operators of H into itself. A representation t of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ in H is an algebra morphism $t: U_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathcal{B}(H)$. A linear subspace $V \subset H$ is called invariant if $t(X)V \subset V$ for all $X \in U_q(\mathfrak{sl}(2,\mathbb{C}))$, and then the restriction $t|_V: U_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathcal{B}(V)$, $t|_V(X) = t(X)|_V$ is a subrepresentation of t. Obviously, $\{0\}$ and H are invariant subspaces of H, and if there are no more invariant subspaces we say that the representation t in H is irreducible. Two irreducible representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$, say $t: U_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathcal{B}(H)$ and $s: U_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathcal{B}(V)$, are said to be equivalent if there exists a linear bijection $T: H \to V$ such that $Tt(X) = s(X)T$ for all $X \in U_q(\mathfrak{sl}(2,\mathbb{C}))$.

Theorem 2.2.1. *For each dimension $N+1$, $N \in \mathbb{Z}_+$, there are four inequivalent irreducible representations. Explicitly, there exists a basis $\{e_0, \dots, e_N\}$ of $\mathbb{C}^{N+1}$ such that they are given by $t(A)\, e_k = \lambda q^{N/2-k}\, e_k$, $t(C)\, e_k = e_{k+1}$, and*

$$t(B)\, e_k = \frac{q^{N+1}\lambda^2(1-q^{-2k}) + q^{-1-N}\lambda^{-2}(1-q^{2k})}{(q-q^{-1})^2}\, e_{k-1}$$

for $\lambda^4 = 1$ with the convention $e_{-1} = 0 = e_{N+1}$.

Proof. Let t be an irreducible representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ in H, and let λ be an eigenvalue of the operator $t(A)$ for the eigenvector v. Then $t(B)^k v$ is an eigenvector for $t(A)$ for the eigenvalue λq^k. Since H is finite dimensional $t(B)^k v = 0$ for k large enough, so we may assume that $t(B)v = 0$. Define $e_0 = v$ and $e_k = t(C)^k e_0$ for $k \in \mathbb{Z}_+$. Let N be the smallest integer such that $e_{N+1} = 0$ and $e_N \neq 0$, then $\mathrm{span}(e_0, \dots, e_N)$ is a non-zero invariant subspace and hence equals H. Then $t(C)e_k = e_{k+1}$ with the convention $e_{N+1} = 0$ and $t(A)e_k = \lambda q^{N/2-k} e_k$ after rescaling λ. Since $t(B)e_k$ is an eigenvector of $t(A)$ for the eigenvalue $\lambda q^{N/2+1-k}$ we have $t(B)e_k = \mu_k e_{k-1}$ for some constant μ_k. The commutation relation for B and C imply

$$\mu_{k+1} - \mu_k = \frac{\lambda^2 q^{N-2k} - \lambda^{-2} q^{2k-N}}{q-q^{-1}} \Longrightarrow$$
$$\mu_k = \frac{q^{N+1}\lambda^2(1-q^{-2k}) + q^{-1-N}\lambda^{-2}(1-q^{2k})}{(q-q^{-1})^2}.$$

This calculation takes into account the initial condition $\mu_0 = 0$. Since we also have the end condition $\mu_N = (\lambda^{-2}q^N - \lambda^2 q^{-N})/(q-q^{-1})$ these two expressions have to be equal, and this leads to the condition $\lambda^4 = 1$. The representations obtained in this way give all finite dimensional representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$.

For different dimensions they are mutually inequivalent, and for the same dimension we see that they are mutually inequivalent since the spectrum of the operator corresponding to A is different. ☑

In the sequel we only use the representations with the spectrum of A contained in $q^{\frac{1}{2}\mathbb{Z}_+}$, or $\lambda = 1$.

§2.3. $*$-structures on $U_q(\mathfrak{sl}(2,\mathbb{C}))$. We have seen that $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is a Hopf algebra, and we now consider how the Hopf algebra structure depends on q and whether $U_q(\mathfrak{sl}(2,\mathbb{C}))$ can be made into a Hopf $*$-algebra. For this purpose we need some special elements of $U_q(\mathfrak{sl}(2,\mathbb{C}))$. We call $0 \neq X \in U_q(\mathfrak{sl}(2,\mathbb{C}))$ *group like* if $\Delta(X) = X \otimes X$. Note that this implies $\varepsilon(X) = 1$. Let X be group like, then we say that $Y \in U_q(\mathfrak{sl}(2,\mathbb{C}))$ is *twisted primitive* with respect to X if $\Delta(Y) = X \otimes Y + Y \otimes S(X)$. Note that this implies $\varepsilon(Y) = 0$ and that the twisted primitive elements form a linear subspace of $U_q(\mathfrak{sl}(2,\mathbb{C}))$. In case we consider the Hopf algebra $U(\mathfrak{g})$, cf. Example 1.1.7, the unit element 1 is the only group like element and the elements of $\mathfrak{g} \subset U(\mathfrak{g})$ are the only twisted primitive elements.

Proposition 2.3.1. *In the Hopf algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ we have*

(i) *A^m, $m \in \mathbb{Z}$, are the group like elements;*
(ii) *$\mathbb{C}(B, C, A - D)$ is the space of twisted primitive elements with respect to A;*
(iii) *$\mathbb{C}(A^m - D^m)$ is the space of twisted primitive elements with respect to A^m for $m \neq 1$.*

For the proof of Proposition 2.3.1 we need the following lemma describing a q-analogue of Newton's binomial formula. For $a \in \mathbb{C}$ we define the q-shifted factorial $(a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i)$ with the empty product equal to 1. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{(q^n;q^{-1})_k}{(q;q)_k} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{2.3.1}$$

Lemma 2.3.2. *Let x, y be elements of an associative algebra satisfying $xy = qyx$, then*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k}.$$

Proof. Use the recurrence

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \tag{2.3.2}$$

and complete induction with respect to n. ☑

Proof of Proposition 2.3.1. Use Lemma 2.3.2 to see that

$$\Delta(C^m) = (A\otimes C + C\otimes D)^m = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^{-2}} C^i A^{m-i} \otimes D^i C^{m-i}$$

and an expression for $\Delta(B^k)$ can be obtained in this way too. So using Lemma 2.1.2 we consider general $X = \sum c_{klm} D^l C^m B^k$ and rewriting each factor in the tensor product into the basis of Lemma 2.1.2 we obtain

$$(2.3.3)\quad \Delta(X) = \sum_{klm}\sum_{j=0}^{k}\sum_{i=0}^{m} c_{klm} \begin{bmatrix} k \\ j \end{bmatrix}_{q^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^{-2}} q^{(i-j)(k-j+m-i)} \\ \times D^{l-m+i-k+j} C^i B^j \otimes D^{l+i+j} C^{m-i} B^{k-j}.$$

To prove (i) we compare (2.3.3) with

$$(2.3.4)\qquad X\otimes X = \sum_{uvw}\sum_{abc} c_{uvw} c_{abc} D^v C^w B^u \otimes D^b C^c B^a.$$

So $w+c=m$, $a+u=k$, $b-v=m+k$. Suppose $m>0$, then (2.3.4) has a non-zero term of the form $D^v C^m B^u \otimes D^b C^m B^a$, which cannot occur in (2.3.3). Hence, $m=0$ and similarly $k=0$. So $X=\sum_l c_l D^l$ and

$$\sum_{l,k} c_l c_k D^l \otimes D^k = \sum_l c_l D^l \otimes D^l$$

implying $c_l = 0$ except for one $l\in\mathbb{Z}$ and the non-zero c_l has to satisfy $c_l^2 = c_l$ or $c_l = 1$.

In the proofs of (ii) and (iii) we have to compare (2.3.3) with the appropriate expressions. These proofs are left as exercises. ☑

Proposition 2.3.1(ii) shows that only for the group like element A the corresponding twisted primitive elements leads to a three-dimensional space. So from now on we consider twisted primitive elements only with respect to A and we consider these elements as the proper analogues of the three-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

Theorem 2.3.3. *$U_q(\mathfrak{sl}(2,\mathbb{C})) \cong U_p(\mathfrak{sl}(2,\mathbb{C}))$ as Hopf algebras if and only if $p=q$ or $p=q^{-1}$.*

Proof. Let $\phi\colon U_q(\mathfrak{sl}(2,\mathbb{C})) \to U_p(\mathfrak{sl}(2,\mathbb{C}))$ be the Hopf algebra isomorphism, then ϕ maps group like elements onto group like elements, so $\phi(D^l) = D^{\tau(l)}$. Since ϕ is an algebra homomorphism, τ is an automorphism of $\mathbb{Z}$. Thus, $\phi(D) = D$ or $\phi(D) = D^{-1} = A$. In the last situation Proposition 2.3.1(ii), (iii) shows that ϕ maps the three-dimensional space of twisted primitive elements with respect to A to the one-dimensional space of twisted primitive elements with respect to D, contradicting ϕ being an isomorphism.

So $\phi(A) = A$, $\phi(D) = D$ and ϕ maps the twisted primitive elements onto the twisted primitive elements (both with respect to A). Now the map $X \mapsto AXD$ maps the space of twisted primitive elements into itself and has eigenvalues 1, q and q^{-1} for respectively the eigenvectors $A - D$, B and C. Hence, $\{1, q, q^{-1}\} = \{1, p, p^{-1}\}$ and thus $p = q$ or $p = q^{-1}$.

From (2.1.1) we see that in case $p = q$ we can define $\phi(B) = \lambda B$, $\phi(C) = \lambda^{-1}C$ for some non-zero $\lambda \in \mathbb{C}$ and that in case $p = q^{-1}$ we can define $\phi(B) = \lambda C$, $\phi(C) = \lambda^{-1}B$ for some non-zero $\lambda \in \mathbb{C}$. It is straightforward to check that ϕ defined in this way gives a Hopf algebra isomorphism. ☑

So we can assume without loss of generality that $|q| \leq 1$. We say that two $*$-structures on a Hopf algebra are equivalent if there exists a Hopf algebra isomorphism of the Hopf algebra onto itself intertwining the two $*$-structures. Otherwise, they are inequivalent $*$-structures.

Theorem 2.3.4. *The list of mutually inequivalent $*$-structures on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is*

(i) $|q| = 1$; $A^* = A$, $B^* = -B$, $C^* = -C$ *and corresponding real form* $U_q(\mathfrak{sl}(2,\mathbb{R}))$,
(ii) $-1 < q < 1$, $q \neq 0$; $A^* = A$, $B^* = C$, $C^* = B$ *and corresponding real form* $U_q(\mathfrak{su}(2))$,
(iii) $-1 < q < 1$, $q \neq 0$; $A^* = A$, $B^* = -C$, $C^* = -B$ *and corresponding real form* $U_q(\mathfrak{su}(1,1))$.

Proof. Suppose $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is a Hopf $*$-algebra, then we can think of $*$ as an antilinear Hopf algebra isomorphism of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ onto $\big(U_{\bar q}(\mathfrak{sl}(2,\mathbb{C}))\big)_{\text{opp}} \cong U_{\bar q^{-1}}(\mathfrak{sl}(2,\mathbb{C})$ as Hopf algebras. The Hopf algebra isomorphism is just interchanging B and C. So we must have $q = \bar q$ or $q = \bar q^{-1}$, so $q \in \mathbb{R}$ or $|q| = 1$. Using the proof of Theorem 2.3.3 we see that in the first case we have to have $B^* = \lambda C$, $C^* = \lambda^{-1}B$ for some non-zero $\lambda \in \mathbb{R}$ and in the second case we have to have $B^* = \lambda B$, $C^* = \lambda^{-1}C$ for $|\lambda| = 1$. The condition on the λ follows from the $*$-operator being an involution. Now use Theorem 2.3.3 again to pick out the inequivalent $*$-structures. ☑

Remark 2.3.5. The names for these real forms are motivated by the fact that in case $q \uparrow 1$ the -1-eigenspace of the corresponding $*$-operator in the Lie algebra, cf. Example 1.2.8, are $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. Note that $\mathfrak{su}(1,1) = C\,\mathfrak{sl}(2,\mathbb{R})\,C^{-1}$ with $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, so $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(1,1)$ are conjugate and hence have the same representation theory. Theorem 2.3.4 shows that this is no longer true in the quantum case. The $*$-operator from Theorem 2.3.4(ii) and corresponding Hopf $*$-algebra is sometimes called the compact real form, since $SU(2)$ is a compact group.

From now on, when considering the real forms $U_q(\mathfrak{su}(2))$ or $U_q(\mathfrak{su}(1,1))$ we always take $0 < q < 1$. This can be done without much loss of generality, since it turns out that the special functions associated to these Hopf $*$-algebras essentially depend on q^2.

It is straightforward to see that for $U_q(\mathfrak{su}(2))$ the finite dimensional representations of Theorem 2.2.1 are unitarisable for $\lambda = \pm 1$. We only consider $\lambda = 1$ and we redefine these representations in the following theorem.

Theorem 2.3.6. *For each spin $l \in \frac{1}{2}\mathbb{Z}_+$ there exists a unique $(2l+1)$-dimensional $*$-representation of $U_q(\mathfrak{su}(2))$ such that the spectrum of A is contained in $q^{\frac{1}{2}\mathbb{Z}_+}$. Equip $\mathbb{C}^{2l+1}$ with orthonormal basis $\{e^l_n\}$, $n = -l, -l+1, \ldots, l$ and denote the representation by t^l. The action of the generators is given by*

$$t^l(A)\, e^l_n = q^{-n} e^l_n, \qquad t^l(D)\, e^l_n = q^n\, e^l_n,$$

$$t^l(B)\, e^l_n = \frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1} - q}\, e^l_{n-1}, \tag{2.3.5}$$

$$t^l(C)\, e^l_n = \frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1} - q}\, e^l_{n+1},$$

where $e^l_{l+1} = 0 = e^l_{-l-1}$.

§2.4. Dual Hopf algebra. Let us now consider the fundamental two-dimensional representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ for spin $l = 1/2$. The matrix elements of $t^{1/2}$ give four linear functionals on $U_q(\mathfrak{sl}(2,\mathbb{C}))$;

$$\begin{cases} t^{1/2}(X)\, e^{1/2}_{-1/2} = \alpha(X)\, e^{1/2}_{-1/2} + \beta(X)\, e^{1/2}_{1/2} \\ t^{1/2}(X)\, e^{1/2}_{1/2} = \gamma(X)\, e^{1/2}_{-1/2} + \delta(X)\, e^{1/2}_{1/2} \end{cases} \iff t^{1/2}(X) = \begin{pmatrix} \alpha(X) & \beta(X) \\ \gamma(X) & \delta(X) \end{pmatrix}.$$

It follows from (2.3.5) that on the basis of Lemma 2.1.2 the linear functionals are given by

$$\begin{gathered} \alpha(D^l C^m B^n) = \delta_{n0}\delta_{m0} q^{-l/2}, \quad \beta(D^l C^m B^n) = \delta_{n1}\delta_{m0} q^{-l/2}, \\ \gamma(D^l C^m B^n) = \delta_{n0}\delta_{m1} q^{l/2}, \quad \delta(D^l C^m B^n) = (\delta_{n0}\delta_{m0} + \delta_{n1}\delta_{m1}) q^{l/2}. \end{gathered} \tag{2.4.1}$$

Theorem 2.4.1. *Let $A_q(SL(2,\mathbb{C}))$ be the complex unital associative subalgebra of the linear dual of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ generated by α, β, γ, δ. Then the following relations hold;*

$$\begin{gathered} \alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \\ \beta\gamma = \gamma\beta, \quad \alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\beta\gamma = 1. \end{gathered} \tag{2.4.2}$$

Then $A_q(SL(2,\mathbb{C}))$ is Hopf algebra. The comultiplication Δ, the counit ε and the antipode S given on the generators by

$$\begin{aligned}\Delta(\alpha) &= \alpha\otimes\alpha+\beta\otimes\gamma, & \Delta(\beta) &= \alpha\otimes\beta+\beta\otimes\delta,\\ \Delta(\gamma) &= \gamma\otimes\alpha+\delta\otimes\gamma, & \Delta(\delta) &= \gamma\otimes\beta+\delta\otimes\delta,\end{aligned}\tag{2.4.3}$$

$$\varepsilon\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\delta & -q^{-1}\beta\\ -q\gamma & \alpha\end{pmatrix}.\tag{2.4.4}$$

$A_q(SL(2,\mathbb{C}))$ has a linear basis formed by the matrix elements

$$t^l_{m,n}\colon X\mapsto \langle t^l(X)e^l_n, e^l_m\rangle.$$

For the proof we need a lemma, which is of interest on its own. First we discuss the notion of a tensor product representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$. If t and s are representations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ in V and W, then we define the tensor product representation $t\otimes s$ of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ in $V\otimes W$ by using the comultiplication Δ, cf. (1.1.5),

$$(t\otimes s)(X)v\otimes w = \sum_{(X)} t(X_{(1)})\,v\otimes s(X_{(2)})\,w,\qquad X\in U_q(\mathfrak{sl}(2,\mathbb{C})).$$

Then we have the following Clebsch-Gordan decomposition.

Lemma 2.4.2. $t^{l_1}\otimes t^{l_2}\cong\bigoplus_{l=|l_1-l_2|}^{l_1+l_2} t^l$.

Proof. Although Lemma 2.4.2 is not concerned with $*$-structures we use the $*$-structure from $U_q(\mathfrak{su}(2))$. Then $t^{l_1}\otimes t^{l_2}$ is a finite-dimensional unitary representation of $U_q(\mathfrak{su}(2))$ and hence completely reducible. Since the spectrum of $(t^{l_1}\otimes t^{l_2})(A)$ is contained in $q^{\frac{1}{2}\mathbb{Z}_+}$ we have a decomposition of the form $t^{l_1}\otimes t^{l_2}=\bigoplus_l m_l\, t^l$ for certain multiplicities m_l. Since $e^{l_1}_n\otimes e^{l_2}_m$ is an eigenvector of the action for A for the eigenvalue q^{-n-m} we can read off the multiplicities from Theorem 2.3.6 by counting eigenvalues for A. ☑

Proof of Theorem 2.4.1. $A_q(SL(2,\mathbb{C}))$ is automatically a Hopf algebra by Definition 1.2.1 if we can show that comultiplication maps into the (algebraic) tensor product $A_q(SL(2,\mathbb{C}))\otimes A_q(SL(2,\mathbb{C}))$. The definition of comultiplication, counit and antipode follow from Definition 1.2.1. Using $t^{1/2}(XY) = t^{1/2}(X)t^{1/2}(Y)$ we find that

$$\langle\Delta(\alpha), X\otimes Y\rangle = \alpha(XY) = \alpha(X)\alpha(Y)+\beta(X)\gamma(Y) = \langle\alpha\otimes\alpha+\beta\otimes\gamma, X\otimes Y\rangle$$

by inspecting the upper left entry. Inspection of the other entries leads to the action of the comultiplication on the other generators. In particular, $A_q(SL(2,\mathbb{C}))$ is a Hopf algebra.

The action of the counit follows by taking $l = m = n = 0$ in (2.4.1). The action of S can be calculated by $S(D^lC^mB^n) = (-q)^{m-n}B^nC^mA^l$, so that $\langle S(\alpha), D^lC^mB^n\rangle = (-q)^{m-n}\alpha(B^nC^mA^l) = (\delta_{n0}\delta_{m0} + \delta_{n1}\delta_{m1})q^{l/2} = \delta(D^lC^mB^n)$ by (2.4.1), and similarly for the other generators. From (1.1.4) we see that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

which implies the relations (2.4.2).

Consider the matrix element $(t^{l_1}\otimes t^{l_2})_{ij;lm} = t^{l_1}_{il}t^{l_2}_{jm}$. Take $l_1 = 1/2$ and use induction with respect to l_2 and Lemma 2.4.2 to see that the matrix elements of all spin representations are contained in $A_q(SL(2,\mathbb{C}))$. Iterating Lemma 2.4.2 shows that each matrix element t^l_{mn} can be written in terms of the generators. In order to prove the linear independence we need the following lemma.

Lemma 2.4.3. *Define $h\colon A_q(SL(2,\mathbb{C}))\to\mathbb{C}$ by $h(1) = 1$, $h(t^l_{mn}) = 0$ for $l > 0$, then h is an invariant functional on $A_q(SL(2,\mathbb{C}))$ and the Schur orthogonality relations hold;*

$$h\big(S(t^l_{mn})t^k_{ij}\big) = \delta_{lk}\delta_{mj}\delta_{in}q^{2(l-n)}\frac{1-q^2}{1-q^{4l+2}}.$$

Proof of Lemma 2.4.3. First observe that $1 = t^0_{00}$ and that 1 cannot be written as a linear combination of matrix elements t^l_{mn} for $l > 0$, since otherwise the trivial representation would occur as a subrepresentation of t^l contradicting its irreducibility. So h is well-defined on the whole of $A_q(SL(2,\mathbb{C}))$. Since $\Delta(t^l_{mn}) = \sum_{k=-l}^{l} t^l_{mk}\otimes t^l_{kn}$ by $t^l(XY) = t^l(X)t^l(Y)$, we immediately see that h is an invariant functional.

In order to prove the orthogonality relations we first observe that

$$\text{(2.4.5)}\qquad \sum_{j=-k}^{k} h\big(S(t^l_{mn})t^k_{ij}\big)\,t^k_{jp} = \sum_{j=-l}^{l} t^l_{mj}\,h\big(S(t^l_{jn})t^k_{ip}\big).$$

This follows from the right invariance of h;

$$h\big(S(t^l_{mn})t^k_{ij}\big)\,1 = (h\otimes id)\big(\Delta(S(t^l_{mn})t^k_{ij})\big) = \sum_{r=-l}^{l}\sum_{s=-k}^{k} h\big(S(t^l_{rn})t^k_{is}\big)\,S(t^l_{mr})t^k_{sj}$$

by Proposition 1.1.5(ii). From (1.1.4) we obtain $\sum_{m=-l}^{l} t^l_{km}\,S(t^l_{mr}) = \varepsilon(t^l_{kr}) = \delta_{kr}$ and this leads to (2.4.5). So the matrix $T^{(n,i)}_{mj} = h\big(S(t^l_{mn})t^k_{ij}\big)$ intertwines t^l and t^k. Hence, it is zero for $k\neq l$ and if $k = l$ we have $T^{(n,i)}_{mj} = c^{(n,i)}\delta_{mj}$ for some constant $c^{(n,i)}\in\mathbb{C}$.

For a representation t of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ in V the contragredient representation t^c of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ in V is defined by $\langle t^c(X)e_n, e_m\rangle = \langle t(S(X))e_m, e_n\rangle$, or in terms of the antipode S of $A_q(SL(2,\mathbb{C}))$, $t^c_{mn} = S(t_{nm})$ for a representation t of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ such that the spectrum of A is contained in $q^{\frac{1}{2}\mathbb{Z}_+}$. Using Theorem 2.3.6 we see that the contragredient of t^l is equivalent to t^l, and consequently $h(S(\xi)) = h(\xi)$ for all $\xi \in A_q(SL(2,\mathbb{C}))$. So we get $T^{(n,i)}_{mj} = h(S(t^k_{ij})S^2(t^l_{mn}))$. Now $S^2(t^l_{mn})$ are the matrix coefficients of the double contragredient representation of t^l and hence there exists an invertible intertwining operator $F \in End(\mathbb{C}^{2l+1})$ such that $(t^l)^{cc}(X)F = Ft^l(X)$ for all $X \in U_q(\mathfrak{sl}(2,\mathbb{C}))$ and hence $S^2(t^l_{mn}) = \sum_{p,r=-l}^{l} F_{mp}t^l_{pr}(F^{-1})_{rn}$. Thus,

$$\begin{aligned}\delta_{kl}\delta_{mj}c^{(n,i)} = T^{(n,i)}_{mj} &= \sum_{p,r=-l}^{l} F_{mp}h\big(S(t^k_{ij})t^l_{pr}\big)(F^{-1})_{rn}\\ &= \delta_{kl}(F^{-1})_{in}\sum_{p=-l}^{l} F_{mp}c^{(j,p)},\end{aligned}$$

or $c^{(n,i)} = c(F^{-1})_{in}$ for some constant $c \in \mathbb{C}$. To determine the constant we observe that $\sum_{m=-l}^{l} S(t^l_{km})t^l_{mr} = \delta_{kr}$ implies $1 = \sum_{m=-l}^{l} c^{(m,m)}$, so $c^{-1} = \mathrm{tr}(F^{-1})$. Summarising,

$$h\big(S(t^l_{mn})\,t^k_{ij}\big) = \delta_{kl}\delta_{mj}\frac{(F^{-1})_{in}}{\mathrm{tr}(F^{-1})}. \tag{2.4.6}$$

In this case we easily show that $Fe^l_k = q^{2k}e^l_k$ is such an intertwiner. Then (2.4.6) and a calculation finish the proof. ☑

Finally, assume that $\sum_{lmn} c_{lmn}t^l_{mn} = 0$, then multiply this element from the left by $S(t^l_{nm})$ and apply the invariant functional h. We get $c_{lmn}h\big(S(t^l_{mn})t^l_{mn}\big) = 0$ from Lemma 2.4.3 and this shows $c_{lmn} = 0$. ☑

Remark 2.4.4. If we let $q \uparrow 1$ in the definition of $A_q(SL(2,\mathbb{C}))$ we obtain a commutative Hopf algebra which is nothing but $\mathrm{Pol}(SL(2,\mathbb{C}))$ by identifying α, β, γ and δ with the coordinate functions. So for $g \in SL(2,\mathbb{C})$ we let $\alpha(g) = g_{11}$, $\beta(g) = g_{12}$, $\gamma(g) = g_{21}$ and $\delta(g) = g_{22}$.

§2.5. More on the dual Hopf algebra. Do the relations in (2.4.2) describe all the relations between the generators of $A_q(SL(2,\mathbb{C}))$? We show that the answer is yes.

Lemma 2.5.1. *Let B be the algebra generated by α, β, γ and δ subject to the relations of (2.4.2), then a linear basis for B is given by $\delta^l\gamma^m\beta^n$, $l,m,n \in \mathbb{Z}_+$, and $\alpha^l\gamma^m\beta^n$, $l \in \mathbb{N}$, $m,n \in \mathbb{Z}_+$.*

Proof. From (2.4.2) it follows that these elements span B. We have to show that they are linearly independent.

Consider the infinite dimensional representation of B in $\ell^2(\mathbb{Z}_+ \times \mathbb{Z})$ with standard orthonormal basis $e_{n,k}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. The action of the generators is given by

$$\alpha\, e_{n,k} = \sqrt{1-q^{2n}}\, e_{n-1,k}, \quad \beta\, e_{n,k} = -q^{n+1}\, e_{n,k-1},$$
$$\gamma\, e_{n,k} = q^n\, e_{n,k+1}, \qquad \delta\, e_{n,k} = \sqrt{1-q^{2n+2}}\, e_{n+1,k}.$$

Using this representation we show that a non-trivial linear combination of such elements cannot give the zero operator. ☑

To show that the same elements also give a basis of $A_q(SL(2,\mathbb{C}))$ we proceed by explicitly calculating the duality for such elements. The exercises contain some hints on how to prove the following theorem.

Theorem 2.5.2. *Define*

$$C^{L,M,N}_{l,m,n} = q^{l(L+M-N)/2} q^{-L(m+n)/2} q^{-m(m-1)/2} q^{-n(n-1)/2} \frac{(q^2;q^2)_n (q^2;q^2)_m}{(1-q^2)^{m+n}},$$

then

$$\langle \alpha^L \gamma^M \beta^N, D^l C^m B^n \rangle = \delta_{Mm}\delta_{Nn} C^{-L,M,N}_{l,m,n}$$

and

$$\langle \delta^L \gamma^M \beta^N, D^l C^m B^n \rangle =$$
$$\begin{cases} q^{(m-M)^2} \begin{bmatrix} L \\ m-M \end{bmatrix}_{q^2} C^{L,M,N}_{l,m,n}, & \text{if } 0 \le m-M = n-N \le L, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 2.5.3. *A linear basis for $A_q(SL(2,\mathbb{C}))$ is given by $\delta^l\gamma^m\beta^n$, $l,m,n \in \mathbb{Z}_+$, and $\alpha^l\gamma^m\beta^n$, $l \in \mathbb{N}$, $m,n \in \mathbb{Z}_+$. In particular, $A_q(SL(2,\mathbb{C}))$ is isomorphic to B defined in Lemma 2.5.1, and (2.4.2) are the only relations in $A_q(SL(2,\mathbb{C}))$.*

Proof. Suppose that in $A_q(SL(2,\mathbb{C}))$ we have

$$0 = \sum c_{LMN} \delta^L \gamma^M \beta^N + \sum c_{-LMN} \alpha^L \gamma^M \beta^N.$$

Let m be the minimal M such that $c_{LMN} \neq 0$, and n the minimal N such that $c_{LmN} \neq 0$. Testing against $D^l C^m B^n$ shows

$$0 = \sum_{L\in\mathbb{Z}} c_{Lmn} C(m,n) q^{-L(m+n)/2} q^{lL/2}, \qquad \forall\, l \in \mathbb{Z},$$

for some non-zero constant $C(m,n)$ by Theorem 2.5.2. So $c_{Lmn} = 0$ for all L. ☑

Corollary 2.5.4. *The duality between $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and $A_q(SL(2,\mathbb{C}))$, considered as an abstract algebra generated by α, β, γ and δ, subject to the relations (2.4.2), defined on the generators by*

$$
\begin{aligned}
&\left\langle A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \left\langle B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
&\left\langle C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \left\langle D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}.
\end{aligned}
\tag{2.5.1}
$$

and extended as Hopf algebra duality, is doubly non-degenerate.

§2.6. Action of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ on $A_q(SL(2,\mathbb{C}))$ and $*$-structures on $A_q(SL(2,\mathbb{C}))$. Now that we have established $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and $A_q(SL(2,\mathbb{C}))$ in duality as Hopf algebras, we may consider the action of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ on $A_q(SL(2,\mathbb{C}))$ as defined in Proposition 1.2.3. It is a simple calculation to give the action on the level of generators using (2.4.3) and (2.5.1). We get in an obvious notation

$$
\begin{aligned}
A.\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} q^{1/2}\alpha & q^{-1/2}\beta \\ q^{1/2}\gamma & q^{-1/2}\delta \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.A = \begin{pmatrix} q^{1/2}\alpha & q^{1/2}\beta \\ q^{-1/2}\gamma & q^{-1/2}\delta \end{pmatrix}, \\
B.\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} 0 & \alpha \\ 0 & \gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.B = \begin{pmatrix} \gamma & \delta \\ 0 & 0 \end{pmatrix}, \\
C.\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} \beta & 0 \\ \delta & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.C = \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix}.
\end{aligned}
\tag{2.6.1}
$$

For $q \uparrow 1$ (2.6.1) corresponds to the action between $U(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathrm{Pol}(SL(2,\mathbb{C}))$ as described in Example 1.2.2.

Corresponding to the three inequivalent $*$-structures on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ described in Theorem 2.3.4 we obtain three $*$-structures on $A_q(SL(2,\mathbb{C}))$ making it into a Hopf $*$-algebra by transposing the $*$-operator from $U_q(\mathfrak{sl}(2,\mathbb{C}))$ to $A_q(SL(2,\mathbb{C}))$ by Definition 1.2.1.

Theorem 2.6.1. *The list of mutually inequivalent $*$-structures on $A_q(SL(2,\mathbb{C}))$ is*

(i) $|q| = 1$; $\alpha^* = \alpha$, $\beta^* = q^{-1}\beta$, $\gamma^* = q\gamma$, $\delta^* = \delta$ *and corresponding real form $A_q(SL(2,\mathbb{R}))$,*
(ii) $-1 < q < 1$, $q \neq 0$; $\alpha^* = \delta$, $\beta^* = -q\gamma$, $\gamma^* = -q^{-1}\beta$, $\delta^* = \alpha$ *and corresponding real form $A_q(SU(2))$,*
(iii) $-1 < q < 1$, $q \neq 0$; $\alpha^* = \delta$, $\beta^* = q\gamma$, $\gamma^* = q^{-1}\beta$, $\delta^* = \alpha$ *and corresponding real form $A_q(SU(1,1))$.*

Proposition 2.6.2. *The invariant functional h on $A_q(SU(2))$ satisfies*

$$h((t^l_{nm})^* t^k_{ij}) = \delta_{lk}\delta_{mj}\delta_{in} q^{2(l-n)} \frac{1-q^2}{1-q^{4l+2}}.$$

In particular, $h: A_q(SU(2)) \to \mathbb{C}$ is a positive linear functional, $h(\xi^*\xi) > 0$ for $0 \neq \xi \in A_q(SU(2))$.

Proof. Since q is real, Lemma 2.4.3 shows that it suffices to prove $S(t^l_{ij}) = (t^l_{ji})^*$. By Theorem 2.3.6 we know that t^l is a $*$-representation of $U_q(\mathfrak{su}(2))$, so

$$\langle (t^l_{ij})^*, X\rangle = \overline{\langle t^l_{ij}, S(X)^*\rangle} = \langle e^l_i, t^l(S(X)^*)e^l_j\rangle = \langle t^l(S(X))e^l_i, e^l_j\rangle = \langle S(t^l_{ji}), X\rangle$$

for all $X \in U_q(\mathfrak{su}(2))$. ☑

Proposition 2.6.2 shows that the Hopf $*$-algebra $A_q(SU(2))$ has the proper q-analogue of the Schur orthogonality relations. It is also a nice $*$-structure since we can give a complete list of mutually inequivalent $*$-representations of $A_q(SU(2))$. It is straightforward to check that the $*$-representations defined in the next theorem are indeed representations.

Theorem 2.6.3. *The following is a complete list of irreducible inequivalent $*$-representations of $A_q(SU(2))$.*

(i) *The one-dimensional $*$-representations π_θ defined by $\pi_\theta(\alpha) = e^{i\theta}$ and $\pi_\theta(\beta) = 0$ for $\theta \in [0, 2\pi)$.*

(ii) *Infinite dimensional $*$-representations acting in the Hilbert space $\ell^2(\mathbb{Z}_+)$. For an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}_+}$ the action of the generators is given by*

$$\pi^\infty_\theta(\alpha)\, e_n = \sqrt{1-q^{2n}}\, e_{n-1}, \quad \pi^\infty_\theta(\gamma)\, e_n = e^{i\theta} q^n\, e_n,$$

with the convention $e_{-1} = 0$.

Sketch of Proof. It is straightforward to see that the representations given in Theorem 2.6.3 are irreducible $*$-representations of $A_q(SU(2))$. Conversely, let π be an irreducible $*$-representation of $A_q(SU(2))$, then it follows from the commutation relations (2.4.2) that the kernel of $\pi(\gamma)$ is an irreducible subspace. Hence, $\pi(\gamma) = 0$, leading to the one-dimensional representations, or $\ker(\pi(\gamma))$ is trivial.

In the last case we can use the Spectral Theorem for the normal operator $\pi(\gamma)$ to show that the spectrum is of the form λq^k, $k \in \mathbb{Z}_+$, for some $\lambda \in \mathbb{C}$. From the commutation relations (2.4.2) it follows that $|\lambda| = 1$, and that $\pi(\alpha)$ and $\pi(\delta)$ are acting as shift operators on the eigenvectors. We then find that π is equivalent to an infinite dimensional representation as in (ii), since the eigenspaces of $\pi(\gamma)$ are one-dimensional. ☑

Remark 2.6.4. For $\lambda \neq 0$ we also define one-dimensional representations of $A_q(SL(2,\mathbb{C}))$ given by

$$\tau_\lambda(\alpha) = \lambda, \quad \tau_\lambda(\beta) = 0 = \tau_\lambda(\gamma), \quad \tau_\lambda(\delta) = \lambda^{-1}.$$

Note that τ_λ is a $*$-representation of $A_q(SU(2))$ if and only if $\lambda = e^{i\theta}$, or $\tau_\lambda = \pi_\theta$, for some $\theta \in [0, 2\pi)$. The counit ε coincides with the special case $\tau_1 = \pi_0$.

Notes and references. The quantised universal enveloping algebra for $\mathfrak{sl}(2,\mathbb{C})$ is the simplest case of a series of quantised universal enveloping algebras. In fact, there is a canonical way to associate to any simple Lie algebra $\mathfrak{g}$ a quantised universal enveloping algebra $U_q(\mathfrak{g})$. The PBW-basis in Lemma 2.1.2 can be proved in various ways, such as by use of the representations in Theorem 2.2.1 or by using the so-called diamond lemma. As is the case for $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$ the representation theory of simple $\mathfrak{g}$ and $U_q(\mathfrak{g})$ is very similar, see [15] and references therein.

The approach to $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and its dual algebra presented here is much inspired by the paper [69] by Masuda et al. The existence of the invariant functional and the corresponding orthogonality relations for the invariant functional have several proofs, see [17], [61], [99], [103].

Theorem 2.6.3 is due to Vaksman and Soibelman [96] and can be used to complete $A_q(SU(2))$ into a C^*-algebra, and so making the connection with Woronowicz's [103] approach to compact quantum groups. See also [43] for the details of the proof of Theorem 2.6.3. For general simple compact quantum groups the $*$-representations have been classified by Soibelman, see references in [15].

Exercises.

1. Finish the proof of Proposition 2.1.1.
2. Verify (2.3.2).
3. Prove (ii) and (iii) of Proposition 2.3.1.
4. Replace in $U_q(\mathfrak{su}(2))$ its generators A, B, C and D by A, $\rho^{-1}B$, $\rho^{-1}C$ and D. Let $\rho\downarrow 0$ and determine the Hopf $*$-algebra structure in the limit case. This Hopf $*$-algebra is denoted by $U_q(\mathfrak{m}(2))$, and is related to the Lie algebra for the group of orientation and distance preserving motions of the Euclidean plane.
5. The proof of Theorem 2.5.2 has to be done in the following stages, which each can be proved using induction and Definition 1.2.1.
   - First show that for $X\in U_q(\mathfrak{sl}(2,\mathbb{C}))$
   $$\langle \delta^L\gamma^M\beta^N, D^lX\rangle = q^{l(L+M-N)/2}\langle \delta^L\gamma^M\beta^N, X\rangle,$$
   $$\langle \alpha^L\gamma^M\beta^N, D^lX\rangle = q^{-l(L+M-N)/2}\langle \alpha^L\gamma^M\beta^N, X\rangle.$$
   - Show that
   $$\langle \delta^L, C^mB^n\rangle = \begin{cases} q^{m(1-L)}\dfrac{(q^{2L};q^{-2})_m(q^2;q^2)_m}{(1-q^2)^{2m}}, & \text{if } m=n\le L,\\ 0, & \text{otherwise,}\end{cases}$$
   $$\langle \alpha^L, C^mB^n\rangle = \delta_{m0}\delta_{n0}.$$
   - Show that
   $$\langle \gamma^M\beta^N, C^mB^n\rangle = \delta_{mM}\delta_{nN}q^{-(n(n-1)/2}q^{-m(m-1)/2}\frac{(q^2;q^2)_m(q^2;q^2)_n}{(1-q^2)^{m+n}}$$

and finish the proof of Theorem 2.5.2.

6. Let $k>0$ and let $\{e_n\}_{n=0}^{\infty}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z}_+)$. Prove that there exists a unitary representation of $U_q(\mathfrak{su}(1,1))$ in $\ell^2(\mathbb{Z}_+)$ such that $A\cdot e_n = q^{k+n}\,e_n$, and $C\cdot e_0 = 0$. Give an explicit expression for the action of B and C on a basis vector. These representations of $U_q(\mathfrak{su}(1,1))$ are *positive discrete series* representations. (Be aware that D is represented by an unbounded operator, since $-1<q<1$.)
7. Prove that the representation in the proof of Lemma 2.5.1 is also a $*$-representation of $A_q(SU(2))$. Decompose this $*$-representation in terms of the irreducible $*$-representations of Theorem 2.6.3.

3. Orthogonal polynomials and basic hypergeometric series

In this lecture we consider first some general theory of orthogonal polynomials and next some explicit examples which can be written in terms of basic hypergeometric series, in particular the famous Askey-Wilson polynomials.

§3.1. Orthogonal polynomials on the real line. Let μ be a non-negative Borel measure on $\mathbb{R}$ such that all moments exist, i.e. $\int x^k\,d\mu(x)<\infty$ for all $k\in\mathbb{Z}_+$, and such that $\mathrm{supp}(\mu)$ contains at least a countably infinite number of points. The polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with degree$(p_n)=n$ and real coefficients are said to be orthogonal polynomials with repect to μ if

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\,d\mu(x)=\delta_{nm}h_n,\qquad h_n>0. \tag{3.1.1}$$

This definition is equivalent to

$$\int_{-\infty}^{\infty} p_n(x)x^m\,d\mu(x)=\delta_{nm}g_n,\qquad g_n\neq 0,\quad 0\le m\le n, \tag{3.1.2}$$

or with x^m replaced by any other polynomial of degree m. The orthogonal polynomials are uniquely determined up to a scalar depending on the degree n. There are two canonical choices; (i) such that the leading coefficient is 1, and then we speak of monic orthogonal polynomials, (ii) such that h_n in (3.1.1) is independent of n and that the leading coefficient is positive and then we assume μ normalised by $m_0=\mu(\mathbb{R})=1$ so that $h_n=1$ and we speak of orthonormal polynomials.

Orthogonal polynomials satisfy a fundamental three-term recurrence relation.

Theorem 3.1.1. *Let p_n be orthogonal polynomials, then*

$$x\,p_n(x)=A_n\,p_{n+1}(x)+B_n\,p_n(x)+C_n\,p_{n-1}(x)$$

with $A_n, B_n, C_n \in \mathbb{R}$ and $A_{n-1}C_n > 0$ for $n \geq 1$.

Proof. $xp_n(x)$ is a polynomial of degree $n+1$, so $xp_n(x) = \sum_{k=0}^{n+1} c_k p_k(x)$. The orthogonality relations (3.1.1) show that

$$h_k\, c_k = \int_{-\infty}^{\infty} p_n(x)\, xp_k(x)\, d\mu(x).$$

Now $h_k > 0$ and the integral is real-valued and zero for $k = 0, \ldots, n-2$ by (3.1.2). We also see that $A_{n-1}h_n = \int p_n(x)xp_{n-1}(x)d\mu(x) = C_n h_{n-1}$, so that $A_{n-1}C_n \geq 0$ since $h_k > 0$. It cannot be zero since $A_n \neq 0$. ☑

On the other hand, the three-term recursion relation together with the initial conditions $p_{-1}(x) = 0$, $p_0(x) = 1$ defines polynomial p_n as polynomials of degree n. Then the converse also holds, and this is commonly called Favard's Theorem.

Theorem 3.1.2. (Favard) *Define polynomials p_n of degree n by*

$$x\, p_n(x) = A_n\, p_{n+1}(x) + B_n\, p_n(x) + C_n\, p_{n-1}(x)$$

with initial conditions $p_{-1}(x) = 0$ and $p_0(x) = 1$. Assume $A_n, B_n, C_n \in \mathbb{R}$ and $A_{n-1}C_n > 0$ for $n \geq 1$, then there exists a non-negative Borel measure μ on $\mathbb{R}$ such that $\{p_n\}_{n=0}^{\infty}$ are orthogonal polynomials with respect to μ.

Remark 3.1.3. (i) Observe that for monic polynomials $A_n = 1$ and that for orthonormal polynomials $A_{n-1} = C_n$.

(ii) Note that the orthogonality measure from Favard's Theorem is not unique. However, under some extra conditions on the coefficients, e.g. the polynomials are orthonormal, i.e. $A_{n-1} = C_n$ and the coefficients A_n, B_n are bounded for $n \to \infty$, the measure μ is unique. In the case mentioned we even have that $\mathrm{supp}(\mu)$ is compact.

Sketch of Proof. By rescaling we may suppose that we deal with

$$x\, p_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x)$$

with initial conditions $p_{-1}(x) = 0$ and $p_0(x) = 1$. In the Hilbert space $\ell^2(\mathbb{Z}_+)$ with orthonormal basis $\{e_n\}_{n=0}^{\infty}$ we define a linear operator J by

$$J\, e_n = a_{n+1}\, e_{n+1} + b_n\, e_n + a_n\, e_{n-1},$$

i.e. J is a tridiagonal operator which are also called Jacobi matrices. We now prove Favard's Theorem under the extra assumption that a_n and b_n are bounded. This implies that J is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}_+)$, hence the Spectral Theorem applies and $J = \int \lambda dE(\lambda)$ for some projection valued Borel measure E on $\mathbb{R}$. Define $\mu(B) = E_{e_0,e_0}(B) = \langle E(B)e_0, e_0\rangle$, then

μ is a non-negative Borel measure on $\mathbb{R}$. Moreover, $\operatorname{supp}(\mu)$ is compact, since $\operatorname{supp}(\mu) \subset \operatorname{supp}(E) \subset \sigma(J)$ and the spectrum of J is compact.

Define the mapping

$$\Lambda: \ell^2(\mathbb{Z}_+) \to L^2(\mu), \qquad e_n \mapsto p_n,$$

where $L^2(\mu) = \{f: \mathbb{R} \to \mathbb{R} \mid \int |f(x)|^2 d\mu(x) < \infty\}$ is a weighted L^2-space. The three-term recurrence relation then implies $\Lambda \circ J = M \circ \Lambda$, where $Mf(x) = xf(x)$ is the multiplication operator. Moreover, Λ is a unitary mapping. Indeed, the polynomials are dense in $L^2(\mu)$ since $\operatorname{supp}(\mu)$ is compact. And from $\Lambda J^n e_0 = x^n$ we get

$$\langle J^n e_0, J^m e_0 \rangle = \int \lambda^{n+m}\, dE_{e_0,e_0}(\lambda) = \int \lambda^{n+m}\, d\mu(\lambda) = \langle x^n, x^m \rangle_{L^2(\mu)}.$$

The unitarity implies

$$\delta_{nm} = \langle e_n, e_m \rangle = \langle \Lambda e_n, \Lambda e_m \rangle_{L^2(\mu)} = \int p_n(x) p_m(x)\, d\mu(x).$$

A similar, but more delicate, construction works if J is unbounded. The deficiency indices are $(0,0)$ or $(1,1)$, in the first case the measure is still unique but might cease to have compact support, and in the second case the orthogonality measure depends on the self-adjoint extension and is no longer unique. ☑

§3.2. Basic hypergeometric series. The series $\sum c_k$ is a *basic hypergeometric series* if c_{k+1}/c_k is a rational function of q^k for a base q. So

$$\frac{c_{k+1}}{c_k} = \frac{(1-a_1q^k)\dots(1-a_rq^k)}{(1-b_1q^k)\dots(1-b_sq^k)} \frac{(-q^k)^{1+s-r}}{1-q^{k+1}} z,$$

and we have the following form for basic hypergeometric series, also known as q-hypergeometric series,

$$ {}_r\varphi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) = {}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, z\right)$$

(3.2.1)
$$= \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k} \frac{\left((-1)^k q^{k(k-1)/2}\right)^{1+s-r} z^k}{(q;q)_k},$$

with the notation for q-shifted factorials

$$(a_1,\dots,a_r;q)_k = (a_1;q)_k \dots (a_r;q)_k, \qquad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i).$$

When dealing with q-hypergeometric series our standard assumption on q is $0 < q < 1$. Then the q-shifted factorials are also well-defined for $k \to \infty$, $(a;q)_\infty = \lim_{k\to\infty}(a;q)_k$.

Since $(q^{-n};q)_k = 0$ for $k > n$ we see that the series (3.2.1) terminates if one of the upper parameters a_i equals q^{-n} for $n \in \mathbb{Z}_+$. If one of the lower parameters equals q^{-N} for $N \in \mathbb{Z}_+$ the series (3.2.1) is not well-defined, unless one of the upper parameters equals q^{-n} for some $n \in \{0, 1, \ldots, N\}$. In case $n = N$ we follow the convention that the series consists of the first $N+1$ terms.

The ratio test shows that for generic values of the parameters the radius of convergence is ∞, 1 or 0 for $r < s+1$, $r = s+1$ or $r > s+1$.

Theorem 3.2.1. (q-binomial theorem) ${}_1\varphi_0(a;-;q,z) = \dfrac{(az;q)_\infty}{(z;q)_\infty}$ *for* $|z| < 1$.

Proof. Let $h_a(z) = {}_1\varphi_0(a;-;q,z)$, which is analytic in the unit disc. Simple calculations show $h_a(z) - h_{aq}(z) = -azh_{aq}(z)$ and $h_a(z) - h_a(qz) = z(1-a)h_{aq}(z)$. Eliminating $h_{aq}(z)$ leads to

$$h_a(z) = \frac{1-az}{1-z}\, h_a(qz) = \frac{(az;q)_n}{(z;q)_n}\, h_a(q^n z).$$

Finally, let $n \to \infty$ and use that $h_a(z)$ is continuous at $z = 0$ and $h_a(0) = 1$. ☑

Corollary 3.2.2.

(i) ${}_1\varphi_0(q^{-n};-;q,q^n z) = (z;q)_n$ *for* $z \in \mathbb{C}$,

(ii) $e_q(z) = {}_1\varphi_0(0;-;q,z) = (z;q)_\infty^{-1}$ *for* $|z| < 1$,

(iii) $E_q(z) = {}_0\varphi_0(-;-;q,-z) = (-z;q)_\infty$ *for* $z \in \mathbb{C}$.

Remark 3.2.3. We call e_q and E_q q-exponential functions. To motivate this terminology we note that $\lim_{q\uparrow 1} e_q(z(1-q)) = e^z = \lim_{q\uparrow 1} E_q(z(1-q))$, which follows formally from the power series representation for e_q and E_q.

Proof. Case (i) and (ii) follow from Theorem 3.2.1 by specialisation of a. Case (iii) follows from replacing z by z/a in Theorem 3.2.1 and letting $a \to \infty$. ☑

Identities for q-hypergeometric series can be obtained by playing around with the q-binomial Theorem thmref3.2.1. As an example we derive Heine's transformation formulae for the ${}_2\varphi_1$-series.

Theorem 3.2.4. (Heine)

$$\begin{aligned}
{}_2\varphi_1(a,b;c;q,z) &= \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\varphi_1(c/b,z;az;q,b), \qquad |z|,|b| < 1,\\
{}_2\varphi_1(a,b;c;q,z) &= \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty}\, {}_2\varphi_1(abz/c,b;bz;q,c/b), \qquad |z|,|c/b| < 1,\\
{}_2\varphi_1(a,b;c;q,z) &= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\varphi_1(c/a,c/b;c;q,abz/c), \qquad |z|,|abz/c| < 1.
\end{aligned}$$

Proof. The second equality follows from the first by applying it twice. The third equality follows from the second on the first, or by a threefold application of the first equality. Hence it suffices to prove the first equality. Write

$$\begin{aligned}
{}_2\varphi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(cq^n;q)_\infty}{(q;q)_n(bq^n;q)_\infty}z^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}(bq^n)^m
\end{aligned}$$

by the q-binomial Theorem 3.2.1

$$\begin{aligned}
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(zq^m)^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_m}{(q;q)_m}b^m\frac{(azq^m;q)_\infty}{(zq^m;q)_\infty}
\end{aligned}$$

by the q-binomial Theorem 3.2.1 again

$$= \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\varphi_1(c/b,z;az;q,b).$$

Interchanging summations is allowed for $|z|, |b| < 1$, since then the double sum is estimated easily by a product of two absolutely convergent series. ☑

Corollary 3.2.5. (q-Saalschütz summation) *For* $n \in \mathbb{Z}_+$

$${}_3\varphi_2\left(\begin{matrix}q^{-n}, a, b\\ c, q^{1-n}ab/c\end{matrix};q,q\right) = \frac{(c/a, c/b;q)_n}{(c, c/(ab);q)_n}.$$

Proof. Consider the last of Heine's transformation formulae in Theorem 3.2.4. Use the q-binomial Theorem 3.2.1 for the quotient of the infinite q-shifted factorials to write it as a power series of z. Now compare the coefficients of z on both sides. Finally, replace a, b by c/a, c/b. ☑

Remark 3.2.6. The series ${}_{r+1}\varphi_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;q,z)$ is called balanced, or Saalschützian, if $b_1\dots b_r = qa_1\dots a_{r+1}$ and $z = q$. Corollary 3.2.5 shows that any balanced ${}_3\varphi_2$-series is summable.

§3.3. Askey-Wilson polynomials.

To formulate the celebrated Askey-Wilson integral we have to introduce the following weight function;

$$w\left(\frac{1}{2}(z+z^{-1})\right) = \frac{(z^2, z^{-2};q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z;q)_\infty}$$

and we use $w(x) = w(x;a,b,c,d|q)$ to stress the dependence on the parameters when needed.

Theorem 3.3.1. *Assume* $|a|, |b|, |c|, |d| < 1$, *then*

$$\int_0^\pi w(\cos\theta; a, b, c, d|q)\, d\theta = \frac{2\pi(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

Theorem 3.3.1 is the key to the Askey-Wilson polynomials. To motivate its introduction define the monomial function $m_k(\cos\theta; a) = (ae^{i\theta}, ae^{-\theta}; q)_k$ and then we consider

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^\pi m_j(\cos\theta; a) m_k(\cos\theta; b) w(\cos\theta; a, b, c, d|q)\, d\theta \\
&= \frac{1}{2\pi}\int_0^\pi w(\cos\theta; aq^j, b^k, c, d|q)\, d\theta \\
&= \frac{(ab;q)_{k+j}(ac, ad; q)_j (bc, bd; q)_k}{(abcd;q)_{k+j}} \frac{(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.
\end{aligned}$$

So we can try to find orthogonal polynomials $r_m(x) = \sum_{j=0}^m c_j m_j(a; x)$ such that r_m is orthogonal to each $m_k(x; b)$ for $0 \le k < m$. Hence the coefficients c_j have to satisfy

$$\sum_{j=0}^m c_j \frac{(abq^k;q)_j}{(abcdq^k;q)_j}(ac, ad; q)_j = \delta_{m,k} g_m \qquad \text{for } 0 \le k \le m. \tag{3.3.1}$$

Take $c_j = (q^{-m}, abcdq^{m-1}; q)_j q^j / (q, ab, ac, ad; q)_j$, then (3.3.1) equals

$${}_3\varphi_2\left(\begin{matrix} q^{-m}, abcdq^{m-1}, abq^k \\ ab,\ abcdq^k \end{matrix}; q, q\right) = \frac{(q^{k-m+1}, cd; q)_m}{(abcdq^k, q^{1-m}/(ab); q)_m}$$

by the q-Saalschütz formula of Corollary 3.2.5. This is zero for $0 \le k < m$ and non-zero for $k = m$, and thus

$$r_m(\cos\theta; a, b, c, d|q) = {}_4\varphi_3\left(\begin{matrix} q^{-m}, abcdq^{m-1}, ae^{i\theta}, ae^{-i\theta} \\ ab,\ ac,\ ad \end{matrix}; q, q\right) \tag{3.3.2}$$

are the orthogonal polynomials with respect to the Askey-Wilson integral of Theorem 3.3.1. Since the value of g_m also follows from this calculation, and since the leading coefficients of r_m and $m_m(x)$ are easily calculable, we can also calculate the norm of r_m with respect to the Askey-Wilson measure.

Theorem 3.3.2. *Define the Askey-Wilson polynomials as*

$$\begin{aligned}
&p_m(\cos\theta; a, b, c, d|q) \\
&= a^{-m}(ab, ac, ad; q)_m\, {}_4\varphi_3\left(\begin{matrix} q^{-m}, abcdq^{m-1}, ae^{i\theta}, ae^{-i\theta} \\ ab,\ ac,\ ad \end{matrix}; q, q\right),
\end{aligned}$$

and assume $|a|, |b|, |c|, |d| < 1$, *then* $p_m(x) = p_m(x; a, b, c, d|q)$ *satisfies*

$$\frac{1}{2\pi}\int_0^\pi p_n(\cos\theta)p_m(\cos\theta)\, w(\cos\theta)\, d\theta = \delta_{nm} h_n$$

with

$$h_n = \frac{(1-q^{n-1}abcd)}{(1-q^{2n-1}abcd)} \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_n} h_0,$$

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

Remark 3.3.3. The factor in front of the ${}_4\varphi_3$-series in Theorem 3.3.2 is chosen such that the squared norm is symmetric in the parameters. Since the weight function possesses this symmetry as well, the Askey-Wilson polynomials are symmetric in its parameters. Using the symmetry in a and b leads to a transformation for ${}_4\varphi_3$-series, which is Sears's transformation formula.

For more general values of the parameters the Askey-Wilson polynomials defined in Theorem 3.3.2 are still orthogonal. The more general result can be obtained by working with contour integration and next use contour shifts and residue calculus.

Proposition 3.3.4. *Let a, b, c and d be real and let all the pairwise products of a, b, c and d be less than 1. Then the Askey-Wilson polynomials $p_n(x) = p_n(x; a, b, c, d|q)$ satisfy the orthogonality relations*

$$\frac{1}{2\pi}\int_0^\pi p_n(\cos\theta)p_m(\cos\theta)w(\cos\theta)\, d\theta + \sum_k p_n(x_k)p_m(x_k)w_k = \delta_{n,m}h_n.$$

The points x_k are of the form $\frac{1}{2}(eq^k + e^{-1}q^{-k})$ for e any of the parameters a, b, c or d with absolute value greater than 1; the sum is over $k \in \mathbb{Z}_+$ such that $|eq^k| > 1$ and w_k is the residue of $z \mapsto w(\frac{1}{2}(z + z^{-1}))$ at $z = eq^k$ minus the residue at $z = e^{-1}q^{-k}$.

The orthogonality relations remain valid for complex parameters a, b, c and d, if they occur in conjugate pairs. If all parameters have absolute value less than 1, the Askey-Wilson orthogonality measure is absolutely continuous, i.e. we are in the situation of Theorem 3.3.2. We use the notation $dm(x) = dm(x; a, b, c, d|q)$ for the normalised orthogonality measure. So for any polynomial p

$$\int_{\mathbb{R}} p(x)\, dm(x) = \frac{1}{h_0}\Big(\frac{1}{2\pi}\int_{-1}^1 p(x)w(x)\frac{dx}{\sqrt{1-x^2}} + \sum_k p(x_k)w_k\Big). \tag{3.3.3}$$

Notes and references. There is an enormous amount of literature available on general orthogonal polynomials, and we only have given a very small portion of the available results. Further introductions can be found in e.g. Chihara [16], Temme [93, Ch. 6], Szegő [92]. More details on this proof of Favard's Theorem 3.1.2 and the relation between orthogonal polynomials and functional analysis can be found in Berezanskiĭ [9, Ch. 7.1], see also Dombrowski [20]. Chihara [16] gives another proof of Favard's Theorem. In case the moment problem is not determined, i.e. there exist more than one orthogonality measure for the corresponding orthogonal polynomials, the analysis becomes much more delicate, see Berg [10] and references given there.

The basic hypergeometric ${}_2\varphi_1$-series was introduced in 1846 by Heine. Since then research has been going on, and a very good account of properties of basic hypergeometric series can be found in the book [27] on this subject by Gasper and Rahman. A number of connections with other fields, such as quantum groups, Lie algebras, number theory, statistical mechanics and other areas in physics are known, see e.g. Andrews [4] for a nice account and references in [27].

The Askey-Wilson integral of Theorem 3.3.1 is due to Askey and Wilson [8], who evaluated the integral by calculating residues and a number of summation formulas. See also [27, §6.1] for another proof and further references. A nice proof of Theorem 3.3.1 is given by Kalnins and Miller [36] using the symmetry in a, b, c and d and a suitable iteration, see [61], and Exercise 3.7, for an adaptation of this method. Another recent elegant proof of the Askey-Wilson measure is given by Berg and Ismail [11].

The Askey-Wilson polynomials, together with their finite discrete counterpart, the so-called q-Racah polynomials, form the top level of the q-analogue of the Askey-scheme. A very useful compendium of properties of orthogonal polynomials both from the Askey scheme and its q-analogue is given by Koekoek and Swarttouw [42].

Exercises.

1. Let p_n be orthonormal polynomials, prove the Christoffel-Darboux formula
$$\sum_{m=0}^{n} p_m(x)p_m(y) = \frac{A_n}{x-y}\Big(p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\Big).$$
What is the resulting identity for $y \to x$?
2. Let p_n be orthogonal polynomials. Prove that p_n has n real simple roots in the convex hull of $\mathrm{supp}(\mu)$. (Hint: Use Exercise 3.1 with $y = x$.)
3. Suppose that μ is a finite discrete non-negative measure with the support containing $N+1$ points. Show that we can still define a finite collection of orthogonal polynomials $\{p_n\}_{n=0}^{N}$. In this case the Jacobi matrix is a self-adjoint $(N+1)\times(N+1)$-matrix whose eigenvalues are the mass points. As

an example, consider the q-Krawtchouk polynomials

$$K_n(x;q^\sigma,N;q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, x, -q^{n-N-\sigma} \\ q^{-N}, 0 \end{matrix} ;q,q\right)$$

and prove the orthogonality relations

$$\frac{q^{N+\sigma}}{(-q^\sigma;q)_{N+1}} \sum_{x=0}^{N} (K_n K_m)(q^{-x};q^\sigma,N;q) w_x(q^\sigma,N) = \delta_{n,m}\left(h_n(q^\sigma,N)\right)^{-1}$$

with

$$w_x(q^\sigma,N) = (-q^{N+\sigma})^x \frac{(q^{-N};q)_x}{(q;q)_x},$$
$$h_n(q^\sigma,N) = \frac{\left(1+q^{2n-N-\sigma}\right)}{(-q^{n-2N-\sigma})^n} \frac{(-q^{-N-\sigma},q^{-N};q)_n}{(q,-q^{1-\sigma};q)_n}.$$

Deduce from this that the dual q-Krawtchouk polynomials

$$\begin{aligned} R_n(q^{-x}-q^{x-N-\sigma};q^\sigma,N;q) &= {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{-x}, -q^{x-N-\sigma} \\ q^{-N}, 0 \end{matrix} ;q,q\right) \\ &= K_x(q^{-n};q^\sigma,N;q) \end{aligned}$$

satisfy the orthogonality relations

$$\begin{aligned} \frac{q^{N+\sigma}}{(-q^\sigma;q)_{N+1}} \sum_{x=0}^{N} (R_n R_m)(q^{-x}-q^{x-N-\sigma};q^\sigma,N;q) h_x(q^\sigma,N) \\ = \delta_{n,m}\left(w_n(q^\sigma,N)\right)^{-1}. \end{aligned}$$

4. Derive the three-term recurrence relation for the Al-Salam and Chihara polynomials s_n, which are Askey-Wilson polynomials with $c=d=0$, so $s_n(x) = s_n(x;a,b|q) = p_n(x;a,b,0,0|q)$;

$$x\,s_n(x) = s_{n+1}(x) + q^n(a+b)\,s_n(x) + (1-q^n)(1-abq^{n-1})\,s_{n-1}(x).$$

5. Define the q-integral by

$$\int_0^b f(x)\,d_qx = (1-q)b\sum_{k=0}^{\infty} f(bq^k)q^k,$$
$$\int_a^b f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx,$$

whenever f is such that the series involved are convergent. Show that $\int_0^b f(x)\,d_qx$ tends to $\int_0^b f(x)\,dx$ as $q \uparrow 1$ for Riemann integrable f. Show that the inverse operator D_q, the q-derivative, is given by $D_q f(x) = (f(x) - f(qx))/(1-q)x$ for $x \neq 0$.

6. Prove Gauß's summation formula;
$$ {}_2\varphi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}, \qquad |c/ab| < 1. $$
(Hint: use the first of Heine's transformation formulae.) Then prove Jackson's transformation formula
$$ {}_2\varphi_1(a,b;c;q,z) = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\varphi_2(a,c/b;c,az;q,bz), \qquad |z| < 1. $$
(Hint: develop $(b;q)_k/(c;q)_k$ in the summand on the left hand side using the terminating Gauß formula, i.e. $a = q^{-k}$.)
7. In this exercise we sketch a proof of the Askey-Wilson integral in Theorem 3.3.1, which is taken from [61] which in turn is motivated by [36]. So $|a|, |b|, |c|, |d| < 1$ and consider $w_{a,b,c,d}(z) = w((z+z^{-1})/2)$ as in §3.3. Define the contour integral
$$ I_{a,b,c,d} = \frac{1}{2\pi i}\oint_{|z|=1} w_{a,b,c,d}(z)\,\frac{dz}{z}. $$
Prove that $I_{a,b,c,d} = (1-abcd)I_{aq,b,c,d}/(1-ab)(1-ac)(1-ad)$. Do this by giving two expressions for
$$ \oint_{|z|=1} \frac{w_{aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}}}{z-z^{-1}}\,\frac{dz}{z} $$
by shifting the contour to $|z| = q^{\pm 1/2}$ and scaling back to $|z| = 1$, and subtracting the result. Then prove as a consequence
$$ I_{a,b,c,d} = \frac{(abcd;q)_\infty}{(ab,ac,ad.bc,bd,cd;q)_\infty} I_{0,0,0,0}. $$
Then prove $I_{1,q^{1/2},-1,-q^{1/2}} = 1$. (Why is this choice okay?) Now finish the proof of Theorem 3.3.1.
8. Formulate Sears's transformation formula, cf. Remark 3.3.3, and prove
$$ {}_3\varphi_2\left(\begin{matrix} q^{-n}, b, c \\ d,\ e \end{matrix}; q, q\right) = \frac{(de/bc;q)_n}{(e;q)_n}(bc/d)^n\, {}_3\varphi_2\left(\begin{matrix} q^{-n}, d/b, d/c \\ d,\ de/bc \end{matrix}; q, q\right). $$
9. Define $\Gamma_q(x) = (q;q)_\infty(1-q)^{1-x}/(q^x;q)_\infty$, show that the first of Heine's transformation formulae of Theorem 3.2.4 can be written as
$$ {}_2\varphi_1(q^a,q^b;q^c;q,z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 t^{b-1}\frac{(tzq^a,tq;q)_\infty}{(tz,tq^{c-b};q)_\infty}\,d_qt. $$
Assuming $\Gamma_q(x) \to \Gamma_x$ as $q \uparrow 1$, what is the corresponding limit as $q \uparrow 1$? See the q-binomial theorem as the analogue of the binomial theorem $\sum_{k=0}^\infty (a)_k z^k/k! = (1-z)^{-a}$, where $(a)_k = a(a+1)\dots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ is a shifted factorial, or Pochhammer symbol.

4. Quantum subgroups and the Haar functional

We discuss a way to consider the analogues of bi-K-invariant functions on $SU(2)$ for arbitrary one-parameter subgroups K. In the group case it is not of much importance which K is chosen, since these groups are all conjugated. However, in the quantum group case it makes a difference. Of particular interest is the Haar functional on bi-K-invariant functions. We also start investigating the analogues of functions on $SU(2)$ behaving like $f(kgh) = \chi_K(k)\chi_H(h)f(g)$ for possibly different one-parameter subgroups K and H of $SU(2)$ having characters χ_K respectively χ_H. For $\chi_K \equiv 1 \equiv \chi_H$ the functions live on $K \setminus SU(2)/H$.

§4.1. Generalised matrix elements. Recall from Proposition 2.3.1 that the twisted primitive elements are the analogues of the Lie algebra. In the group case we have that $f \in \mathrm{Pol}(G)$ is left and right K-invariant for the one-parameter group $K = \exp(tX)$ if and only if $X.f = 0 = f.X$ with the action of $U(\mathfrak{g})$ on $\mathrm{Pol}(G)$ as in Example 1.2.2. Such functions are called spherical.

For the quantum $SL(2,\mathbb{C})$ group we already know that

$$(D-A).t^l_{ij} = \sum_{k=-l}^{l} t^l_{ik}\, t^l_{kj}(D-A) = (q^j - q^{-j})t^l_{ij}$$

and similarly $t^l_{ij}.(D-A) = (q^i - q^{-i})t^l_{ij}$. So for $l \in \mathbb{Z}_+$ we may consider t^l_{00} as the spherical functions on $A_q(SL(2,\mathbb{C}))$ with respect to the subgroup 'generated' by $D-A$. We now want to do this for more general twisted primitive elements. The $*$-structure of the real form $U_q(\mathfrak{su}(2))$ and $A_q(SU(2))$ is needed.

For $\sigma \in \mathbb{R}$ we define

$$X_\sigma = iq^{\frac{1}{2}}B - iq^{-\frac{1}{2}}C - \frac{q^\sigma - q^{-\sigma}}{q - q^{-1}}(A - D) \in U_q(\mathfrak{su}(2)). \tag{4.1.1}$$

We also define

$$X_\infty = D - A = \lim_{\sigma\to\infty}(q^{-1} - q)q^\sigma X_\sigma = \lim_{\sigma\to-\infty}(q - q^{-1})q^{-\sigma}X_\sigma, \tag{4.1.2}$$

recovering the case of the previous paragraph as a special case. From (2.1.2) and Theorem 2.6.1(ii)

$$\Delta(X_\sigma) = A \otimes X_\sigma + X_\sigma \otimes D, \qquad S(X_\sigma) = -X_\sigma, \qquad (X_\sigma A)^* = X_\sigma A. \tag{4.1.3}$$

Note that $X_\sigma A$ is self-adjoint, so we consider it as the analogue of the element $\begin{pmatrix} \sigma & i \\ -i & -\sigma \end{pmatrix}$ in $i\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$ instead of an element of the real form $\mathfrak{su}(2)$. Instead of considering X_σ we can take a self-adjoint element in the space of primitive elements, but we need this definition in order to be able to prove certain multiplicative properties in the dual algebra later.

Fundamental for what follows is that we can explicitly calculate the eigenvectors and eigenvalues of each self-adjoint matrix $t^l(X_\sigma A)$.

Proposition 4.1.1. *The self-adjoint operator $t^l(X_\sigma A)$ has an orthonormal basis of eigenvectors $v^{l,j}(\sigma)=\sum_{n=-l}^{l} v_n^{l,j}(\sigma)e_n^l$ corresponding to the eigenvalue*

$$\lambda_j(\sigma)=\frac{q^{-2j-\sigma}-q^{\sigma+2j}+q^{\sigma}-q^{-\sigma}}{q-q^{-1}},\qquad j=-l,-l+1,\dots,l.$$

The coefficients $v_n^{l,j}(\sigma)$ are explicitly known by

$$v_n^{l,j}(\sigma)=C^{l,j}(\sigma)i^{n-l}q^{\sigma(l-n)}q^{\frac{1}{2}(l-n)(l-n-1)}\left(\frac{(q^{4l};q^{-2})_{l-n}}{(q^2;q^2)_{l-n}}\right)^{1/2}$$
$$\times R_{l-n}(q^{2j-2l}-q^{-2j-2l-2\sigma};q^{2\sigma},2l;q^2),$$

where R_{l-n} is a dual q-Krawtchouk polynomial, cf. Exercise 3.3, and the constant is given by

$$C^{l,j}(\sigma)=$$
$$q^{l+j}\begin{bmatrix}2l\\ l-j\end{bmatrix}_{q^2}^{1/2}\left(\frac{1+q^{-4j-2\sigma}}{1+q^{-2\sigma}}\right)^{1/2}\left((-q^{2-2\sigma};q^2)_{l-j}(-q^{2+2\sigma};q^2)_{l+j}\right)^{-1/2}.$$

Proof. The relation $t^l(X_\sigma A)\sum_{m=-l}^{l} c_m e_m^l=\lambda\sum_{m=-l}^{l} c_m e_m^l$ leads to a (finite) three-term recurrence relation for the coefficients c_m by (4.1.1) and Theorem 2.3.6. Comparison with the three-term recurrence relation for the dual q-Krawtchouk polynomials $R_n(y)=R_n(y;q^\sigma,N;q)$

$$y\,R_n(y)=$$
$$(1-q^{n-N})\,R_{n+1}(y)+(q^{-N}-q^{-N-\sigma})q^n\,R_n(y)-(1-q^n)q^{-N-\sigma}\,R_{n-1}(y)$$

for $y=q^{-x}-q^{x-N-\sigma}$, $x\in\{0,\dots,N\}$, gives the value for the coefficients and for the eigenvalues. The normalisation follows from the orthogonality relations for the q-Krawtchouk polynomials, cf. Exercise 3.3. ☑

Since the vectors $v^{l,j}(\sigma)$ are orthonormal in a finite dimensional space, we have the orthogonality relations

$$\text{(4.1.4)}\qquad \sum_{n=-l}^{l} v_n^{l,j}(\sigma)\overline{v_n^{l,i}(\sigma)}=\delta_{i,j},\qquad \sum_{i=-l}^{l} v_m^{l,i}(\sigma)\overline{v_n^{l,i}(\sigma)}=\delta_{n,m}.$$

The first part of (4.1.4) is equivalent to the orthogonality relations for the q-Krawtchouk polynomials, and the second part of (4.1.4) is equivalent to the orthogonality relations for the dual q-Krawtchouk polynomials, cf. Exercise 3.3. With this orthonormal basis for $\mathbb{C}^{2l+1}$ we can define generalised matrix elements.

Lemma 4.1.2. *In $A_q(SU(2))$ we define*

$$a^l_{i,j}(\tau,\sigma)(X) = \langle t^l(X)v^{l,j}(\sigma), v^{l,i}(\tau)\rangle, \qquad \sigma,\tau\in\mathbb{R},\ i,j=-l,-l+1,\dots,l,$$

then

$$a^l_{i,j}(\tau,\sigma) = \sum_{n,m=-l}^{l} v^{l,j}_m(\sigma)\overline{v^{l,i}_n(\tau)}t^l_{n,m} \in A_q(SU(2)),$$

and

$$(X_\sigma A).a^l_{i,j}(\tau,\sigma) = \lambda_j(\sigma)\, a^l_{i,j}(\tau,\sigma) \quad\text{and}\quad a^l_{i,j}(\tau,\sigma).(X_\tau A) = \lambda_i(\tau)\, a^l_{i,j}(\tau,\sigma).$$

Proof. The first statement is obvious from Proposition 4.1.1 and to prove the last we observe that $\langle Y, X.\xi\rangle = \langle YX,\xi\rangle$, so that for all $Y\in U_q(\mathfrak{su}(2))$

$$\begin{aligned}\langle Y,(X_\sigma A).a^l_{i,j}(\tau,\sigma)\rangle &= \langle t^l(Y)t^l(X_\sigma A)v^{l,j}(\sigma), v^{l,i}(\tau)\rangle\\ &= \lambda_j(\sigma)\langle t^l(Y)v^{l,j}(\sigma), v^{l,i}(\tau)\rangle = \lambda_j(\sigma)\langle Y, a^l_{i,j}(\tau,\sigma)\rangle.\end{aligned}$$

The other case is treated similarly using $(X_\tau A)^* = X_\tau A$. ☑

We want the action of X_σ and X_τ to be more symmetric than in Lemma 4.1.2. For this we define $b^l_{i,j}(\tau,\sigma) = A.a^l_{i,j}(\tau,\sigma)$, then
(4.1.5)

$$X_\sigma.b^l_{i,j}(\tau,\sigma) = \lambda_j(\sigma)\, D.b^l_{i,j}(\tau,\sigma) \quad\text{and}\quad b^l_{i,j}(\tau,\sigma).X_\tau = \lambda_i(\tau)\, b^l_{i,j}(\tau,\sigma).D$$

since $(X.\xi).Y = X.(\xi.Y)$ by the coassociativity of Δ. Explicitly,

$$b^l_{i,j}(\tau,\sigma) = \sum_{n,m=-l}^{l} v^{l,j}_m(\sigma)\overline{v^{l,i}_n(\tau)}q^{-m}\, t^l_{n,m} \in A_q(SU(2)). \tag{4.1.6}$$

The case $l = 1/2$ is of particular interest and we write

$$b^{1/2}(\tau,\sigma) = \frac{1}{\sqrt{(1+q^{2\sigma})(1+q^{2\tau})}}\begin{pmatrix}\alpha_{\tau,\sigma}, & \beta_{\tau,\sigma}\\ \gamma_{\tau,\sigma}, & \delta_{\tau,\sigma}\end{pmatrix},$$

or explicitly

$$\begin{aligned}\alpha_{\tau,\sigma} &= q^{1/2}\alpha - iq^{\sigma-1/2}\beta + iq^{\tau+1/2}\gamma + q^{\sigma+\tau-1/2}\delta,\\ \beta_{\tau,\sigma} &= -q^{\sigma+1/2}\alpha - iq^{-1/2}\beta - iq^{\sigma+\tau+1/2}\gamma + q^{\tau-1/2}\delta,\\ \gamma_{\tau,\sigma} &= -q^{\tau+1/2}\alpha + iq^{\tau+\sigma-1/2}\beta + iq^{1/2}\gamma + q^{\sigma-1/2}\delta,\\ \delta_{\tau,\sigma} &= q^{\tau+\sigma+1/2}\alpha + iq^{\tau-1/2}\beta - iq^{\sigma+1/2}\gamma + q^{-1/2}\delta.\end{aligned}$$

The following two propositions are fundamental in giving explicit formulas for $b^l_{i,j}(\tau,\sigma)$ and hence for the generalised matrix elements $a^l_{i,j}(\tau,\sigma)$.

Definition 4.1.3. *$\xi \in A_q(SU(2))$ is a (τ,σ)-spherical element if*

$$X_\sigma.\xi = 0 \quad \text{and} \quad \xi.X_\tau = 0.$$

Proposition 4.1.4. (i) *Let $\xi \in A_q(SU(2))$ be a (τ,σ)-spherical element and let $\eta \in A_q(SU(2))$ satisfy*

$$X_\sigma.\eta = \lambda\, D.\eta \quad \text{and} \quad \eta.X_\tau = \mu\,\eta.D. \tag{4.1.7}$$

for $\lambda,\mu \in \mathbb{C}$. Then $\eta\xi$ satisfies (4.1.7) for the same λ, μ. Moreover, if $\lambda,\mu \in \mathbb{R}$, then $\eta^*\eta$ is a (τ,σ)-spherical element. In particular, the space of (τ,σ)-spherical element forms a $*$-subalgebra of $A_q(SU(2))$.
(ii) *If $\eta \in span(t^l_{nm})$ satisfies (4.1.7) for arbitrary $\lambda,\mu \in \mathbb{C}$ and η is non-zero, then $\lambda = \lambda_j(\sigma)$, $\mu = \lambda_i(\tau)$ for some $i,j \in \{-l,-l+1,\ldots,l\}$ and η is a multiple of $b^l_{i,j}(\tau,\sigma)$.*

Proof. To prove (i) we first note that by Definition 1.2.1 and Proposition 1.2.3 we have in general for Hopf algebras in duality

$$u.(ab) = \sum_{(u)} (u_{(1)}.a)(u_{(2)}.b), \qquad \Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}, \tag{4.1.8}$$

with the notation of (1.1.5).

First consider $\eta\xi$, then

$$X_\sigma.(\eta\xi) = (A.\eta)(X_\sigma.\xi) + (X_\sigma.\eta)(D.\xi) = \lambda(D.\eta)(D.\xi) = \lambda\, D.(\eta\xi),$$

since X_σ is twisted primitive and D group like. Similarly we prove $(\eta\xi).X_\tau = \mu\,(\eta\xi).D$.

To prove the other statement of (i) we observe that in general for Hopf $*$-algebras in duality the left action satisfies $u.a^* = (S(u)^*.a)^*$. Now proceed as before to obtain

$$X_\sigma.(\eta^*\eta) = (A.\eta^*)(X_\sigma.\eta) + (X_\sigma.\eta^*)(D.\eta) = (\lambda - \bar{\lambda})(D.\eta)^*(D.\eta),$$

since $S(A)^* = D$ and $S(X_\sigma)^* = -X_\sigma$. This yields zero for $\lambda \in \mathbb{R}$. Similarly we prove $(\eta^*\eta).X_\tau = 0$ for real μ.

The proof of (ii) follows immediately from the linear basis of $A_q(SU(2))$ given in Theorem 2.4.1 and the multiplicity of each eigenvalue of $t^l(X_\sigma A)$ being one. ☑

Proposition 4.1.5. *Let η satisfy* (4.1.7) *with $\lambda = \lambda_j(\sigma)$ and $\mu = \lambda_i(\tau)$, then*
(i) $\alpha_{\tau+2i,\sigma+2j}\eta$ *satisfies* (4.1.7) *with $\lambda = \lambda_{j-1/2}(\sigma)$ and $\mu = \lambda_{i-1/2}(\tau)$,*
(ii) $\beta_{\tau+2i,\sigma+2j}\eta$ *satisfies* (4.1.7) *with $\lambda = \lambda_{j+1/2}(\sigma)$ and $\mu = \lambda_{i-1/2}(\tau)$,*
(iii) $\gamma_{\tau+2i,\sigma+2j}\eta$ *satisfies* (4.1.7) *with $\lambda = \lambda_{j-1/2}(\sigma)$ and $\mu = \lambda_{i+1/2}(\tau)$,*
(iv) $\delta_{\tau+2i,\sigma+2j}\eta$ *satisfies* (4.1.7) *with $\lambda = \lambda_{j+1/2}(\sigma)$ and $\mu = \lambda_{i+1/2}(\tau)$.*

Remark 4.1.6. Proposition 4.1.5 gives a way to express the corner elements, i.e. $b^l_{ij}(\tau,\sigma)$ with $l = \max(|i|,|j|)$, in terms of products of $\alpha_{\tau,\sigma}$, $\beta_{\tau,\sigma}$, $\gamma_{\tau,\sigma}$, $\delta_{\tau,\sigma}$, but with integer shifts in the parameters τ and σ.

Sketch of proof. Take $\xi \in A_q(SU(2))$ arbitrary for the moment, then as in the proof of Proposition 4.1.4,

$$X_\sigma.(\xi\eta) = [\lambda A.\xi + X_\sigma.\xi](D.\eta), \quad (\xi\eta).X_\tau = [\mu\,\xi.A + \xi.X_\tau](\eta.D).$$

So, if ξ satisfies

$$(4.1.9) \qquad [\lambda A.\xi + X_\sigma.\xi] = \lambda_1 D.\xi, \qquad [\mu\,\xi.A + \xi.X_\tau] = \mu_1\,\xi.D$$

for some $\lambda_1, \mu_1 \in \mathbb{C}$, then we get

$$X_\sigma.(\xi\eta) = \lambda_1\, D.(\xi\eta), \qquad \text{and} \qquad (\xi\eta).X_\tau = \mu_1\,(\xi\eta).D.$$

Next we assume $\xi = a\alpha + b\beta + c\gamma + d\delta$ for some complex constants a, b, c and d. Using (2.6.1) we see that the first requirement of (4.1.9) leads to $M\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $M\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for a 2×2-matrix depending on λ, λ_1 and σ. From $\det(M) = 0$ it follows that for $\lambda = \lambda_j(\sigma)$ we have $\lambda_1 = \lambda_{j\pm 1/2}$. Similarly, the second requirement of (4.1.9) leads to linear equations of a similar form and again a determinant condition implies $\mu = \lambda_i(\tau)$ and $\mu_1 = \lambda_{i\pm 1/2}(\tau)$. Combining each of the cases leads to a solution for a, b, c and d up to a scalar. ☑

Since $\lambda_i(\sigma) = 0 \Leftrightarrow i = 0$ we see from Proposition 4.1.4(ii) that there is no (τ,σ)-spherical element in $\mathrm{span}(t^{1/2}_{nm})$ and that there is a one-dimensional space of (τ,σ)-spherical elements in $\mathrm{span}(t^1_{nm})$. Take such an element and keep only the non-constant terms to find the following (τ,σ)-spherical element;

$$\rho_{\tau,\sigma} = \frac{1}{2}\big(\alpha^2 + \delta^2 + q\gamma^2 + q^{-1}\beta^2 + i(q^{-\sigma} - q^{\sigma})(q\delta\gamma + \beta\alpha) \\ - i(q^{-\tau} - q^{\tau})(\delta\beta + q\gamma\alpha) + (q^{-\sigma} - q^{\sigma})(q^{-\tau} - q^{\tau})\beta\gamma\big).$$

Proposition 4.1.7. *The $*$-subalgebra of $A_q(SU(2))$ of (τ,σ)-spherical elements is generated by the self-adjoint element $\rho_{\tau,\sigma}$. Moreover,*

$$\begin{aligned}
\beta_{\tau+1,\sigma-1}\gamma_{\tau,\sigma} &= 2q^{\tau+\sigma}\rho_{\tau,\sigma} - q^{2\sigma-1} - q^{2\tau+1},\\
\gamma_{\tau-1,\sigma+1}\beta_{\tau,\sigma} &= 2q^{\tau+\sigma}\rho_{\tau,\sigma} - q^{2\sigma+1} - q^{2\tau-1},\\
\alpha_{\tau+1,\sigma+1}\delta_{\tau,\sigma} &= 2q^{\tau+\sigma+1}\rho_{\tau,\sigma} + 1 + q^{2\sigma+2\tau+2},\\
\delta_{\tau-1,\sigma-1}\alpha_{\tau,\sigma} &= 2q^{\tau+\sigma-1}\rho_{\tau,\sigma} + 1 + q^{2\sigma+2\tau-2}.
\end{aligned}$$

Proof. Indeed, $\rho^*_{\tau,\sigma} = \rho_{\tau,\sigma}$ for the $*$-operator of Theorem 2.6.1(ii). From Proposition 4.1.4(ii) and $\lambda_i(\sigma) = 0$ if and only if $i = 0$, it follows that a linear basis for the space of (τ,σ)-spherical elements is given by $b^l_{00}(\tau,\sigma)$, $l \in \mathbb{Z}_+$. From Lemma 2.4.2 we see that $\rho^l_{\tau,\sigma}$ has a non-zero component in $\mathrm{span}(t^l_{nm})$. Since $\mathrm{span}(t^l_{nm})$ is invariant under the left action of X_σ and under the right action of X_τ it follows by induction on l that $b^l_{00}(\tau,\sigma)$ is a polynomial of degree l in $\rho_{\tau,\sigma}$. From Proposition 4.1.5 it follows that each product on the left hand side is a (τ,σ)-spherical element, and by considering the degrees it has to be a polynomial of degree 1 in $\rho_{\tau,\sigma}$ by the first part. It remains to calculate the leading and constant coefficient by comparing the coefficients of α^2 and the unit 1 on both sides. ☑

Corollary 4.1.8. *The following relations hold in $A_q(SL(2,\mathbb{C}))$;*

$$\begin{aligned}
\alpha_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau-1,\sigma-1}\alpha_{\tau,\sigma}, & \beta_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau-1,\sigma+1}\beta_{\tau,\sigma},\\
\gamma_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau+1,\sigma-1}\gamma_{\tau,\sigma}, & \delta_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau+1,\sigma+1}\delta_{\tau,\sigma}.
\end{aligned}$$

Proof. The proofs of these statements are all similar. To prove the first, multiply the last equation of Proposition 4.1.7 by $\alpha_{\tau,\sigma}$ and use the third equation in the left hand side. Cancelling terms proves the first statement. ☑

Remark 4.1.9. We define $\rho_{\tau,\infty}$, $\rho_{\infty,\sigma}$ and $\rho_{\infty,\infty}$ as limit cases of $\rho_{\tau,\sigma}$ as follows;

$$\rho_{\infty,\sigma} = \lim_{\tau\to\infty} 2q^{\sigma+\tau-1}\rho_{\tau,\sigma}, \qquad \rho_{\tau,\infty} = \lim_{\sigma\to\infty} 2q^{\sigma+\tau-1}\rho_{\tau,\sigma},$$

and $\rho_{\infty,\infty} = \lim_{\sigma\to\infty}\rho_{\infty,\sigma} = \lim_{\tau\to\infty}\rho_{\tau,\infty} = -\gamma\gamma^*$. Since we take $0 < q < 1$, this is just replacing $q^\infty = 0$. The corresponding factorisations of Proposition 4.1.7 remain valid under these limit transitions, since we can take limits in $\alpha_{\tau,\sigma}$, etcetera. In case $\sigma = \tau \to \infty$, Proposition 4.1.7 reduces to part of the relations (2.4.2) in $A_q(SU(2))$.

§4.2. The Haar functional restricted to (τ,σ)-spherical elements. Recall that the invariant functional or Haar functional is a positive linear functional on $A_q(SU(2))$, which is defined by $h(t^l_{mn}) = \delta_{l0}$.

Lemma 4.2.1. *$h(\delta^l\gamma^m\beta^n) = h(\alpha^l\gamma^m\beta^n) = 0$ unless $l = 0$ and $n = m$ and*

$$h(\gamma^m\beta^m) = (-q)^m \frac{1-q^2}{1-q^{2m+2}}.$$

Proof. The invariance of h implies $h(a.X) = h(a)\varepsilon(X) = h(X.a)$. Use this with $X = A$, and $X = D$ (so that the left and right action is a homomorphism because A and D are group like), (2.6.1) and $a = \delta^l\gamma^m\beta^n$ and $a = \alpha^l\gamma^m\beta^n$ to see that the Haar functional applied to these basis elements gives zero unless $l = 0$, $m = n$.

In this case we take $X = C$ and we calculate, using the analogue of (4.1.8) for the right action,

$$\begin{aligned}
&\delta\gamma(-q^{-1}\beta\gamma)^n.C \\
&= \big(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.A\big)\big(-q^{-1}\beta\gamma.C\big) + \big(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.C\big)\big(-q^{-1}\beta\gamma.D\big) \\
&= \big(q^{-1}\delta\gamma(-q^{-1}\beta\gamma)^{n-1}\big)\big(-q^{-1/2}\beta\alpha\big) + \big(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.C\big)\big(-q^{-1}\beta\gamma\big) \\
&= q^{-1/2-2n}(-q^{-1}\beta\gamma)^n(1+q^{-1}\beta\gamma) + \big(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.C\big)\big(-q^{-1}\beta\gamma\big)
\end{aligned}$$

and together with the initial condition for $n = 0$ we get

$$= \frac{q^{-1/2-2n}}{1-q^2}\big((1-q^{2n+2})(-q^{-1}\beta\gamma)^n - (1-q^{2n+4})(-q^{-1}\beta\gamma)^{n+1}\big).$$

Apply h and $\varepsilon(C) = 0$ to get a two-term recurrence relation, which is uniquely solved with the initial condition $h(1) = 1$. ☑

Corollary 4.2.2. *Let D be the self-adjoint positive diagonal operator $\ell^2(\mathbb{Z}_+)$ with orthonormal basis $\{e_n\}_{n=0}^\infty$ defined by $D\colon e_n \mapsto q^{2n}e_n$,*

$$h(a) = \frac{(1-q^2)}{2\pi}\int_0^{2\pi} \operatorname{tr}\big(D\pi_\phi^\infty(a)\big)\,d\phi, \qquad a \in A_q(SL(2,\mathbb{C}))$$

Proof. Check that the right hand side coincides with h for $a = \delta^l\gamma^m\beta^n$ or $a = \alpha^l\gamma^m\beta^n$. ☑

Remark 4.2.3. The trace operation in Corollary 4.2.2 is well-defined due to the appearance of D. Since $D^{1/2}$ is a Hilbert-Schmidt operator, so is $\pi_\phi^\infty(a)D^{1/2}$, since $\pi_\phi^\infty(a)$ is a bounded operator. The trace of the product of two Hilbert-Schmidt operators is well-defined. Moreover, $\operatorname{tr}\big(D\pi_\phi^\infty(a)\big) = \operatorname{tr}\big(\pi_\phi^\infty(a)D\big)$ and it is independent of the choice of the basis. The trace can be estimated by the product of the Hilbert-Schmidt norms of $D^{1/2}$ and $\pi_\phi^\infty(a)D^{1/2}$, i.e. $\operatorname{tr}\big(D\pi_\phi^\infty(a)\big)| \leq \| a \| /(1-q^2)$, so that the function in Corollary 4.2.2 is integrable. Here $\| a \|$, $a \in A_q(SU(2))$, denotes the supremum of the operator norm $\pi_\theta^\infty(a)$ as θ varies, where we use the notation of Theorem 2.6.3.

The next theorem is the key in determining the generalised matrix elements of the previous section in terms of q-special functions.

Theorem 4.2.4. *The Haar functional on the subalgebra generated by the self-adjoint element $\rho_{\tau,\sigma}$ is given by*

$$h(p(\rho_{\tau,\sigma})) = \int_{\mathbb{R}} p(x)\, dm(x;a,b,c,d|q^2)$$

for any polynomial p. Here $a=-q^{\sigma+\tau+1}$, $b=-q^{-\sigma-\tau+1}$, $c=q^{\sigma-\tau+1}$, $d=q^{-\sigma+\tau+1}$ and $dm(x;a,b,c,d|q^2)$ denotes the normalised Askey-Wilson measure.

The proof is based on Corollary 4.2.2 and the spectral theory of Jacobi matrices and their connection with orthonormal polynomials. The proof of the general statement is of some computational complexity. We first consider the case $\sigma\to\infty$ of Theorem 4.2.4. This limit case is also needed in the proof of the general statement. We then illustrate the procedure for the general case in a less computational case, namely the Haar functional restricted to the co-central elements. In these two cases the calculations are much simpler, and the proof of the general case is contained in the exercises. The limiting case $\sigma\to\infty$ of Theorem 4.2.4 is phrased in terms of the q-integral introduced in Exercise 3.5.

Theorem 4.2.5. *The Haar functional on the subalgebra generated by the self-adjoint element $\rho_{\tau,\infty}$ is given by the q-integral*

$$h\big(p(\rho_{\tau,\infty})\big) = \frac{1}{1+q^{2\tau}}\int_{-1}^{q^{2\tau}} p(x)\, d_{q^2}x$$

for any polynomial p.

To prove Theorem 4.2.5 from a spectral analysis of $\pi_\theta^\infty(\rho_{\tau,\infty})$ we use the following proposition.

Proposition 4.2.6. *$\ell^2(\mathbb{Z}_+)$ has an orthogonal basis of eigenvectors v_λ^θ, where $\lambda=-q^{2k}$, $k\in\mathbb{Z}_+$, and $\lambda=q^{2\tau+2k}$, $k\in\mathbb{Z}_+$, for the eigenvalue λ of the self-adjoint operator $\pi_\theta^\infty(\rho_{\tau,\infty})$. The squared norm is given by*

$$\langle v_\lambda^\theta, v_\lambda^\theta\rangle = q^{-2k}(q^2;q^2)_k(-q^{2-2\tau};q^2)_k(-q^{2\tau};q^2)_\infty, \qquad \lambda=-q^{2k},$$
$$\langle v_\lambda^\theta, v_\lambda^\theta\rangle = q^{-2k}(q^2;q^2)_k(-q^{2+2\tau};q^2)_k(-q^{-2\tau};q^2)_\infty, \qquad \lambda=q^{2\tau+2k}.$$

Moreover, $v_\lambda^\theta=\sum_{n=0}^\infty i^n e^{in\theta}p_n(\lambda)\,e_n$ with the polynomial $p_n(\lambda)$ defined by

$$\begin{aligned} p_n(\lambda) &= \frac{q^{-n\tau}q^{n(n-1)/2}}{\sqrt{(q^2;q^2)_n}}\,{}_2\varphi_1(q^{-2n},q^{2\tau}/\lambda;0;q;-q^2\lambda) \\ &= \frac{(-1)^n q^{n\tau}q^{n(n-1)/2}}{\sqrt{(q^2;q^2)_n}}\,{}_2\varphi_1(q^{-2n},-1/\lambda;0;q;q^{2-2\tau}\lambda). \end{aligned} \tag{4.2.1}$$

Remark 4.2.7. The basis described in Proposition 4.2.6 induces an orthogonal decomposition of the representation space $\ell^2(\mathbb{Z}_+)=V_1^\theta\oplus V_2^\theta$, where V_1^θ is the subspace with basis $v^\theta_{-q^{2k}}$, $k \in \mathbb{Z}_+$, and V_2^θ is the subspace with basis $v^\theta_{q^{2\tau+2k}}$, $k \in \mathbb{Z}_+$.

Sketch of Proof. Consider the Al-Salam and Carlitz polynomials

$$\begin{aligned} U_n^{(a)}(x;q) &= (-a)^n q^{n(n-1)/2}\, {}_2\varphi_1(q^{-n}, x^{-1}; 0; q, qx/a) \\ &= (-1)^n q^{n(n-1)/2}\, {}_2\varphi_1(q^{-n}, a/x; 0; q, qx). \end{aligned}$$

The equality follows from the second of Heine's transformation formulae of Theorem 3.2.4. Take $b = q^{-n}$ and reverse the order of summation on the right hand side before letting $c \to 0$. The Al-Salam and Carlitz polynomials are orthogonal polynomials satisfying the three-term recurrence relation

$$x\, U_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)\, U_n^{(a)}(x;q) - aq^{n-1}(1-q^n)\, U_{n-1}^{(a)}(x;q).$$

Compare this with the action of $\rho_{\tau,\infty}$ in $\ell^2(\mathbb{Z}_+)$,

$$\begin{aligned} &\pi_\theta^\infty(\rho_{\tau,\infty})\, e_n = \\ &iq^{\tau+n}e^{i\theta}\sqrt{1-q^{2n+2}}\, e_{n+1} - (1-q^{2\tau})q^{2n}\, e_n - iq^{\tau+n-1}e^{-i\theta}\sqrt{1-q^{2n}}\, e_{n-1}, \end{aligned}$$

to see that v^θ_λ as defined in Proposition 4.2.6 is indeed formally an eigenvector for $\pi_\theta^\infty(\rho_{\tau,\infty})$ for the eigenvalue λ. For $\lambda = -q^{2k}$ or $\lambda = q^{2\tau+2k}$, $k \in \mathbb{Z}_+$, it is easy to show that $v^\theta_\lambda \in \ell^2(\mathbb{Z}_+)$. The orthogonality follows, since $\pi_\theta^\infty(\rho_{\tau,\infty})$ is self-adjoint.

It remains to calculate the squared norm and to prove the completeness in $\ell^2(\mathbb{Z}_+)$. The squared norm can be calculated using the fact that we already have established orthogonality and some easy calculations using Theorem 3.2.1 and Corollary 3.2.2(iii). The completeness follows, since the dual orthogonality relations also hold, which are in fact the orthogonality relations for the Al-Salam and Carlitz polynomials. ☑

Proof of Theorem 4.2.5. We calculate the trace with respect to the orthogonal basis of eigenvectors described in Proposition 4.2.6;

$$\begin{aligned} &\mathrm{tr}\Big(D\pi_\theta^\infty(p(\rho_{\tau,\infty}))\Big) = \\ &\sum_{k=0}^\infty p(-q^{2k}) \frac{\langle Dv^\theta_{-q^{2k}}, v^\theta_{-q^{2k}}\rangle}{\langle v^\theta_{-q^{2k}}, v^\theta_{-q^{2k}}\rangle} + \sum_{k=0}^\infty p(q^{2\tau+2k}) \frac{\langle Dv^\theta_{q^{2\tau+2k}}, v^\theta_{q^{2\tau+2k}}\rangle}{\langle v^\theta_{q^{2\tau+2k}}, v^\theta_{q^{2\tau+2k}}\rangle}. \end{aligned}$$

So it remains to calculate the matrix coefficients on the diagonal of the operator D with respect to this basis. This can be done by a straightforward calculation, or by use of the so-called q-Charlier polynomials, cf. Exercise 4.3. The result is

$$\begin{aligned} \langle Dv^\theta_{-q^{2k}}, v^\theta_{-q^{2k}}\rangle &= (-q^{2\tau+2};q^2)_\infty (q^2;q^2)_k (-q^{2-2\tau};q^2)_k, \\ \langle Dv^\theta_{q^{2\tau+2k}}, v^\theta_{q^{2\tau+2k}}\rangle &= (-q^{2-2\tau};q^2)_\infty (q^2;q^2)_k (-q^{2+2\tau};q^2)_k, \end{aligned}$$

Thus

$$\operatorname{tr}\Big(D\pi_\theta^\infty(p(\rho_{\tau,\infty}))\Big) = \frac{1}{1+q^{2\tau}}\Big(\sum_{k=0}^\infty p(-q^{2k})q^{2k} + q^{2\tau}\sum_{k=0}^\infty p(q^{2\tau+2k})q^{2k}\Big).$$

This expression is independent of θ, so that we obtain from Corollary 4.2.2

$$h(p(\rho_{\tau,\infty})) = \frac{1-q^2}{1+q^{2\tau}}\Big(\sum_{k=0}^\infty p(-q^{2k})q^{2k} + q^{2\tau}\sum_{k=0}^\infty p(q^{2\tau+2k})q^{2k}\Big),$$

which is precisely the definition of the q-integral. ☑

The proof of Theorem 4.2.5 is straightforward, since the representation space $\ell^2(\mathbb{Z}_+)$ has an orthonormal basis of eigenvectors of $\pi_\theta^\infty(\rho_{\tau,\infty})$ for each θ, so that the spectrum of $\pi_\theta^\infty(\rho_{\tau,\infty})$ is purely discrete.

To prove the general case we have to invoke Jacobi matrices, as in §3.1. We can show that $\pi_\theta^\infty(\rho_{\tau,\sigma})$ on each of the components of the decomposition $\ell^2(\mathbb{Z}_+) = V_1^\theta \oplus V_2^\theta$ of Remark 4.2.7 is represented by a Jacobi matrix, although $\pi_\theta^\infty(\rho_{\tau,\sigma})$ in the standard basis is represented by a five-term recurrence operator. On each of the components we can determine the corresponding orthogonality measure using Al-Salam and Chihara polynomials, see Exercise 3.4. Then the Poisson kernel for the Al-Salam and Chihara polynomials comes into play, which is a (very-well poised) ${}_8\varphi_7$-series, on each of these components. Finally, the results have to be matched using Bailey's formula for the sum of two of such ${}_8\varphi_7$-series of the correct form. In Exercises 4.4–7 more explicit hints are given. In order to show how this works we consider the case of the Haar functional on the cocentral elements, in which all ingredients are contained but in which the computations are much simpler.

Theorem 4.2.8. *The Haar functional on the subalgebra generated by the self-adjoint element $\alpha+\alpha^*$ is given by the integral*

$$h\big(p((\alpha+\alpha^*)/2)\big) = \frac{2}{\pi}\int_{-1}^1 p(x)\sqrt{1-x^2}\,dx$$

for any polynomial p.

Proof. Consider

$$2\pi_\theta^\infty\big((\alpha+\alpha^*)/2\big)e_n = \sqrt{1-q^{2n}}\,e_{n-1} + \sqrt{1-q^{2n+2}}\,e_{n+1}. \tag{4.2.2}$$

So the operator $\pi_\theta^\infty\big((\alpha+\alpha^*)/2\big)$ is represented by a Jacobi matrix with respect to the standard basis of $\ell^2(\mathbb{Z}_+)$.

Take $a=b=0$ in the Al-Salam and Chihara polynomials of Exercise 3.4 to obtain Rogers's continuous q-Hermite polynomials $H_n(x|q)$ satisfying

(4.2.3)

$$2x\,H_n(x|q) = H_{n+1}(x|q) + (1-q^n)\,H_{n-1}(x|q),\quad H_{-1}(x|q)=0,\ H_0(x|q)=1.$$

The continuous q-Hermite polynomials satisfy the orthogonality relations

$$\int_0^\pi H_n(\cos\phi|q)H_m(\cos\phi|q)w(\cos\phi|q)\,d\phi = \delta_{nm}\frac{2\pi(q;q)_n}{(q;q)_\infty},$$

with $w(\cos\phi|q) = (e^{2i\phi}, e^{-2i\phi};q)_\infty$, by taking $a = b = c = d = 0$ in the Askey-Wilson orthogonality measure of Theorem 3.3.2.

Compare (4.2.2) with the three-term recurrence relation (4.2.3) to see that the orthonormal polynomials associated to the Jacobi matrix $\pi_\theta^\infty((\alpha+\alpha^*)/2)$ are the orthonormal continuous q-Hermite polynomials $H_n(x|q^2)/\sqrt{(q^2;q^2)_n}$.

We use the spectral theory of Jacobi matrices as in the proof of Theorem 3.1.2 with $p_n(x) = H_n(x|q^2)/\sqrt{(q^2;q^2)_n}$ as the orthonormal polynomials and the absolutely continuous measure $dm(x|q^2) = (2\pi)^{-1}(q^2;q^2)_\infty w(x|q^2)(1-x^2)^{-1/2}dx$ on $[-1,1]$ as the (normalised) orthogonality measure. So we obtain a unitary mapping Λ, intertwining $\pi_\theta^\infty((\alpha+\alpha^*)/2)$ on $\ell^2(\mathbb{Z}_+)$ with the multiplication operator M on $L^2([-1,1], dm(x|q^2))$. Hence,

$$\begin{aligned}
\mathrm{tr}\Big(D\pi_\theta^\infty(p((\alpha+\alpha^*)/2))\Big) &= \sum_{n=0}^\infty q^{2n}\langle \pi_\theta^\infty(p((\alpha+\alpha^*)/2))\,e_n, e_n\rangle \\
&= \sum_{n=0}^\infty q^{2n}\langle \Lambda\pi_\theta^\infty(p((\alpha+\alpha^*)/2))\,e_n, \Lambda e_n\rangle \\
&= \sum_{n=0}^\infty q^{2n}\int_{-1}^1 p(x)\left|(\Lambda e_n)(x)\right|^2 dm(x|q^2) \\
&= \sum_{n=0}^\infty q^{2n}\int_{-1}^1 p(x)\frac{\left(H_n(x|q^2)\right)^2}{(q^2;q^2)_n}\,dm(x|q^2) \\
&= \int_{-1}^1 p(x)P_{q^2}(x,x|q^2)\,dm(x|q^2),
\end{aligned}$$

where $P_t(x;y|q)$ is the Poisson kernel for the continuous q-Hermite polynomials. Explicitly, P is given by

$$\begin{aligned}
P_t(\cos\phi,\cos\psi|q) &= \sum_{n=0}^\infty \frac{H_n(\cos\phi|q)H_n(\cos\psi|q)t^n}{(q;q)_n} \\
&= \frac{(t^2;q)_\infty}{(te^{i\phi+i\psi}, te^{i\phi-i\psi}, te^{-i\phi+i\psi}, te^{-i\phi-i\psi};q)_\infty}, \qquad |t|<1.
\end{aligned}$$

Interchanging summation and integration is easily justified.

From the explicit expression for the Poisson kernel we get

$$w(x|q^2)P_{q^2}(x,x|q^2) = \frac{4(1-x^2)}{(1-q^2)(q^2;q^2)_\infty},$$

so that

$$\operatorname{tr}\Big(D\pi_\theta^\infty\big(p((\alpha+\alpha^*)/2)\big)\Big) = \frac{2}{\pi(1-q^2)}\int_{-1}^{1} p(x)\sqrt{1-x^2}\,dx.$$

Since this is independent of the parameter θ of the infinite dimensional representation, the proof follows from Corollary 4.2.2. ☑

Notes and references. The introduction of these infinitesimally generated 'subgroups' of the quantum $SU(2)$ group is due to Koornwinder [60]. This paper has been very influential for the development of the relation between q-special functions and quantum groups. The rest of §4.1 has been taken from an unpublished announcement [76] by Noumi and Mimachi, of which an even shorter announcement [75] has appeared, see also [79], and from [45], [50], [51]. The (dual) q-Krawtchouk polynomials and the element of $X_\sigma A$ also naturally occur when determining the spherical functions on the Hecke algebra of type B_n with respect to the parabolic subalgebra corresponding to A_{n-1}, see [52].

Lemma 4.2.1 is due to Woronowicz [103], but its proof is taken from Noumi and Mimachi [80], and Corollary 4.2.2 is due to Vaksman and Soibelman [96]. See Dunford and Schwartz [22, Ch. XI, §6] for more details on Hilbert-Schmidt operators and trace class operators. In fact, $\| a \|$, $a \in A_q(SU(2))$, is a C^*-norm, and we can complete $A_q(SU(2))$ into a C^*-algebra in which $A_q(SU(2))$ is a dense subalgebra. This corresponds to Woronowicz's approach to the quantum $SU(2)$ group, [103], [104]. The key Theorem 4.2.4 is due to Koornwinder [60], but the proof sketched here is taken from Koelink and Verding [54]. The details of the last part of the proof of Proposition 4.2.6 can be found in [48]. Theorem 4.2.8 is due to Woronowicz [103], with a different proof. The Poisson kernel is known from the work by Rogers (1894-6) on the continuous q-Hermite polynomials, see [5], [13].

Exercises.

1. Fill in the proof of Proposition 4.1.5.
2. Fill in the gaps of the last paragraph of the proof of Proposition 4.2.6. Use the orthogonality relations for the Al-Salam and Carlitz polynomials;

$$\int_a^1 U_m^{(a)}(q;x)U_n^{(a)}(q;x)(qx,qx/a;q)_\infty\,d_qx = \\ \delta_{nm}(-a)^n(1-q)q^{n(n-1)/2}(q;q)_n(q,a,1/a;q)_\infty$$

   assuming $a<0$ for the measure to be non-negative.
3. Define the discrete orthogonality measure μ by

$$\int_{\mathbb{R}} p(x)\,d\mu(x) = \sum_{n=0}^{\infty}\frac{q^{2n\tau}q^{n(n-1)}}{(q^2;q^2)_n}p(q^{-2n}),$$

and define the q-Charlier polynomials

$$c_n(x;a;q) = {}_2\varphi_1(q^{-n}, x; 0; q, -q^{n+1}/a).$$

Prove from Proposition 4.2.6

$$\int_{\mathbb{R}} c_k(x;q^{2\tau};q^2)c_l(x;q^{2\tau};q^2)\,d\mu(x) = \delta_{k,l}q^{-2k}(q^2,-q^{2-2\tau};q^2)_k(-q^{2\tau};q^2)_\infty.$$

Assuming $k \geq l$, prove that

$$\begin{aligned}\langle Dv^\theta_{-q^{2k}}, v^\theta_{-q^{2l}}\rangle &= \int_{\mathbb{R}} \frac{1}{x} c_k(x;q^{2\tau};q^2)c_l(x;q^{2\tau};q^2)\,d\mu(x)\\ &= c_l(0;q^{2\tau};q^2)\int_{\mathbb{R}} \frac{1}{x} c_k(x;q^{2\tau};q^2)\,d\mu(x),\end{aligned}$$

and calculate the last integral explicitly using the summation formulas of §3.

4. Use the notation $v^\theta_\lambda(q^\tau) = v^\theta_\lambda$ for the orthogonal basis of Proposition 4.2.6 in order to stress the τ-dependence. Prove that

$$\begin{aligned}\pi^\infty_\theta(\alpha_{\tau,\infty})v^\theta_\lambda(q^\tau) &= e^{i\theta}iq^{1/2-\tau}(1+\lambda)v^\theta_{\lambda/q^2}(q^{\tau-1}),\\ \pi^\infty_\theta(\beta_{\tau,\infty})v^\theta_\lambda(q^\tau) &= e^{-i\theta}iq^{1/2}v^\theta_\lambda(q^{\tau-1}),\\ \pi^\infty_\theta(\gamma_{\tau,\infty})v^\theta_\lambda(q^\tau) &= e^{i\theta}iq^{1/2}(q^{2\tau}-\lambda)v^\theta_\lambda(q^{\tau+1}),\\ \pi^\infty_\theta(\delta_{\tau,\infty})v^\theta_\lambda(q^\tau) &= -e^{-i\theta}iq^{1/2+\tau}v^\theta_{\lambda q^2}(q^{\tau+1})\end{aligned}$$

and from this that

$$\begin{aligned}&2\pi^\infty_\theta(\rho_{\tau,\sigma})v^\theta_\lambda =\\ &qe^{-2i\theta}v^\theta_{\lambda q^2} + q^{-1}e^{2i\theta}(1-q^{-2\tau}\lambda)(1+\lambda)v^\theta_{\lambda/q^2} + \lambda q^{1-\tau}(q^{-\sigma}-q^{\sigma})v^\theta_\lambda.\end{aligned}$$

Use Proposition 4.1.7 and that $\alpha_{\tau,\sigma}$ can be written as a linear combination of $\alpha_{\tau,\infty}$ and $\beta_{\tau,\infty}$, and similarly for the other elements.

5. $\pi^\infty_\theta(\rho_{\tau,\sigma})$ preserves the orthogonal decomposition of Remark 4.2.7, and is given by a Jacobi matrix on each component. Let

$$D = \begin{pmatrix} D^\theta_{1,1} & D^\theta_{1,2}\\ D^\theta_{2,1} & D^\theta_{2,2}\end{pmatrix}$$

be the corresponding decomposition of D. So that

$$\mathrm{tr}\big(D\pi^\infty_\theta(p(\rho_{\tau,\sigma}))\big) = \mathrm{tr}_{V^\theta_1}\big(D^\theta_{11}\pi^\infty_\theta(p(\rho_{\tau,\sigma}))\big) + \mathrm{tr}_{V^\theta_2}\big(D^\theta_{22}\pi^\infty_\theta(p(\rho_{\tau,\sigma}))\big).$$

Denote by w_m^θ, $m\in\mathbb{Z}_+$, the orthonormal basis of V_1^θ obtained by normalising $v_{-q^{2m}}^\theta$, $m\in\mathbb{Z}_+$, and by u_m^θ, $m\in\mathbb{Z}_+$, the orthonormal basis of V_2^θ obtained by normalising $v_{q^{2\tau+2m}}^\theta$, $m\in\mathbb{Z}_+$. Show that the corresponding orthonormal polynomials can be given in terms of orthonormal Al-Salam and Chihara polynomials,

$$h_n(x;s,t|q)=(q,-q/s^2;q)^{-1/2}\,s_n(x;q^{1/2}t/s,-q^{1/2}/st|q),$$

with the notation of Exercise 3.4. And prove that

$$\mathrm{tr}_{V_1^\theta}(D_{11}^\theta\pi_\theta^\infty(p(\rho_{\tau,\sigma})))=\int_{\mathbb{R}}p(x)\sum_{n=0}^\infty\sum_{m=0}^\infty\langle Dw_n^\theta,w_m^\theta\rangle\times$$
$$h_n(x;q^\tau,q^\sigma|q^2)h_m(x;q^\tau,q^\sigma|q^2)e^{2i(m-n)\theta}\,dm(x;q^{1+\sigma-\tau},-q^{1-\sigma-\tau},0,0|q^2)$$

and similarly

$$\mathrm{tr}_{V_2^\theta}(D_{22}^\theta\pi_\theta^\infty(p(\rho_{\tau,\sigma})))=\int_{\mathbb{R}}p(x)\sum_{n=0}^\infty\sum_{m=0}^\infty\langle Du_n^\theta,u_m^\theta\rangle\times$$

$$h_n(x;q^{-\tau},q^{-\sigma}|q^2)h_m(x;q^{-\tau},q^{-\sigma}|q^2)e^{2i(m-n)\theta}\,dm(x;q^{1-\sigma+\tau},-q^{1+\sigma+\tau},0,0|q^2)$$

using the notation for the normalised orthogonality measure for Askey-Wilson polynomials, cf. (3.3.3).

6. Prove that $h(p(\rho_{\tau,\sigma}))=$
$\frac{1-q^2}{1+q^{2\tau}}\int_{\mathbb{R}}p(x)P_{q^2}(x,x;q^{1+\sigma-\tau},-q^{1-\sigma-\tau}|q^2)\,dm(x;q^{1+\sigma-\tau},-q^{1-\sigma-\tau},0,0|q^2)+$
$\frac{1-q^2}{1+q^{-2\tau}}\int_{\mathbb{R}}p(x)P_{q^2}(x,x;q^{1-\sigma+\tau},-q^{1+\sigma+\tau}|q^2)\,dm(x;q^{1-\sigma+\tau},-q^{1+\sigma+\tau},0,0|q^2)$,
where

$$P_t(x,y;a,b|q)=\sum_{k=0}^\infty t^k\frac{s_k(x;a,b|q)s_k(y;a,b|q)}{(q,ab;q)_k}$$

is the Poisson kernel for the Al-Salam–Chihara polynomials.

7. Using the standard notation for very-well poised ${}_8\varphi_7$-series, cf. [27, Ch. 2],

$${}_8W_7(a;b,c,d,e,f;q,z)={}_8\varphi_7\left(\begin{matrix}a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ f\\ \sqrt{a},-\sqrt{a},qa/b,qa/c,qa/d,qa/e,qa/f\end{matrix};q,z\right),$$

the Poisson kernel for the Al-Salam and Chihara polynomials has been evaluated by Askey, Rahman and Suslov, see [7, (14.8)],

$$P_t(\cos\theta,\cos\psi;a,b|q)=\frac{(ate^{i\theta},ate^{-i\theta},bte^{i\psi},bte^{-i\psi},t;q)_\infty}{(te^{i\theta+i\psi},te^{i\theta-i\psi},te^{i\psi-i\theta},te^{-i\psi-i\theta},abt;q)_\infty}$$
$$\times\,{}_8W_7(\frac{abt}{q};t,be^{i\theta},be^{-i\theta},ae^{i\psi},ae^{-i\psi};q,t).$$

Use Bailey's formula, cf. [27, (2.11.7)],

$$\frac{1}{(b/a;q)_\infty}\,{}_8W_7(a;b,c,d,e,f;q,q)+\frac{(aq,c,d,e;q)_\infty}{(a/b;q)_\infty(aq/c,aq/d,aq/e,aq/f;q)_\infty}$$
$$\times\frac{(f,bq/c,bq/d,bq/e,bq/f;q)_\infty}{(bc/a,bd/a,be/a,bf/a,b^2q/a;q)_\infty}\,{}_8W_7(\frac{b^2}{a};b,\frac{bc}{a},\frac{bd}{a},\frac{be}{a},\frac{bf}{a};q,q)$$
$$=\frac{(aq,aq/(cd),aq/(ce),aq/(cf),aq/(de),aq/(df),aq/(ef);q)_\infty}{(aq/c,aq/d,aq/e,aq/f,bc/a,bd/a,be/a,bf/a;q)_\infty}$$

to finish the proof of Theorem 4.2.4. The absolutely continuous part of the measure has to be treated differently from possible discrete mass points.

5. Askey-Wilson polynomials and generalised matrix elements

We have now developed all the necessary ingredients for the interpretation of Askey-Wilson polynomials on the quantum $SU(2)$ group. In this section we first show that orthogonal polynomials are of importance in describing generalised matrix elements. Next these polynomials are explicitly calculated in terms of Askey-Wilson polynomials.

§5.1. Generalised matrix elements and orthogonal polynomials. First we establish a relation between generalised matrix elements and orthogonal polynomials.

Theorem 5.1.1. *For fixed $i,j\in\frac{1}{2}\mathbb{Z}_+$ such that $i-j\in\mathbb{Z}$, there exists a system of orthogonal polynomials $(p_k)_{k\in\mathbb{Z}_+}$ of degree k such that for $l\geq m=\max(|i|,|j|)$, $l-m\in\mathbb{Z}_+$,*

$$b^l_{i,j}(\tau,\sigma)=b^m_{i,j}(\tau,\sigma)p_{l-m}(\rho_{\tau,\sigma}).$$

Proof. We first prove that an expression of this form exists. Consider for any polynomial s_{l-m} of degree $l-m$ the expression $b^m_{i,j}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma})$. If we decompose this product with respect to the linear basis the decomposition of $A_q(SU(2))$ of Theorem 2.4.1 we get

$$b^m_{i,j}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma})=\sum_{k=|2m-l|}^{l}b^k,\qquad b^k\in\operatorname{span}(t^k_{nm}),$$

where the upper and lower bound follow from Lemma 2.4.2. The mappings $X.$ and $.X$ preserve $\operatorname{span}(t^k_{nm})$. Proposition 4.1.4(i) shows that the left hand side satisfies (4.1.7) with $\lambda=\lambda_j(\sigma)$ and $\mu=\lambda_i(\tau)$. Consequently, each b^k has to satisfy (4.1.7) with $\lambda=\lambda_j(\sigma)$ and $\mu=\lambda_i(\tau)$. Proposition 4.1.4(ii) implies that $b^k = 0$ for $k < m$ and $b^k = c_k b^k_{i,j}(\tau,\sigma)$ for $k \geq m$ and some constants c_k. Hence,

$$b^m_{i,j}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=m}^{l} c^k b^k_{i,j}(\tau,\sigma).$$

Since both sides contain the same degree of freedom, the existence of such polynomials follows once we know that the mapping $s_{l-m} \mapsto b^m_{i,j}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma})$ is injective. This can be seen by applying the one-dimensional $*$-representation π_θ of $A_q(SU(2))$, cf. Theorem 2.6.3, and use of the explicit expression of $b^m_{i,j}(\tau,\sigma)$, cf. (4.1.6). Then use $\pi_\theta(\rho_{\tau,\sigma}) = \cos 2\theta$ and $\pi_\theta(t^l_{nm}) = \delta_{nm}e^{-2in\theta}$, which follows by observing that $\pi_\theta(\xi) = \langle A^{2i\theta/\ln q}, \xi\rangle$ holds (formally) for $\xi \in A_q(SU(2))$.

For $l, k \geq m$, $l-m, k-m \in \mathbb{Z}_+$ we have $b^l_{i,j}(\tau,\sigma) \in \mathrm{span}(t^l_{nm})$, $b^k_{i,j}(\tau,\sigma) \in \mathrm{span}(t^k_{nm})$, so that the Schur orthogonality relations for the Haar functional h, cf. Theorem 2.6.2, imply

$$h\Big(\big(b^l_{i,j}(\tau,\sigma)\big)^* b^k_{i,j}(\tau,\sigma)\Big) = \delta_{k,l}h_l, \qquad h_l > 0. \tag{5.1.1}$$

Now $\big(b^m_{i,j}(\tau,\sigma)\big)^* b^m_{i,j}(\tau,\sigma) = w_m(\rho_{\tau,\sigma})$ for some polynomial w_m of degree $2m$, by Proposition 4.1.4(i) and 4.1.7, and the Clebsch-Gordan series of Lemma 2.4.2. Hence,

$$h\big(\,\bar{p}_{l-m}(\rho_{\tau,\sigma})p_{k-m}(\rho_{\tau,\sigma})w_m(\rho_{\tau,\sigma})\big) = \delta_{l,k}h_l, \qquad h_l > 0,$$

since we have already established the existence of such polynomials and since $\rho^*_{\tau,\sigma} = \rho_{\tau,\sigma}$. Consequently, by Theorem 4.2.4 the polynomials p_{l-m}, $l-m \in \mathbb{Z}_+$, form a system of orthogonal polynomials with respect to the measure $w_m(x)\,dm(x;a,b,c,d \mid q^2)$ on $\mathbb{R}$, where $dm(x;a,b,c,d \mid q^2)$ is the measure described in Theorem 4.2.4. ☑

§5.2. Generalised matrix elements and Askey-Wilson polynomials. Combining Theorem 5.1.1, and in particular the description of the orthogonality measure, with Theorem 4.2.4 shows that it suffices to calculate

$$w_m(\cos\theta) = \big|\pi_{\theta/2}\big(b^m_{i,j}(\tau,\sigma)\big)\big|^2, \qquad m = \max(|i|,|j|),$$

to find the orthogonality measure for the orthogonal polynomials from Theorem 5.1.1. The first step is the following proposition, where we use the one-dimensional representations τ_λ defined in Remark 2.6.4.

Proposition 5.2.1. *For $i,j \in \frac{1}{2}\mathbb{Z}_+$, $i-j \in \mathbb{Z}$ and $m = \max(|i|,|j|)$ we have*
(i) *In case $m = i$ or $-i \leq j \leq i$,*

$$\tau_\lambda\big(b^i_{i,j}(\tau,\sigma)\big) = C^{i,j}(\sigma)C^{i,i}(\tau)q^{-i}\lambda^{-2i}(\lambda^2 q^{1+\tau-\sigma};q^2)_{i-j}(-\lambda^2 q^{1+\tau+\sigma};q^2)_{i+j}.$$

(ii) *In case* $m=j$ *or* $-j\le i\le j$,

$$\tau_\lambda\big(b^j_{i,j}(\tau,\sigma)\big)= C^{j,i}(\tau)C^{j,j}(\sigma)q^{-j}\lambda^{-2j}(\lambda^2q^{1+\sigma-\tau};q^2)_{j-i}(-\lambda^2q^{1+\tau+\sigma};q^2)_{i+j}.$$

(iii) *In case* $m=-i$ *or* $i\le j\le -i$,

$$\tau_\lambda\big(b^{-i}_{i,j}(\tau,\sigma)\big)= C^{-i,-j}(-\sigma)C^{-i,-i}(-\tau)q^i\lambda^{2i}(\lambda^2q^{1-\tau+\sigma};q^2)_{j-i}(-\lambda^2q^{1-\tau-\sigma};q^2)_{-i-j}.$$

(iv) *In case* $m=-j$ *or* $j\le i\le -j$,

$$\tau_\lambda\big(b^{-j}_{i,j}(\tau,\sigma)\big)= C^{-j,-i}(-\tau)C^{-j,-j}(-\sigma)q^j\lambda^{2j}(\lambda^2q^{1-\sigma+\tau};q^2)_{i-j}(-\lambda^2q^{1-\tau-\sigma};q^2)_{-i-j}.$$

Proof. First we observe that the function $\tau_\lambda\big(b^l_{i,j}(\tau,\sigma)\big)$ of λ satisfies the symmetry relations

$$\begin{aligned}\tau_\lambda\big(b^l_{i,j}(\tau,\sigma)\big)&=\overline{\tau_{\bar\lambda}\big(b^l_{j,i}(\sigma,\tau)\big)}\\ &=\tau_\lambda\big(b^l_{-j,-i}(-\sigma,-\tau)\big)\\ &=\overline{\tau_{\bar\lambda}\big(b^l_{-i,-j}(-\tau,-\sigma)\big)}.\end{aligned}\tag{5.2.1}$$

This follows from $\tau_\lambda(t^l_{nm})=\delta_{nm}\lambda^{-2n}$ and $v^{l,j}_n(\sigma)=\overline{v^{l,-j}_n(-\sigma)}$, which in turn follows from $C^{l,j}(\sigma)=C^{l,-j}(-\sigma)$ and Exercise 3.8.

So it suffices to prove the first statement. Use Proposition 4.1.5 to find

$$b^l_{l,m}(\tau,\sigma)=C^{l,m}(\sigma)C^{l,l}(\tau)q^{\sigma(m-l)}q^{\frac12(l-m)(l-m-1)} \times\prod_{k=0}^{l+m-1}\delta_{\tau+2l-1-k,\sigma+2m-1-k}\prod_{j=0}^{l-m-1}\gamma_{\tau+l-m-1-j,\sigma-l+m+1+j}.$$

Apply τ_λ to find

$$\begin{aligned}\tau_\lambda\big(b^l_{l,m}(\tau,\sigma)\big)&=C^{l,m}(\sigma)C^{l,l}(\tau)q^{\sigma(m-l)}q^{\frac12(l-m)(l-m-1)}\\ &\times\prod_{k=0}^{l+m-1}(q^{\tau+\sigma+2l+2m-2-2k+1/2}\lambda+q^{-1/2}\lambda^{-1})\\ &\times\prod_{j=0}^{l-m-1}(-q^{\tau+l-m-1-j+1/2}\lambda+q^{\sigma-l+m+1+j-1/2}\lambda^{-1})\\ &=C^{l,m}(\sigma)C^{l,l}(\tau)q^{-l}\lambda^{-2l}(\lambda^2q^{1+\tau-\sigma};q^2)_{l-m}(-\lambda^2q^{1+\tau+\sigma};q^2)_{l+m}\end{aligned}$$

and this proves (i). ☑

Remark 5.2.2. We can lift the symmetry relations of (5.2.1) to the algebra $A_q(SU(2))$, as follows. Let $\Psi: A_q(SU(2)) \to A_q(SU(2))$ be the algebra isomorphism obtained by interchanging β and γ. It follows from Corollary 2.5.3 that this is well-defined. From (2.4.3) we see that Ψ is an anti-coalgebra isomorphism. Hence, Ψ gives a coalgebra-isomorphism and anti-isomorphism of $U_q(\mathfrak{sl}(2,\mathbb{C}))$, which is just interchanging B and C. Using this, Theorem 2.3.6 and Lemma 2.1.2, then implies $\Psi(t^l_{nm}) = t^l_{mn}$. Similarly, let $\Phi: A_q(SU(2)) \to A_q(SU(2))$ be the anti-algebra isomorphism obtained by interchanging α and δ. Again Corollary 2.5.3 implies that it is well-defined. From (2.4.3) we see that Φ is an anti-coalgebra isomorphism. Hence, Φ gives a anti-coalgebra isomorphism and anti-isomorphism of $U_q(\mathfrak{sl}(2,\mathbb{C}))$, which is just interchanging A and D. Using this, Theorem 2.3.6 and Lemma 2.1.2, then implies $\Phi(t^l_{nm}) = t^l_{-m,-n}$. Combining gives $\Phi\circ\Psi(t^l_{nm}) = t^l_{-n,-m}$, where $\Phi\circ\Psi$ is interchanging β and γ and α and δ.

We can now prove the main result of this section in which we relate Askey-Wilson polynomials to generalised matrix elements. The Askey-Wilson polynomials involve four continuous parameters. In the following theorem we establish an interpretation of the Askey-Wilson polynomials with two continuous and two discrete parameters. We rewrite the Askey-Wilson polynomials using the following notation;

$$p_n^{(\alpha,\beta)}(x;s,t|q) = p_n(x;q^{1/2}t/s, q^{1/2+\alpha}s/t, -q^{1/2}/(st), -stq^{1/2+\beta}|q).$$

In this way the Askey-Wilson polynomials are q-analogues of the Jacobi polynomials. The case $s=t=1$ goes under the name of continuous q-Jacobi polynomials. Note also that the Al-Salam and Chihara polynomials with the parametrisation as in Exercise 4.5 can be considered as the corresponding Hermite polynomial of $p_n^{(\alpha,\beta)}(x;s,t|q)$, i.e. they can be obtained by letting $\alpha=\beta\to\infty$.

Theorem 5.2.3. *For $i,j\in\frac{1}{2}\mathbb{Z}_+$, $i-j\in\mathbb{Z}$ and $l-m\in\mathbb{Z}_+$, $m=\max(|i|,|j|)$ we have*

(i) *In case $m=i$ or $-i\le j\le i$:*

$$b^l_{i,j}(\tau,\sigma) = d^l_{i,j}(\tau,\sigma)b^i_{i,j}(\tau,\sigma)p_{l-i}^{(i-j,i+j)}(\rho_{\tau,\sigma};q^\tau,q^\sigma\mid q^2).$$

(ii) *In case $m=j$ or $-j\le i\le j$:*

$$b^l_{i,j}(\tau,\sigma) = d^l_{j,i}(\sigma,\tau)b^j_{i,j}(\tau,\sigma)p_{l-j}^{(j-i,i+j)}(\rho_{\tau,\sigma};q^\sigma,q^\tau\mid q^2).$$

(iii) *In case $m=-i$ or $i\le j\le -i$:*

$$b^l_{i,j}(\tau,\sigma) = d^l_{-i,-j}(-\tau,-\sigma)b^{-i}_{i,j}(\tau,\sigma)p_{l+i}^{(j-i,-i-j)}(\rho_{\tau,\sigma};q^{-\tau},q^{-\sigma}\mid q^2).$$

(iv) *In case* $m = -j$ *or* $j \le i \le -j$:

$$b^l_{i,j}(\tau,\sigma) = d^l_{-j,-i}(-\sigma,-\tau) b^{-j}_{i,j}(\tau,\sigma) p^{(i-j,-i-j)}_{l+j}(\rho_{\tau,\sigma}; q^{-\sigma}, q^{-\tau} \mid q^2).$$

Here the constant is given by

$$d^l_{i,j}(\tau,\sigma) = \frac{C^{l,j}(\sigma) C^{l,i}(\tau)}{C^{i,j}(\sigma) C^{i,i}(\tau)} \frac{q^{i-l}}{(q^{4l}; q^{-2})_{l-i}}.$$

Proof. The explicit form for the orthogonality measure given in the proof of Theorem 5.1.1, the symmetry relations of (5.2.1) and $\pi_{\theta/2}(\rho_{\tau,\sigma}) = \cos\theta$ being independent of σ, τ, show that it suffices to prove the first statement.

From the explicit form of the Askey-Wilson weight measure, cf. Proposition 3.3.4, we immediately get

$$(az, a/z; q)_r \, dm(x; a, b, c, d|q) = \frac{(ab, ac, ad; q)_r}{(abcd; q)_r} dm(aq^r, b, c, d|q)$$

for $r \in \mathbb{Z}_+$, $x = (z + z^{-1})/2$. A double application shows that for $r, s \in \mathbb{Z}_+$, $x = (z + z^{-1})/2$,

$$(az, a/z; q)_r (dz, d/z; q)_s \, dm(x; a, b, c, d|q) = \\ (ab, ac; q)_r (bd, cd; q)_s \frac{(ad; q)_{r+s}}{(abcd; q)_{r+s}} dm(aq^r, b, c, dq^s|q).$$

Hence, Proposition 5.2.1(i) and Theorem 4.2.4 imply that the looked-for polynomials are multiples of the Askey-Wilson polynomials

$$p_{l-i}(\rho_{\tau,\sigma}; -q^{\sigma+\tau+1+2i+2j}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{\tau-\sigma+1+2i-2j} | q^2),$$

which we rewrite in the shorthand notation.

It remains to calculate the constant. We apply the one-dimensional $*$-representation $\pi_{\theta/2}$ to both sides of (i), and next we compare the coefficient of $e^{-il\theta}$ on both sides. The coefficient of $e^{-il\theta}$ on the left hand side is

$$v^{l,i}_l(\sigma) \overline{v^{l,j}_l(\tau)} q^{-l} = C^{l,j}(\sigma) C^{l,i}(\tau) q^{-l}.$$

The coefficient of $e^{-i(l-i)\theta}$ of p_{l-i} is $(q^{2l+2i+2}; q^2)_{l-i} = (q^{4l}; q^{-2})_{l-i}$, so that the coefficient of $e^{-il\theta}$ on the right hand side equals $C^{i,j}(\sigma) C^{i,i}(\tau) q^{-i} (q^{4l}; q^{-2})_{l-i}$, from which we obtain the value for $d^l_{i,j}(\tau,\sigma)$. ☑

Since the generalised matrix elements $a^l_{ij}(\tau,\sigma)$, see Lemma 4.1.2, are obtained from $b^l_{ij}(\tau,\sigma)$ by applying the simple algebra isomorphism $D.$, we also have explicit expressions for the generalised matrix elements in terms of Askey-Wilson polynomials.

§5.3. Limit cases. Theorem 5.2.3 remains valid for the limiting cases $\sigma \to \infty$ or $\tau \to \infty$ or even $\sigma = \tau \to \infty$, cf. Remark 4.1.9. If we let either σ or τ tend to infinity, we see a similar expression with the Askey-Wilson polynomials replaced by the big q-Jacobi polynomials, which are defined by

$$P_n^{(\alpha,\beta)}(x;c,d;q) = {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, xq^{\alpha+1}/c \\ q^{\alpha+1}, -q^{\alpha+1}d/c \end{matrix}; q, q\right). \tag{5.3.1}$$

This is due to the limit transition

$$\begin{aligned}\lim_{a\to 0}(a\sqrt{cq/d})^n(-q/a^2;q)_n\, p_n^{(\alpha,\beta)}\Big(\frac{xq^{1/2}}{2a\sqrt{cd}}; a, \sqrt{\frac{c}{d}}\Big|q\Big)& \\ = (q^{\alpha+1}, -q^{\beta+1}c/d;q)_n\, P_n^{(\alpha,\beta)}(x;c,d;q),&\end{aligned}$$

which follows by taking term-wise limits in the ${}_4\varphi_3$-series expression for the Askey-Wilson polynomials. This limit transition is motivated from the definition of $\rho_{\tau,\infty}$ and $\rho_{\infty,\sigma}$ in Remark 4.1.9.

For $\sigma = \tau \to \infty$ we need the limit transition of the Askey-Wilson polynomials to little q-Jacobi polynomials defined by

$$\begin{aligned} p_n^{(\alpha,\beta)}(x;q) &= {}_2\varphi_1(q^{-n}, q^{n+\alpha+\beta+1}; q^{\alpha+1}; q, qx) \\ &= \frac{(q^{-n-\beta};q)_n}{(q^{\alpha+1};q)_n}\, {}_3\varphi_2\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, xq^{\beta+1} \\ q^{\beta+1}, 0 \end{matrix}; q, q\right). \end{aligned} \tag{5.3.2}$$

Using the ${}_3\varphi_2$-series representation for the little q-Jacobi polynomials we can prove the limit transition

$$\begin{aligned}\lim_{a\to 0}(-q^{1/2+\beta}a^2)^n\, p_n^{(\alpha,\beta)}\Big(\frac{xq^{1/2}}{2a^2}; a, a\Big|q\Big)& \\ = (-1)^n q^{n\beta} q^{n(n-1)/2}(q^{\alpha+1};q)_n\, p_n^{(\alpha,\beta)}(x;q).&\end{aligned}$$

This limit transition is motivated from the limit transition of $\rho_{\tau,\sigma}$ to $\rho_{\infty,\infty}$ in Remark 4.1.9. This limit case is stated separately. Note that it suffices to do the calculations in the first case and then use the symmetry relations of Remark 5.2.2 and the commutation relations in $A_q(SU(2))$.

Corollary 5.3.1.

$$\begin{aligned}
t^l_{n,m} &= c^l_{n,m}\delta^{n+m}\gamma^{n-m}\, p^{(n-m,n+m)}_{l-n}(-q^{-1}\beta\gamma;q^2), && (n \ge m \ge -n),\\
t^l_{n,m} &= c^l_{m,n}\delta^{n+m}\beta^{m-n}\, p^{(m-n,m+n)}_{l-m}(-q^{-1}\beta\gamma;q^2), && (m \ge n \ge -m),\\
t^l_{n,m} &= c^l_{-n,-m}\beta^{m-n}\alpha^{-m-n}\, p^{(m-n,-n-m)}_{l+n}(-q^{2m+2n-1}\beta\gamma;q^2), && (-n \ge m \ge n),\\
t^l_{n,m} &= c^l_{-m,-n}\gamma^{n-m}\alpha^{-m-n}\, p^{(n-m,-n-m)}_{l+m}(-q^{2m+2n-1}\beta\gamma;q^2), && (-m \ge n \ge m),
\end{aligned}$$

with $p_k^{(\alpha,\beta)}(x;q^2)$ a little q-Jacobi polynomial and

$$c^l_{n,m} = \begin{bmatrix} l-m \\ n-m \end{bmatrix}_{q^2}^{1/2} \begin{bmatrix} l+n \\ n-m \end{bmatrix}_{q^2}^{1/2} q^{-(n-m)(l-n)}$$

Notes and references. The main Theorem 5.2.3 is due to Koornwinder [60] and was already announced in 1990, see [57], in the case $l \in \mathbb{Z}_+$ and $i = j = 0$, i.e. in the case of (τ, σ)-spherical elements. This has led to Koelink [45] in which the case $j = 0$ is calculated in order to be able to give a quantum group theoretic proof of the addition formula for continuous q-Legendre polynomials. Noumi and Mimachi [75], [76] then announced Theorem 5.2.3 in general. There are some other ways of proving Theorem 5.2.3, notably by using the Casimir operator leading to the q-difference equation for Askey-Wilson polynomials [60] or by the Clebsch-Gordan decomposition leading to the three-term recurrence relation for the Askey-Wilson polynomials. This proof uses the techniques of [45], see also [50]. The symmetry relations in Remark 5.2.2 can already be found in [56].

The limit transitions of the Askey-Wilson polynomials to big and little q-Jacobi polynomials is taken from Koornwinder [60]. See [27], [42], [61] for more information and further references on the big and little q-Jacobi polynomials, which were originally introduced by Andrews and Askey. Corollary 5.3.1 is one of the first known interactions between q-special functions and quantum groups due to Koornwinder [56], Masuda et al. [70] and Vaksman and Soibelman [96]. A number of other special cases of Theorem 5.2.3 in the case $j = 0$ have been considered before, see Noumi and Mimachi [77], [78], [80].

Exercises.

1. Prove that h_l of (5.1.1) can be written as
$$\frac{(1-q^2)q^{2l}}{1-q^{4l+2}} \tau_{\sqrt{q}}\big(b^l_{j,j}(\sigma,\sigma)\big)\tau_{\sqrt{q}}\big(b^l_{i,i}(\tau,\tau)\big).$$
2. Use Proposition 5.2.1 to derive the following generating function for the dual q-Krawtchouk polynomials of Exercise 3.3;
$$\sum_{n=0}^{N} t^n q^{n(N+\sigma)/2} \frac{(q^{-N};q)_n}{(q;q)_n} R_n(q^{-x} - q^{x-N-\sigma}; q^\sigma, N; q)$$
$$= (-tq^{-(N+\sigma)/2};q)_x (tq^{(\sigma-N)/2};q)_{N-x}$$
for $N \in \mathbb{N}$, $x \in \{0, \ldots, N\}$.
3. Prove the limit transitions of the Askey-Wilson polynomials to the big and little q-Jacobi polynomials.
4. Prove the second equality in (5.3.2).
5. Prove the orthogonality relations for the little q-Jacobi polynomials;
$$\int_0^1 p_n^{(\alpha,\beta)}(x;q) p_m^{(\alpha,\beta)}(x;q)\, t^\alpha \frac{(qt;q)_\infty}{(q^{\beta+1};q)_\infty}\, d_q t = \delta_{nm} h_n$$
and calculate the squared norm h_n.

6. Addition formulas for Askey-Wilson polynomials

As an application of the interpretation of Askey-Wilson polynomials on the quantum $SU(2)$ group established in the previous section we derive two addition formulas for the q-Legendre polynomial $p_n^{(0,0)}(\cdot; q^\tau, q^\sigma | q^2)$ in this section.

§6.1. Abstract addition formulas. Since t^l defines a unitary representation of the $*$-algebra $U_q(\mathfrak{su}(2))$, we obtain the properties

$$\Delta(t^l_{nm}) = \sum_{k=-l}^{l} t^l_{nk} \otimes t^l_{km}, \quad \varepsilon(t^l_{nm}) = \delta_{nm}, \quad S(t^l_{nm}) = (t^l_{mn})^*,$$

$$\sum_{p=-l}^{l} t^l_{ip}(t^l_{jp})^* = \delta_{ij} = \sum_{p=-l}^{l} (t^l_{pi})^* t^l_{pj}$$

which are easily verified by testing against appropriate elements of $U_q(\mathfrak{su}(2))$. The analogous statements for the generalised matrix elements is the following.

Proposition 6.1.1. *The elements* $a^l_{i,j}(\tau,\sigma)$, $\sigma,\tau \in \mathbb{R}\cup\{\infty\}$, $i,j = -l, -l+1, \ldots, l$, $l \in \frac{1}{2}\mathbb{Z}_+$, *defined in Lemma 4.1.2, satisfy*

$$\Delta(a^l_{i,j}(\tau,\sigma)) = \sum_{p=-l}^{l} a^l_{i,p}(\tau,\mu) \otimes a^l_{p,j}(\mu,\sigma), \quad \forall\, \mu \in \mathbb{R}\cup\{\infty\},$$

$$\left(a^l_{i,j}(\tau,\sigma)\right)^* = S\left(a^l_{j,i}(\sigma,\tau)\right), \qquad \varepsilon\left(a^l_{i,j}(\tau,\sigma)\right) = \langle v^{l,j}(\sigma), v^{l,i}(\tau)\rangle,$$

$$\sum_{p=-l}^{l} a^l_{i,p}(\tau,\mu)\left(a^l_{j,p}(\sigma,\mu)\right)^* = \langle v^{l,j}(\sigma), v^{l,i}(\tau)\rangle = \sum_{p=-l}^{l} \left(a^l_{p,i}(\mu,\tau)\right)^* a^l_{p,j}(\mu,\sigma).$$

Proof. These statements are proved by testing against appropriate elements. Firstly,

$$\begin{aligned}
\langle X\otimes Y, \Delta(a^l_{i,j}(\tau,\sigma))\rangle &= \langle XY, a^l_{i,j}(\tau,\sigma)\rangle = \langle t^l(X)t^l(Y)v^{l,j}(\sigma), v^{l,i}(\tau)\rangle \\
&= \sum_{p=-l}^{l} \langle t^l(X)v^{l,p}(\mu), v^{l,i}(\tau)\rangle\langle t^l(Y)v^{l,j}(\sigma), v^{l,p}(\mu)\rangle \\
&= \sum_{p=-l}^{l} \langle X, a^l_{i,p}(\tau,\mu)\rangle\langle Y, a^l_{p,j}(\mu,\sigma)\rangle, \qquad \forall\, X,Y \in U_q(\mathfrak{su}(2)),
\end{aligned}$$

by developing $t^l(Y)v^{l,j}(\sigma)$ in the orthonormal basis $\{v^{l,p}(\mu)\}_{p=-l,\ldots,l}$. The next statement follows from

$$\begin{aligned}
\langle X, (a^l_{i,j}(\tau,\sigma))^*\rangle &= \overline{\langle S(X)^*, (a^l_{i,j}(\tau,\sigma))\rangle} = \overline{\langle t^l(S(X)^*)v^{l,j}(\sigma), v^{l,i}(\tau)\rangle} \\
&= \langle t^l(S(X))v^{l,i}(\tau), v^{l,j}(\sigma)\rangle = \langle S(X), a^l_{j,i}(\sigma,\tau)\rangle = \langle X, S(a^l_{j,i}(\sigma,\tau))\rangle
\end{aligned}$$

for arbitrary $X \in U_q(\mathfrak{su}(2))$ and

$$\varepsilon\big(a^l_{i,j}(\tau,\sigma)\big) = \langle 1, a^l_{i,j}(\tau,\sigma)\rangle = \langle v^{l,j}(\sigma), v^{l,i}(\tau)\rangle.$$

To prove the last statement we apply $(id \otimes S)$ to the first statement and we use the Hopf algebra axiom $m \circ (id \otimes S) = \eta \circ \varepsilon$, cf. Definition 1.1.4. The first equality then follows from the second property. The second equality is proved similarly using the map $m \circ (S \otimes id)$. ☑

Corollary 6.1.2. *For $l \in \mathbb{Z}_+$, $\sigma, \tau \in \mathbb{R}\cup\{\infty\}$ the action of Δ on (τ,σ)-spherical elements is given by*

$$\Delta\big(b^l_{00}(\tau,\sigma)\big) = \sum_{n=-l}^{l} \big(D.b^l_{0n}(\tau,\mu)\big) \otimes b^l_{n0}(\mu,\sigma), \qquad \forall\, \mu \in \mathbb{R}\cup\{\infty\}.$$

Proof. Use Proposition 6.1.1, $a^l_{ij}(\tau,\sigma) = D.b^l_{ij}(\tau,\sigma)$ and $\Delta \circ Z. = (id \otimes Z.) \circ \Delta$. This follows from

$$\langle \Delta(Z.\xi), X \otimes Y\rangle = \langle \xi, XYZ\rangle = \langle \Delta X \otimes YZ\rangle = \langle (id \otimes Z.)\Delta(\xi), X \otimes Y\rangle,$$

where we use $\langle Z.\xi, X\rangle = \langle \xi, XZ\rangle$. ☑

Corollary 6.1.2 is the starting point for the derivation of an addition formula for a two-parameter family of Askey-Wilson polynomials, and in this sense we may call it an abstract addition formula. Most of the ingredients of the proof have already been established. But let us first note that applying the one-dimensional representation $\tau_{\sqrt{\lambda}} \otimes \tau_{\sqrt{\nu}}$ to the identity in $A_q(SU(2)) \otimes A_q(SU(2))$ of Corollary 6.1.2 leads to the addition formula

(6.1.1)

$$\begin{aligned}
(q^2;q^2)_l q^{-l} p_l^{(0,0)}(\xi(\lambda\nu); q^\tau, q^\sigma \,|\, q^2) &= \frac{p_l^{(0,0)}(\xi(\lambda); q^\mu, q^\tau \,|\, q^2)\, p_l^{(0,0)}(\xi(\nu); q^\mu, q^\sigma \,|\, q^2)}{(-q^{2-2\mu}, -q^{2+2\mu}; q^2)_l} \\
&+ \sum_{p=1}^{l} \frac{(1+q^{4p+2\mu})(q^2;q^2)_{l+p}(\lambda\nu)^{-p}(\lambda q^{\mu-\tau}, -\lambda q^{\tau+\mu}, \nu q^{\mu-\sigma}, -\nu q^{\mu+\sigma}; q^2)_p}{(1+q^{2\mu})(q^2;q^2)_{l-p}(-q^{2-2\mu};q^2)_{l-p}(-q^{2+2\mu};q^2)_{l+p}} \\
&\qquad \times p_{l-p}^{(p,p)}(\xi(\lambda); q^\mu, q^\tau \,|\, q^2)\, p_{l-p}^{(p,p)}(\xi(\nu); q^\mu, q^\sigma \,|\, q^2) \\
&+ \sum_{p=1}^{l} \frac{(1+q^{4p-2\mu})(q^2;q^2)_{l+p}(\lambda\nu)^{-p}(\lambda q^{\tau-\mu}, -\lambda q^{-\tau-\mu}, \nu q^{\sigma-\mu}, -\nu q^{-\mu-\sigma}; q^2)_p}{(1+q^{-2\mu})(q^2;q^2)_{l-p}(-q^{2+2\mu};q^2)_{l-p}(-q^{2-2\mu};q^2)_{l+p}} \\
&\qquad \times p_{l-p}^{(p,p)}(\xi(\lambda); q^{-\mu}, q^{-\tau} \,|\, q^2)\, p_{l-p}^{(p,p)}(\xi(\nu); q^{-\mu}, q^{-\sigma} \,|\, q^2),
\end{aligned}$$

with $\xi(\lambda) = \frac{1}{2}(q^{-1}\lambda + q\lambda^{-1})$. The one-dimensional representation $\tau_{\sqrt{\lambda}} \otimes \tau_{\sqrt{\nu}}$ has a very large kernel, so we may expect that a more general addition formula than (6.1.1) holds. This is indeed the case, and we proceed to derive this for the special case $\mu \to \infty$.

§6.2. Suitable basis for the right hand side. Our plan is to derive an addition formula from Corollary 6.1.2 for the case $\mu \to \infty$. We first determine the explicit form of the terms in the right hand side of Corollary 6.1.2.

Lemma 6.2.1. *For $l \in \mathbb{Z}_+$ fixed and $n \in \{0, \dots, l\}$ we have*

$$\begin{aligned}
b^l_{n,0}(\infty,\sigma) &= C_n(\sigma)\big(\delta_{\infty,\sigma-1}\gamma_{\infty,\sigma}\big)^n P^{(n,n)}_{l-n}(\rho_{\infty,\sigma}; q^{2\sigma}, 1; q^2),\\
b^l_{0,n}(\tau,\infty) &= C_n(\tau)\big(\delta_{\tau-1,\infty}\beta_{\tau,\infty}\big)^n P^{(n,n)}_{l-n}(\rho_{\tau,\infty}; q^{2\tau}, 1; q^2),\\
b^l_{-n,0}(\infty,\sigma) &= C_n(\sigma)\big(\beta_{\infty,\sigma-1}\alpha_{\infty,\sigma}\big)^n P^{(n,n)}_{l-n}(\rho_{\infty,\sigma}; q^{2\sigma+2n}, q^{2n}; q^2),\\
b^l_{0,-n}(\tau,\infty) &= C_n(\tau)\big(\gamma_{\tau-1,\infty}\alpha_{\tau,\infty}\big)^n P^{(n,n)}_{l-n}(\rho_{\tau,\infty}; q^{2\tau+2n}, q^{2n}; q^2)
\end{aligned}$$

with the constant given by

$$C_n(\sigma) = q^{(l-n)(l-n-1)/2} q^{-\sigma l} \frac{(q^{2n+2}, -q^{2\sigma-2l}; q^2)_{l-n}}{\sqrt{(q^2, q^{2l+2n+2}; q^2)_{l-n}}}.$$

Proof. Use Theorem 5.2.3 and the limit transition of the Askey-Wilson polynomials to big q-Jacobi polynomials in §5.3 to obtain this form. The form of the factors in front follows from Proposition 4.1.5. ☑

Theorem 5.2.3 and Lemma 6.2.1 give explicit expressions in term of orthogonal polynomials for all the elements in the identity in the non-commutative algebra $A_q(SU(2)) \otimes A_q(SU(2))$ of Corollary 6.1.2 for $\mu \to \infty$. Although identities in non-commuting variables have their charm and are useful, we want to obtain an identity in commuting variables. In order to do so we use the $*$-representations of $A_q(SU(2))$ described in Theorem 2.6.3. Now any one-dimensional $*$-representation leads to a trivial identity, so we have to consider infinite dimensional $*$-representations. Since the result is independent of the θ of π^∞_θ, we may restrict ourselves to $\theta = 0$, and we denote $\pi = \pi^\infty_0$. The idea is to let the right hand side act on suitable eigenvectors of the self-adjoint operator $\pi(\rho_{\tau,\infty}) \otimes \pi(\rho_{\infty,\sigma})$, and to determine the (generalised) eigenvectors of the self-adjoint operator $\pi \otimes \pi(\Delta(\rho_{\tau,\sigma}))$ in terms of the eigenvectors of $\pi(\rho_{\tau,\infty}) \otimes \pi(\rho_{\infty,\sigma})$.

Proposition 6.2.2. *$\ell^2(\mathbb{Z}_+)$ has an orthogonal basis of eigenvectors $v_\lambda(q^\tau)$, where $\lambda = -q^{2n}$, $n \in \mathbb{Z}_+$, $\lambda = q^{2\tau+2n}$, $n \in \mathbb{Z}_+$, for the eigenvalue λ of the self-adjoint operator $\pi(\rho_{\tau,\infty})$. For $\lambda = -q^{2n}$, $\lambda = q^{2\tau+2n}$, $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
\pi(\alpha_{\tau,\infty})v_\lambda(q^\tau) &= iq^{1/2-\tau}(1+\lambda)v_{\lambda/q^2}(q^{\tau-1}),\\
\pi(\beta_{\tau,\infty})v_\lambda(q^\tau) &= iq^{1/2}v_\lambda(q^{\tau-1}),\\
\pi(\gamma_{\tau,\infty})v_\lambda(q^\tau) &= iq^{1/2}(q^{2\tau}-\lambda)v_\lambda(q^{\tau+1}),\\
\pi(\delta_{\tau,\infty})v_\lambda(q^\tau) &= -iq^{1/2+\tau}v_{\lambda q^2}(q^{\tau+1}).
\end{aligned}$$

Proof. The first statement is Proposition 4.2.6 for $\theta = 0$, and the second statement is Exercise 4.4 for $\theta = 0$. This can be proved as follows. First observe that the result for $\pi(\beta_{\tau,\infty})$ and $\pi(\delta_{\tau,\infty})$ imply the result for $\pi(\gamma_{\tau,\infty})$ and $\pi(\alpha_{\tau,\infty})$ by Proposition 4.2.6 and the case $\sigma \to \infty$ of Proposition 4.1.7. Let us prove the statement for $\pi(\delta_{\tau,\infty})$. Take $\sigma \to \infty$ in Corollary 4.1.8 using Remark 4.1.9, to see $\delta_{\tau,\infty}\rho_{\tau,\infty} = q^{-2}\rho_{\tau+1,\infty}\delta_{\tau,\infty}$. By the first part of the proposition we must have $\pi(\delta_{\tau,\infty})v_\lambda(q^\tau) = Cv_{\lambda q^2}(q^{\tau+1})$ for some constant C. Using the normalisation $\langle v_\lambda(q^\tau), e_0\rangle = 1$ we get

$$C = \langle \pi(\delta_{\tau,\infty})v_\lambda(q^\tau), e_0\rangle = \langle v_\lambda(q^\tau), \pi(\delta_{\tau,\infty^*})e_0\rangle = \langle v_\lambda(q^\tau), \pi(iq^{\tau+1/2}\gamma + q^{-1/2}\alpha)e_0\rangle = -iq^{\tau+1/2}.$$

The statement for $\pi(\beta_{\tau,\infty})$ is proved analogously. ☑

Proposition 6.2.3. *$\ell^2(\mathbb{Z}_+)$ has an orthogonal basis of eigenvectors $v_\lambda(q^\sigma)$, where $\lambda = -q^{2n}$, $n \in \mathbb{Z}_+$, and $\lambda = q^{2\sigma+2n}$, $n \in \mathbb{Z}_+$, for the eigenvalue λ of the self-adjoint operator $\pi(\rho_{\infty,\sigma})$. Moreover,*

$$\langle v_\lambda(q^\sigma), v_\lambda(q^\sigma)\rangle = q^{-2n}(q^2;q^2)_n(-q^{2-2\sigma};q^2)_n(-q^{2\sigma};q^2)_\infty, \qquad \lambda = -q^{2n},$$
$$\langle v_\lambda(q^\sigma), v_\lambda(q^\sigma)\rangle = q^{-2n}(q^2;q^2)_n(-q^{2+2\sigma};q^2)_n(-q^{-2\sigma};q^2)_\infty, \qquad \lambda = q^{2\sigma+2n}.$$

For $\lambda = -q^{2n}$, $\lambda = q^{2\sigma+2n}$, $n \in \mathbb{Z}_+$, we have

$$\pi(\alpha_{\infty,\sigma})v_\lambda(q^\sigma) = iq^{1/2-\sigma}(1+\lambda)v_{\lambda/q^2}(q^{\sigma-1}),$$
$$\pi(\beta_{\infty,\sigma})v_\lambda(q^\sigma) = iq^{1/2}(q^{2\sigma}-\lambda)v_\lambda(q^{\sigma+1}),$$
$$\pi(\gamma_{\infty,\sigma})v_\lambda(q^\sigma) = iq^{1/2}v_\lambda(q^{\sigma-1}),$$
$$\pi(\delta_{\infty,\sigma})v_\lambda(q^\sigma) = -iq^{1/2+\sigma}v_{\lambda q^2}(q^{\sigma+1}).$$

Proof. Use $-q\pi(\gamma) = \pi(\beta)$ to find $\pi(\rho_{\tau,\sigma}) = \pi(\rho_{\sigma,\tau})$, $\pi(\alpha_{\tau,\sigma}) = \pi(\alpha_{\sigma,\tau})$, $\pi(\beta_{\tau,\sigma}) = \pi(\gamma_{\sigma,\tau})$, $\pi(\gamma_{\tau,\sigma}) = \pi(\beta_{\sigma,\tau})$ and $\pi(\delta_{\tau,\sigma}) = \pi(\delta_{\sigma,\tau})$, to reduce to Propositions 4.2.6 and 6.2.2. ☑

Corollary 6.2.4. *For $n \in \{0,\dots,l\}$ we have*

$$\pi(b^l_{n,0}(\infty,\sigma))v_\lambda(q^\sigma) = C_n(\sigma)q^{n\sigma}P^{(n,n)}_{l-n}(\lambda;q^{2\sigma},1;q^2)\,v_{\lambda q^{2n}}(q^\sigma),$$
$$\pi(D.b^l_{0,n}(\tau,\infty))v_\mu(q^\tau) = C_n(\tau)q^{n(\tau+1)}P^{(n,n)}_{l-n}(\mu;q^{2\tau},1;q^2)\,v_{\mu q^{2n}}(q^\tau)$$

and

$$\pi(b^l_{-n,0}(\infty,\sigma))v_\lambda(q^\sigma) = C_n(\sigma)(-1)^n q^{n(\sigma-1)}(-\lambda,\lambda q^{-2\sigma};q^{-2})_n \times P^{(n,n)}_{l-n}(lq^{-2n};q^{2\sigma},1;q^2)\,v_{\lambda q^{-2n}}(q^\sigma),$$
$$\pi(D.b^l_{0,-n}(\tau,\infty))v_\mu(q^\tau) = C_n(\tau)(-1)^n q^{n(\tau-2)}(-\mu,\mu q^{-2\tau};q^{-2})_n \times P^{(n,n)}_{l-n}(\mu q^{-2n};q^{2\tau},1;q^2)\,v_{\mu q^{-2n}}(q^\tau).$$

Proof. Use $D.b^{1/2}_{ij}(\tau,\infty)=q^j b^{1/2}_{ij}(\tau,\infty)$, so $D.\rho_{\tau,\infty}=\rho_{\tau,\infty}$, and

$$P_n^{(\alpha,\beta)}(Ax;Ac,Ad;q)=P_n^{(\alpha,\beta)}(x;c,d;q)$$

for $A>0$ and Propositions 6.2.2 and 6.2.3 and Lemma 6.2.1. ☑

§6.3. Suitable basis for the left hand side. The basis $v_\mu(q^\tau)\otimes v_\lambda(q^\sigma)$ of $\ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)$ is very well suited for the action of the right hand side under $\pi\otimes\pi$, cf. Corollary 6.2.4. Next we study $\pi\otimes\pi\big(\Delta(\rho_{\tau,\sigma})\big)$ with respect to this basis.

Lemma 6.3.1.

$$\begin{aligned}2(\pi\otimes\pi)\Delta(\rho_{\tau,\sigma})v_\mu(q^\tau)\otimes v_\lambda(q^\sigma)&=q^2 v_{\mu q^2}(q^\tau)\otimes v_{\lambda q^2}(q^\sigma)+\\ &q^{-2}(1+\lambda)(1+\mu)(1-\lambda q^{-2\sigma})(1-\mu q^{-2\tau})v_{\mu q^{-2}}(q^\tau)\otimes v_{\lambda q^{-2}}(q^\sigma)\\ +\Big(\lambda q^{1-\sigma}(q^{-\tau}-q^\tau)+\mu q^{1-\tau}&(q^{-\sigma}-q^\sigma)+\lambda\mu q^{1-\tau-\sigma}(1+q^2)\Big)v_\mu(q^\tau)\otimes v_\lambda(q^\sigma).\end{aligned}$$

Proof. Use

$$\Delta\big(b^{1/2}_{ij}(\tau,\sigma)\big)=\sum_{n=-1/2}^{1/2}q^n b^{1/2}_{in}(\tau,\infty)\otimes b^{1/2}_{nj}(\infty,\sigma),$$

cf. Corollary 6.1.2, to get

$$\begin{aligned}\Delta(\beta_{\tau+1,\sigma-1}\gamma_{\tau,\sigma})&=q^{-1}\alpha_{\tau+1,\infty}\gamma_{\tau,\infty}\otimes\beta_{\infty,\sigma-1}\alpha_{\infty,\sigma}+\alpha_{\tau+1,\infty}\delta_{\tau,\infty}\otimes\beta_{\infty,\sigma-1}\gamma_{\infty,\sigma}\\ &+\beta_{\tau+1,\infty}\gamma_{\tau,\infty}\otimes\delta_{\infty,\sigma-1}\alpha_{\infty,\sigma}+q\beta_{\tau+1,\infty}\delta_{\tau,\infty}\otimes\delta_{\infty,\sigma-1}\gamma_{\infty,\sigma}.\end{aligned}$$

Now the lemma follows from Propositions 4.1.7, 6.2.2 and 6.2.3. ☑

Lemma 6.3.1 implies that each subspace of the form

$$\operatorname{span}\{v_{\mu q^{2m}}(q^\tau)\otimes v_{\lambda q^{2m}}(q^\sigma)\mid m\in\mathbb{Z}_+\}$$

with either $\mu\in\{-1,q^{2\tau}\}$ or $\lambda\in\{-1,q^{2\sigma}\}$ is invariant under $(\pi\otimes\pi)\Delta(\rho_{\tau,\sigma})$. We pick one, say $\mu=-1$, $\lambda=-q^{2p}$ for $p\in\mathbb{Z}_+$ fixed, and we call this subspace $W\cong\ell^2(\mathbb{Z}_+)$. Using $\mu=-1$, $\lambda=-q^{2p}$ and in Lemma 6.3.1 we obtain a three-term recurrence relation, which can be solved using a sub-class of the Askey-Wilson polynomials. If viewed as q-Jacobi polynomials as in §5.2, the polynomials we need are the Laguerre case of the q-Jacobi polynomials, i.e. we let $\beta\to\infty$. So we define

$$l_n^{(\alpha)}(x;s,t|q)=p_n(x;q^{1/2+\alpha}s/t,q^{1/2}t/s,-q^{1/2}/(st),0|q).$$

From the three-term recurrence relation for the Askey-Wilson polynomials we get

$$2xl_n(x) = l_{n+1}(x) + (1-q^n)(1-q^{\alpha+n})(1+q^n s^{-2})(1+q^{n+\alpha}t^{-2})l_{n-1}(x)$$
$$+q^n\Big((t-t^{-1})q^{1/2}s^{-1} + (s-s^{-1})q^{1/2+\alpha}t^{-1} + (1+q)q^{1/2+n+\alpha}s^{-1}t^{-1}\Big)l_n(x),$$

where $l_n(x) = l_n^{(\alpha)}(x;s,t|q)$. The orthogonality measure for the q-Laguerre polynomials follows from (3.3.3) and we denote the normalised orthogonality measure by

$$dm^{(\alpha)}(\cdot;s,t|q) = dm(\cdot;q^{1/2+\alpha}s/t, q^{1/2}t/s, -q^{1/2}/(st), 0|q).$$

Proposition 6.3.2. *Define* $\Lambda\colon W \to L^2(dm^{(p)}(\cdot;q^\tau,q^\sigma|q^2))$ *by*

$$v_{-q^{2m}}(q^\tau)\otimes v_{-q^{2m+2p}}(q^\sigma) \mapsto$$
$$q^{-2m-p}\sqrt{(q^2,-q^{2-2\sigma};q^2)_p(-q^{2\tau},-q^{2\sigma};q^2)_\infty}\; l_m^{(p)}(\cdot;q^\tau,q^\sigma|q^2),$$

then Λ is a unitary operator intertwining $(\pi\otimes\pi)\Delta(\rho_{\tau,\sigma})$ on W with the multiplication operator on $L^2(dm^{(p)}(\cdot;q^\tau,q^\sigma|q^2))$.

Proof. $(\pi\otimes\pi)\Delta(\rho_{\tau,\sigma})$ is a Jacobi matrix on W and the recurrence relation can be matched with the three-term recurrence relation for the q-Laguerre polynomials $l_m^{(p)}(\cdot;q^\tau,q^\sigma|q^2)$, so we can use the technique of the proof of Theorem 3.1.2. The constants involved follow from Propositions 3.3.4, 4.2.6 and 6.2.3. ☑

§6.4. Addition formula for Askey-Wilson polynomials. We now have sufficient information on the action of both sides of Corollary 6.1.2 for $\mu\to\infty$.

Theorem 6.4.1. *The following addition formula holds for $l,m,p\in\mathbb{Z}_+$, $\sigma,\tau\in\mathbb{R}$, $x\in\mathbb{C}$;*

$$p_l^{(0,0)}(x;q^\tau,q^\sigma|q^2)l_m^{(p)}(x;q^\tau,q^\sigma|q^2) =$$
$$\sum_{n=0}^{l} D^{n,l}(\tau,\sigma)P_{l-n}^{(n,n)}(-q^{2m};q^{2\tau},1;q^2)P_{l-n}^{(n,n)}(-q^{2m+2p};q^{2\sigma},1;q^2)l_{m+n}^{(p)}(x;q^\tau,q^\sigma|q^2)$$
$$+\sum_{n=1}^{l} D^{n,l}(\tau,\sigma)(q^{2m},q^{2m+2p},-q^{2m+2p-2\sigma},-q^{2m-2\tau};q^{-2})_n\times$$
$$P_{l-n}^{(n,n)}(-q^{2m-2n};q^{2\tau},1;q^2)P_{l-n}^{(n,n)}(-q^{2m+2p-2n};q^{2\sigma},1;q^2)l_{m-n}^{(p)}(x;q^\tau,q^\sigma|q^2),$$

with the constant given by

$$D^{n,l}(\tau,\sigma) =$$
$$(-q^{2\sigma-2l}, -q^{2\tau-2l}; q^2)_{l-n}\frac{(q^{2(l-n+1)}; q^2)_n}{(q^2;q^2)_n}(q^{2(n+1)}; q^2)_l q^{(l-n)(l-n-2\sigma-2\tau)}.$$

Proof. Let $\pi\otimes\pi$ act on Corollary 6.1.2, $\mu\to\infty$, and restrict the action to W, which is also invariant under the action of the right hand side by Corollary 6.2.4, as it should be. Let the resulting operator identity act on $v_{-q^{2m}}(q^\tau)\otimes v_{-q^{2m+2p}}(q^\sigma)$ and apply Λ of Proposition 6.3.2 to get as an identity in $L^2(dm^{(p)}(\cdot;q^\tau,q^\sigma|q^2))$;

$$q^{-2m-p}\sqrt{(q^2,-q^{2-2\sigma};q^2)_p(-q^{2\tau},-q^{2\sigma};q^2)_\infty}\, p_l^{(0,0)}(x;q^\tau,q^\sigma|q^2) l_m^{(p)}(x;q^\tau,q^\sigma|q^2)$$
$$= \Lambda\Big(p_l^{(0,0)}(\pi\otimes\pi(\Delta(\rho_{\tau,\sigma}));q^\tau,q^\sigma|q^2)v_{-q^{2m}}(q^\tau)\otimes v_{-q^{2m+2p}}(q^\sigma)\Big)$$
$$= \sum_{n=0}^{l} C_n(\sigma)C_n(\tau)q^{n(\tau+\sigma+1)}P_{l-n}^{(n,n)}(-q^{2m};q^{2\tau},1;q^2)P_{l-n}^{(n,n)}(-q^{2m+2p};q^{2\sigma},1;q^2)$$
$$\times\Lambda\big(v_{-q^{2m+2n}}(q^\tau)\otimes v_{-q^{2m+2p+2n}}(q^\sigma)\big)+$$
$$\sum_{n=1}^{l} C_n(\sigma)C_n(\tau)q^{n(\tau+\sigma-3)}(q^{2m},q^{2m+2p},-q^{2m+2p-2\sigma},-q^{2m-2\tau};q^{-2})_n$$
$$\times P_{l-n}^{(n,n)}(-q^{2m-2n};q^{2\tau},1;q^2)P_{l-n}^{(n,n)}(-q^{2m+2p-2n};q^{2\sigma},1;q^2)$$
$$\times\Lambda\big(v_{-q^{2m-2n}}(q^\tau)\otimes v_{-q^{2m+2p-2n}}(q^\sigma)\big).$$

Use Proposition 6.3.2 and simplify to get the results, which holds for all $x\in\mathbb{C}$ by continuity. ☑

Remark 6.4.2. **It seems that in deriving Theorem 6.4.1 there is some arbitrariness in the choice of the subspace W of $\ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)$, to which we have restricted our attention. Apart from the choice of $p\in\mathbb{Z}_+$ there are 8 of such possible choices for W, which lead to 8 of such addition formulas. These are obviously four by four equivalent by interchanging τ and σ and noting that the q-Legendre polynomial involved is also invariant under such a change. The remaining four types of addition formulas differ, since the big q-ultraspherical polynomials are evaluated at other points of the spectrum. However, from**

(6.4.1)
$$P_n^{(\alpha,\beta)}(-x;c,d;q) = (-q^{\alpha-\beta}d/c)^n\frac{(q^{\beta+1},-q^{\beta+1}c/d;q)_n}{(q^{\alpha+1},-q^{\alpha+1}d/c;q)_n}P_n^{(\beta,\alpha)}(x;d,c;q),$$

(which is a direct consequence of the orthogonality relations) and the trivial relation $P_n^{(\alpha,\beta)}(Ax;Ac,Ad;q) = P_n^{(\alpha,\beta)}(x;c,d;q)$, for $A>0$, we see that

(6.4.2)
$$P_{l-n}^{(n,n)}(-q^m;q^\tau,1;q) = (-q^{-\tau})^{l-n}\frac{(-q^{n+1+\tau};q)_{l-n}}{(-q^{n+1-\tau};q)_{l-n}}P_{l-n}^{(n,n)}(q^{m-\tau};q^{-\tau},1;q).$$

If we use this transformation for the big q-Jacobi polynomials in Theorem 6.4.1 and we next change x into $-x$ using $p_n^{(\alpha,\beta)}(-x;s,t|q)=(-1)^n p_n^{(\alpha,\beta)}(x;-s,t|q)$ for the q-Legendre polynomial, $\alpha=\beta=0$, and for the q-Laguerre polynomial, $\beta\to\infty$, and we change q^τ to $-q^{-\tau}$, then we obtain the same addition formula as if we had started off with the space W spanned by the vectors type $v_{q^{2\tau+2m}}(q^\tau)\otimes v_{-q^{2m+2p}}(q^\sigma)$, $m\in\mathbb{Z}_+$. A similar approach can be used for the other big q-ultraspherical polynomial, and a combination of both shows that Theorem 6.4.1 contains all the other possibilities by symmetry considerations.

Using the orthogonality relations for the q-Laguerre polynomials, cf. Proposition 3.3.4, we can pick out the term for $n=0$ from the right hand side of Theorem 6.4.1 and we obtain the following product formula for big q-Legendre polynomials.

Corollary 6.4.3. *For $l,n\in\mathbb{Z}_+$, $0\le n\le l$ we have the product formula*

$$P_l^{(0,0)}(-q^{2m};q^{2\tau},1;q^2)P_l^{(0,0)}(-q^{2(m+p)};q^{2\sigma},1;q)=\\ \frac{1}{C}\int_{\mathbb{R}} p_l^{(0,0)}(x;q^\tau,q^\sigma|q^2)\Big(l_m^{(p)}(x;q^\tau,q^\sigma|q^2)\Big)^2\,dm^{(p)}(x;q^\tau,q^\sigma|q^2),$$

with

$$C=D^{0,l}(\tau,\sigma)(q^2,q^{2+2p},-q^{2-2\tau},-q^{2+2p-2\sigma};q)_m.$$

Product formulae for big q-Legendre polynomials at other points of the spectrum are obtained from (6.4.2).

Notes and references. There are much more applications of the interpretation of Askey-Wilson polynomials on the quantum $SU(2)$ group than just addition formulas, see e.g. [50]. The addition formula for the Askey-Wilson polynomials in (6.1.1) is due to Noumi and Mimachi [75], see also [45], [50]. The addition formula derived in this section is taken from [47], see also [51]. The motivation for choosing $\mu\to\infty$ is that we still can obtain a polynomial identity. In [51] a very general identity, i.e. coming from Corollary 6.1.2 for general σ, τ and μ, is derived, but for this a non-symmetric Poisson kernel for Al-Salam and Chihara polynomials is needed. The result from [51] contains Theorem 6.4.1 as a limit case. Some applications of identities for q-special functions in non-commuting variables, of which Lemma 2.3.2 is a simple example, as well as further references can be found in Koornwinder [62].

The case $\sigma=\tau=\mu=0$ of Corollary 6.1.2 has been used in [45] to derive the addition formula for the continuous q-Legendre polynomial, which is a special case of the addition formula for the continuous q-ultraspherical polynomials analytically proved by Rahman and Verma [84]. From the case $\sigma=\tau=\mu\to\infty$ of Corollary 6.1.2 Koornwinder [58] has proved an addition formula for the little q-Legendre polynomial, which can be obtained from Theorem 6.4.1 as the limit case $\sigma=\tau\to\infty$. Also the addition formula for big q-Legendre polynomials of [48] is a special case of Theorem 6.4.1. The limit case $q\uparrow 1$ of Theorem 6.4.1 to the addition formula for Legendre polynomials, cf. Exerxise 6.1, is highly non-trivial. It uses the asymptotic behaviour of a class of orthogonal polynomials including the q-Laguerre polynomials, see Van Assche and Koornwinder [97] for the general theorem and [47] for the explicit application to this case. In [97] it is also shown how the weak convergence of orthogonality measures can be used to show that the product formulas as the one in Corollary 6.4.3 tend to the product formula for the Legendre polynomials, cf. [47] for the details for this case.

Exercises.

1. Let $R^{(\alpha,\beta)}$ be the Jacobi polynomial normalised by $R^{(\alpha,\beta)}(1) = 1$, i.e. $R^{(\alpha,\beta)}(x) =$

$$\sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k}{k!\,(\alpha+1)_k}\left(\frac{1-x}{2}\right)^k = {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right).$$

The Legendre polynomial $R_n^{(0,0)}$ satisfies the addition formula

$$R_l^{(0,0)}(xy+t\sqrt{(1-x^2)(1-y^2)}) = R_l^{(0,0)}(x)R_l^{(0,0)}(y) +$$
$$2\sum_{m=1}^{l} \frac{(l+m)!}{(l-m)!(m!)^2} 2^{-2m}\left(\sqrt{(1-x^2)(1-y^2)}\right)^m R_{l-m}^{(m,m)}(x)R_{l-m}^{(m,m)}(y)T_m(t),$$

where $T_m(\cos\theta) = \cos m\theta$ is the Chebyshev polynomial of the first kind. Prove that for $q \uparrow 1$ in (6.1.1) leads to the addition formula for Legendre polynomials.

2. Derive an addition formula for the little q-Legendre polynomial. Do this by redoing the proof of Theorem 6.4.1 using Corollary 5.3.1, or by taking suitable limits in Theorem 6.4.1.
3. Given the q-integral

$$\int_{-d}^{c} \frac{(qx/c, -qx/d; q)_\infty}{(q^{\alpha+1}x/c, -q^{\beta+1}x/d; q)_\infty}\, d_qx =$$
$$(1-q)c\frac{(q, -d/c, -qc/d, q^{\alpha+\beta+2}; q)_\infty}{(q^{\alpha+1}, q^{\beta+1}, -q^{\beta+1}c/d, -q^{\alpha+1}d/c; q)_\infty},$$

prove the orthogonality relations for the big q-Jacobi polynomials

$$\int_{-d}^{c} P_n^{(\alpha,\beta)}(x;c,d;q)P_m^{(\alpha,\beta)}(x;c,d;q)\frac{(qx/c, -qx/d; q)_\infty}{(q^{\alpha+1}x/c, -q^{\beta+1}x/d; q)_\infty}\, d_qx = \delta_{nm}h_n$$

Calculate h_n and prove (6.4.1).

7. Convolution theorem for Al-Salam and Chihara polynomials

In this section we consider the $*$-operator on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ leading to the real form $U_q(\mathfrak{su}(1,1))$, see Theorem 2.3.4. Then we show how we can interpret Al-Salam and Chihara polynomials as overlap coefficients similarly as q-Krawtchouk polynomials for the real form $U_q(\mathfrak{su}(2))$. This interpretation can be used to find a very general convolution theorem for the Al-Salam and Chihara polynomials.

§7.1. Positive discrete series representations of $U_q(\mathfrak{su}(1,1))$. In Exercise 2.6 the positive discrete series representations had to be calculated. We recall the result. For every $k>0$ the representation π_k of $U_q(\mathfrak{su}(1,1))$ acting in $\ell^2(\mathbb{Z}_+)$ is given by

$$\pi_k(A)\,e_n = q^{k+n}\,e_n, \qquad \pi_k(C)\,e_n = q^{1/2-k-n}\,\frac{\sqrt{(1-q^{2n})(1-q^{4k+2n-2})}}{q-q^{-1}}\,e_{n-1},$$

where $\{e_n\}_{n=0}^{\infty}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z}_+)$. The action of B follows from $B=-C^*$,

$$\pi_k(B)\,e_n = q^{-1/2-k-n}\,\frac{\sqrt{(1-q^{2n+2})(1-q^{4k+2n})}}{q^{-1}-q}\,e_{n+1}.$$

Note that $\pi_k(D)$ is an unbounded operator, but that $\pi_k(A),\pi_k(B),\pi_k(C)\in\mathcal{B}(\ell^2(\mathbb{Z}_+))$. The operators that we consider will be bounded. Obviously, π_k is an irreducible unitary (i.e. $*$-)representation of $U_q(\mathfrak{su}(1,1))$.

Lemma 7.1.1.

$$\pi_{k_1}\otimes\pi_{k_2}\cong\bigoplus_{j=0}^{\infty}\pi_{k_1+k_2+j}$$

Sketch of Proof. The left hand side is a unitary representation of $U_q(\mathfrak{su}(1,1))$, hence completely reducible. The decomposition follows by counting lowest weight vectors, i.e. considering the kernel of C. See Exercise 7.1-2 for more details. ☑

Although the basis of the representation space $\ell^2(\mathbb{Z}_+)$ of π_k is independent of k we use the notation e_n^k to stress the k-dependence. Then Lemma 7.1.1 implies that there exists a unitary matrix mapping the orthogonal basis $e_{n_1}^{k_1}\otimes e_{n_2}^{k_2}$ onto $e_n^{k_1+k_2+j}$ intertwining the action of $U_q(\mathfrak{su}(1,1))$ on both sides. The matrix elements of this unitary mapping are the Clebsch-Gordan coefficients.

Lemma 7.1.2. *The Clebsch-Gordan coefficients are defined by*

$$e_n^k=\sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2,n}^{k_1,k_2,k}\,e_{n_1}^{k_1}\otimes e_{n_2}^{k_2},$$

where $k = k_1 + k_2 + j$ for $j \in \mathbb{Z}_+$. The sum is finite; $n_1 + n_2 = n + j$. The Clebsch-Gordan coefficients are normalised by $\langle e_0^k, e_0^{k_1} \otimes e_j^{k_2}\rangle > 0$.

Proof. Let A act on both sides to get

$$\sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2,n}^{k_1,k_2,k}\, q^{n+k}\, e_{n_1}^{k_1} \otimes e_{n_2}^{k_2} = q^{n+k}\, e_n^k$$

$$= \sum_{n_1,n_2=0}^{\infty} C_{n_1,n_2,n}^{k_1,k_2,k}\, q^{k_1+n_1+k+2+n_2}\, e_{n_1}^{k_1} \otimes e_{n_2}^{k_2},$$

so $C_{n_1,n_2,n}^{k_1,k_2,k}$ is zero unless $n + k = k_1 + n_1 + k_2 + n_2$ or $n_1 + n_2 = n + j$.

Using the action of B and C we can derive recurrence relations for the Clebsch-Gordan coefficients. Using $\Delta(C) = A \otimes C + C \otimes D$ we get

$$(7.1.1)\quad q^{1/2-k-n}\sqrt{(1-q^{2n})(1-q^{4k+2n-2})}\, C_{n_1,n_2,n-1}^{k_1,k_2,k} =$$
$$q^{k_1+n_1-1/2-k_2-n_2}\sqrt{(1-q^{2n_2+2})(1-q^{4k_2+2n_2})}\, C_{n_1,n_2+1,n}^{k_1,k_2,k} =$$
$$q^{-1/2-k_1-n_1-k_2-n_2}\sqrt{(1-q^{2n_1+2})(1-q^{4k_1+2n_1})}\, C_{n_1+1,n_2,n}^{k_1,k_2,k}.$$

Using the action of B we get a similar recursion for $C_{n_1,n_2,n+1}^{k_1,k_2,k}$ showing that the Clebsch-Gordan coefficients are completely determined by $C_{n_1,n_2,0}^{k_1,k_2,k}$. Take $n = 0$ in (7.1.1) to find a two-term recurrence which can be solved by iteration;

$$C_{n_1,n_2,0}^{k_1,k_2,k} = (-1)^{n_1} q^{2n_1k_1} q^{n_1(n_1-1)} \sqrt{\frac{(q^{2j}, q^{4k_2+2j-2}; q^{-2})_{n_1}}{(q^2, q^{4k_1}; q^2)_{n_1}}}\, C_{0,j,0}^{k_1,k_2,k}$$

where $n_1 + n_2 = j$, $k = k_1 + k_2 + j$. Since the transition matrix is unitary we have

$$1 = \sum_{n_1+n_2=j} |C_{n_1,n_2,0}^{k_1,k_2,k}|^2 = |C_{0,j,0}^{k_1,k_2,k}|^2\, {}_2\varphi_1\left(\begin{matrix} q^{-2j}, q^{2-2j-4k_2} \\ q^{4k_1}\end{matrix}; q^2, q^{4k_1+4k_2+4j-2}\right)$$
$$= |C_{0,j,0}^{k_1,k_2,k}|^2 \frac{(q^{4k_1+4k_2+2j-2}; q^2)_j}{(q^{4k_1}; q^2)_j}$$

by the Gauß summation formula of Exercise 3.6. This determines $C_{0,j,0}^{k_1,k_2,k}$ up to a phase factor, and given the normalisation $\langle e_0^k, e_0^{k_1} \otimes e_j^{k_2}\rangle > 0$ we have $C_{0,j,0}^{k_1,k_2,k} > 0$ and its value as well as the value of $C_{n_1,n_2,n}^{k_1,k_2,k}$ are completely determined. ☑

Remark 7.1.3. The value obtained for $C_{0,j,0}^{k_1,k_2,k}$ and hence for $C_{n_1,n_2,0}^{k_1,k_2,k}$, together with the three-term recurrence relation obtained from the action of B completely determine the Clebsch-Gordan coefficients. This can be used to find an explicit expression in terms of q-hypergeometric series. We come back to the explicit expression in §7.3.

§7.2. Twisted primitive elements and eigenvectors. From Proposition 2.3.1(ii) we have a description of the twisted primitive elements in $U_q(\mathfrak{sl}(2,\mathbb{C}))$. We now choose

$$Y_s = q^{1/2}B - q^{-1/2}C + \frac{s^{-1}+s}{q^{-1}-q}(A-D)$$

in the space of twisted primitive elements. Then Y_sA is a self-adjoint element in $U_q(\mathfrak{su}(1,1))$ for $s \in \mathbb{R}\setminus\{0\}$, or $s \in \mathbb{T}$, the unit circle. We study the bounded self-adjoint operator $\pi_k(Y_sA)$. In order to formulate the following result, we recall the Al-Salam and Chihara polynomials defined in Exercise 3.4. By S_n we denote the orthonormal Al-Salam and Chihara polynomials;

$$S_n(x;a,b|q) = \frac{1}{\sqrt{(q,ab;q)_n}}\, s_n(x;a,b|q)$$

and corresponding normalised orthogonality measure

$$dm(x;a,b|q) = dm(x;a,b,0,0|q),$$

cf. Exercise 3.4 and (3.3.3). We also use the notation $\mu(x) = (x+x^{-1})/2 = \mu(x^{-1})$ for $x \neq 0$ in this section.

Proposition 7.2.1. *Define* $\Lambda\colon \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}, dm(\cdot\,; q^{2k}s, q^{2k}/s|q^2))$ *by*

$$\Lambda\colon e_n^k \mapsto S_n(\cdot\,; q^{2k}s, q^{2k}/s|q^2),$$

then Λ *is a unitary mapping intertwining* $\pi_k(Y_sA)$ *with* $2(M-\mu(s))/(q^{-1}-q)$, *where* M *is the multiplication map on* $L^2(\mathbb{R}, dm(\cdot\,; q^{2k}s, q^{2k}/s|q^2))$.

Remark 7.2.2. Proposition 7.2.1 says that formally

$$v^k(x) = \sum_{n=0}^{\infty} S_n(\mu(x); q^{2k}s, q^{2k}/s|q^2)\, e_n^k$$

is an eigenvector of the bounded self-adjoint operator $\pi_k(Y_sA)$ for the eigenvalue

$$\lambda_x = \frac{x+x^{-1}-s-s^{-1}}{q^{-1}-q} = 2\frac{\mu(x)-\mu(s)}{q^{-1}-q}.$$

The spectrum of $\pi_k(Y_sA)$ is $\{\lambda_x \mid \mu(x) \in \operatorname{supp}(dm(\cdot\,; q^{2k}s, q^{2k}/s|q^2))\}$.

Proof. We have

$$\begin{aligned}\big((q^{-1}-q)\pi_k(Y_sA) + s + s^{-1}\big)e_n &= \sqrt{(1-q^{2n+2})(1-q^{4k+2n})}\, e_{n+1} \\ &\quad + q^{2k+2n}(s+s^{-1})\, e_n + \sqrt{(1-q^{2n})(1-q^{4k+2n-2})}\, e_{n-1},\end{aligned}$$

and comparing with the three-term recurrence relation for the orthonormal Al-Salam and Chihara polynomials $S_n(x) = S_n(x; a, b|q)$, cf. Exercise 3.4,

$$\begin{aligned} 2x\, S_n(x) &= a_{n+1}\, S_{n+1}(x) + q^n (a+b)\, S_n(x) + a_n\, S_{n-1}(x), \\ a_n &= \sqrt{(1-abq^{n-1})(1-q^n)}. \end{aligned} \tag{7.2.1}$$

leads to the result, cf. proof of Theorem 3.1.2. ☑

Next we consider the action of $Y_s A$ in the tensor product representation $\pi_{k_1} \otimes \pi_{k_2}$ acting in $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$. This can be done using orthogonal polynomials in two variables.

Proposition 7.2.3. *Define* $\Upsilon \colon \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}^2, dm(x,y))$, *where*

$$dm(x,y) = dm(x; q^{2k_1} w, q^{2k_1}/w|q^2)\, dm(y; q^{2k_2} s, q^{2k_2}/s|q^2), \qquad y = \mu(w),$$

by

$$\Upsilon \colon e^{k_1}_{n_1} \otimes e^{k_2}_{n_2} \mapsto S_{n_1}(x; q^{2k_1} w, q^{2k_1}/w|q^2)\, S_{n_2}(y; q^{2k_2} s, q^{2k_2}/s|q^2)$$

then Υ *is a unitary mapping intertwining* $\pi_{k_1} \otimes \pi_{k_2}(\Delta(Y_s A))$ *with* $2(M_x - \mu(s))/(q^{-1} - q)$, *where* M_x *is multiplication by* x *in* $L^2(dm(x,y))$.

Remark 7.2.4. (i) Put, $y = \mu(w)$,

$$R_{l,m}(x,y) = S_l(x; q^{2k_1} w, q^{2k_1}/w|q^2)\, S_m(y; q^{2k_2} s, q^{2k_2}/s|q^2)$$

then $R_{l,m}$ are orthonormal polynomials in two variables of degree l in x and $l+m$ in y;

$$\iint R_{l,m}(x,y)\, R_{r,s}(x,y)\, dm(x,y) = \delta_{lr}\,\delta_{ms},$$

as a straightforward consequence of the orthogonality relations of the Al-Salam and Chihara polynomials.

(ii) Proposition 7.2.3 states that formally the vector $w(x;y) =$

$$\begin{aligned} &\sum_{n_1,n_2=0}^{\infty} S_{n_1}(\mu(x); q^{2k_1} y, q^{2k_1}/y|q^2)\, S_{n_2}(\mu(y); q^{2k_2} s, q^{2k_2}/s|q^2)\, e^{k_1}_{n_1} \otimes e^{k_2}_{n_2} \\ &\qquad = \sum_{n_1=0}^{\infty} S_{n_1}(\mu(x); q^{2k_1} y, q^{2k_1}/y|q^2)\, e^{k_1}_{n_1} \otimes v^{k_2}(y) \end{aligned}$$

is an eigenvector of $\pi_{k_1} \otimes \pi_{k_2}(\Delta(Y_s A))$ for the eigenvalue λ_x. This last observation is essentially the way to obtain Proposition 7.2.3, since $\Delta(Y_s A) = A^2 \otimes Y_s A + Y_s A \otimes 1$ acts as a three-term recurrence operator in $e^{k_1}_{n_1} \otimes v^{k_2}(y)$.

Proof. We use $\Delta(Y_s A) = A^2 \otimes Y_s A + Y_s A \otimes 1$ and Proposition 7.2.1 to define for fixed y the map $\Lambda \colon \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ by

$$\Lambda \colon e^{k_1}_{n_1} \otimes e^{k_2}_{n_2} \mapsto S_{n_2}(y; q^{2k_2} s, q^{2k_2}/s|q^2)\, e^{k_1}_{n_1}$$

to obtain the recurrence in n_1

$$\Lambda\Big((q^{-1}-q)\big(\pi_{k_1}\otimes\pi_{k_2}(\Delta(Y_sA))+s+s^{-1}\Big)\,e^{k_1}_{n_1}\otimes e^{k_2}_{n_2}=$$
$$S_{n_2}(y;q^{2k_2}s,q^{2k_2}/s|q^2)\Big(q^{2n_1}\big((s+s^{-1})q^{2k_1}+\lambda_y q^{2k_1}(q^{-1}-q)\big)\,e^{k_1}_{n_1}$$
$$+\sqrt{(1-q^{2n_1+2})(1-q^{4k_1+2n})}\,e^{k_1}_{n_1+1}+\sqrt{(1-q^{2n_1})(1-q^{4k_1+2n_1-1})}\,e^{k_1}_{n_1-1}\Big)$$

Use the explicit expression for λ_y as in Remark 7.2.2 and the three-term recurrence relation (7.2.1) to obtain the result. ☑

By Lemma 7.1.2 the representation space $\ell^2(\mathbb{Z}_+)\otimes\ell^2(\mathbb{Z}_+)$ has another orthonormal basis. We now calculate the action of $\Upsilon\, e^k_n$.

Proposition 7.2.5. *Let $k=k_1+k_2+j$ for $j\in\mathbb{Z}_+$, and $x=\mu(z)$ then*

$$(\Upsilon\, e^k_n)(x,y)=S_n(x;q^{2k}s,q^{2k}/s|q^2)(\Upsilon\, e^k_0)(x,y),$$
$$(\Upsilon\, e^k_0)(x,y)=C\,p_j(y;q^{2k_1}z,q^{2k_1}/z,q^{2k_2}s,q^{2k_2}/s|q^2),$$
$$C^{-1}=\sqrt{(q^2,q^{4k_1},q^{4k_2},q^{4k_1+4k_2+2j-2};q^2)_j}\,.$$

Proof. By Proposition 7.2.3 Υ intertwines $\pi_{k_1}\otimes\pi_{k_2}(\Delta(Y_sA))$ with $2(M_x-\mu(s))/(q^{-1}-q)$, and by Lemmas 7.1.1 and 7.1.2 the action of $\pi_{k_1}\otimes\pi_{k_2}(\Delta(Y_sA))$ on e^k_n is $\pi_k(Y_sA)$, so

$$2\frac{x-\mu(s)}{q^{-1}-q}\,\Upsilon e^k_n(x,y)=\Upsilon(\pi_k(Y_sA)\,e^k_n)(x,y)$$

and we obtain the three-term recurrence relation as in Proposition 7.2.1, but with initial conditions $\Upsilon e^k_{-1}=0$ and $\Upsilon e^k_0(x,y)$ some polynomial in two variables x and y. Hence, the first statement follows.

Now $\Upsilon e^k_n(x,y)$ is a polynomial in two variables, and since Υ is unitary we have the orthogonality relations $\delta_{nm}\delta_{kl}=\langle\Upsilon e^k_n,\Upsilon e^l_m\rangle=$

$$\int S_n(x;q^{2k}s,q^{2k}/s|q^2)S_m(x;q^{2l}s,q^{2l}/s|q^2)\int\Upsilon e^k_0(x,y)\Upsilon e^l_0(x,y)\,dm(x,y),$$

by our first observation. Since the orthogonality measure for the Al-Salam and Chihara polynomials is unique (i.e. the corresponding moment problem is determined), we find that for $k=l$ the inner integral must give the orthogonality measure $dm(x;q^{2k}s,q^{2k}/s|q^2)$ as measures on $\mathbb{R}$ with respect to x. Take $k\neq l$, then the inner integral as function of x gives zero when integrated against an arbitrary polynomial. Since the support of the measure is compact, the polynomials are dense in the corresponding L^2-space and the inner integral is zero for $k\neq l$. Since $\Upsilon e^{k_1}_{n_1}\otimes e^{k_2}_{n_2}(x,y)$ is a polynomial of degree n_1+n_2 in y, see

Remark 7.2.4, the Clebsch-Gordan decomposition of Lemma 7.1.2 implies that $\Upsilon e_0^k(x,y)$ is a polynomial of degree j in y, where $k = k_1 + k_2 + j$ for $j \in \mathbb{Z}_+$, say $\Upsilon e_0^k(x,y) = p_j(y)$. Let $l = k_1 + k_2 + i$ for $i \in \mathbb{Z}_+$, then we obtain the orthogonality relations

$$\int_y p_j(y)p_i(y)\,dm(x,y) = \delta_{ij}\,dm(x;q^{2k}s,q^{2k}/s|q^2), \quad \text{for almost all } x.$$

We now assume for ease of presentation that $dm(x,y)$ is absolutely continuous, the general case being proved similarly. This is the case if we take $q^{2k_2} < |s| < q^{-2k_2}$ since $k_1, k_2 > 0$. Put $x = \cos\theta$, $y = \cos\psi$, and use Theorem 3.3.2 and (3.3.3), to find

$$\frac{1}{2\pi}\int_0^\pi p_i(\cos\psi)p_j(\cos\psi)\frac{(e^{\pm 2i\psi},e^{\pm 2i\theta};q^2)_\infty}{(q^{2k_2}se^{\pm i\psi},q^{2k_2}e^{\pm i\psi}/s,q^{2k_1}e^{\pm i\psi\pm i\theta};q^2)_\infty}d\psi =$$
$$\delta_{ij}\frac{(q^{4k_1+4k_2+4j};q^2)_\infty}{(q^2,q^{4k_1},q^{4k_2};q^2)_\infty}\frac{(e^{\pm 2i\theta};q^2)_\infty}{(q^{2k_1+2k_2+2j}se^{\pm i\theta},q^{2k_1+2k_2+2j}e^{\pm i\theta}/s;q^2)_\infty}$$

for almost all θ. The $\pm$-signs mean that we take all possible combinations in the infinite q-shifted factorials. Cancelling the $(e^{\pm 2i\theta};q^2)_\infty$ on both sides and comparing the result with Theorem 3.3.2 we see that p_j is a multiple of $p_j(\cdot;q^{2k_1}e^{i\theta},q^{2k_1}e^{-i\theta},q^{2k_2}s,q^{2k_2}/s|q^2)$. The (real) constant in front follows up to a sign by comparing the squared norms. To determine the sign we recall the normalisation of Lemma 7.1.2,

$$0 < \langle e_0^k, e_0^{k_1}\otimes e_j^{k_2}\rangle = \langle \Upsilon e_0^k, \Upsilon e_0^{k_1}\otimes e_j^{k_2}\rangle =$$
$$C\iint p_j(y;q^{2k_1}z,q^{2k_1}/z,q^{2k_2}s,q^{2k_2}/s|q^2)S_j(y;q^{2k_2}s,q^{2k_2}/s|q^2)\,dm(x,y).$$

The Askey-Wilson polynomials are orthogonal with respect to the integration over y, and the Al-Salam and Chihara polynomial has positive leading coefficient, so the inner integral is positive and the remaining measure over x is positive as well. Hence the double integral is positive and we conclude that $C > 0$. ☑

Remark 7.2.6. (i) The proof shows that the polynomials, $x = \mu(z)$,

$$P_{l,m}(x,y) =$$
$$S_l(x;q^{2k_1+2k_2+2m}s,q^{2k_1+2k_2+2m}/s|q^2)\,p_m(y;q^{2k_1}z,q^{2k_1}/z,q^{2k_2}s,q^{2k_2}/s|q^2),$$

of degree m in y and $l+m$ in x are orthogonal with respect to the measure $dm(x,y)$;

$$\iint P_{l,m}(x,y)P_{r,s}(x,y)\,dm(x,y) = \delta_{lr}\delta_{ms}(q^2,q^{4k_1},q^{4k_2},q^{4k_1+4k_2+2m-2};q^2)_m.$$

(ii) Υe_0^k can also be calculated explicitly using the Clebsch-Gordan coefficients, so, $y=\mu(w)$,

$$\Upsilon e_0^k(x,y)=\sum_{n_1+n_2=j} C^{k_1,k_2,k}_{n_1,n_2,0}\, S_{n_1}(x;q^{2k_1}w,q^{2k_1}/w|q^2)\, S_{n_2}(y;q^{2k_2}s,q^{2k_2}/s|q^2)$$

by Lemma 7.1.2 and Proposition 7.2.3. This can be evaluated directly by using the ${}_3\varphi_2$-series representation for the Al-Salam and Chihara polynomials and the explicit expression for $C^{k_1,k_2,k}_{n_1,n_2,0}$ derived in the proof of Lemma 7.1.2, interchanging summations and using summation formulas of §3. The sum can also be evaluated by viewing it as a convolution, and using suitable generating functions for the Al-Salam and Chihara and Askey-Wilson polynomials.

§7.3. Convolution theorem for Al-Salam and Chihara polynomials. The convolution formula for the Al-Salam and Chihara polynomials is obtained by applying Υ to Lemma 7.1.2 using the results of Propositions 7.2.3 and 7.2.5. The results holds as an identity in a weighted L^2-space, but since it is a polynomial identity it holds for all x, y.

Lemma 7.3.1. *For $x=\mu(z)$, $y=\mu(w)$, and $k=k_1+k_2+j$ we have*

$$\sum_{n_1+n_2=n+j} C^{k_1,k_2,k}_{n_1,n_2,n}\, S_{n_1}(x;q^{2k_1}w,q^{2k_1}/w|q^2)\, S_{n_2}(y;q^{2k_2}s,q^{2k_2}/s|q^2) = \frac{S_n(x;q^{2k}s,q^{2k}/s|q^2)p_j(y;q^{2k_1}z,q^{2k_1}/z,q^{2k_2}s,q^{2k_2}/s|q^2)}{\sqrt{(q^2,q^{4k_1},q^{4k_2},q^{4k_1+4k_2+2j-2};q^2)_j}}.$$

We have not yet calculated the Clebsch-Gordan coefficients explicitly, but we can now use Lemma 7.3.1 to determine $C^{k_1,k_2,k}_{n_1,n_2,n}$ by specialising to a generating function for the Clebsch-Gordan coefficients. The result is phrased in terms of q-Hahn polynomials, which are defined as follows:

$$Q_n(q^{-x},a,b,N;q)={}_3\varphi_2\left(\begin{matrix}q^{-n},q^{-x},abq^{n+1}\\ aq,\ q^{-N}\end{matrix};q,q\right).$$

Lemma 7.3.2. *With $n_1+n_2=n+j$ we get*

$$C^{k_1,k_2,k_1+k_2+j}_{n_1,n_2,n}=C\,Q_j(q^{-2n_1};q^{4k_1-2},q^{4k_2-2},n+j;q^2),$$

with the constant C given by

$$\frac{q^{2n_1k_1-2n(k+j)}(q^2;q^2)_{n+j}\sqrt{(q^{4k_1};q^2)_{n_1}(q^{4k_2};q^2)_{n_2}(q^{4k_1};q^2)_j}}{\sqrt{(q^2;q^2)_n(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_j(q^{4k_1+4k_2+4j};q^2)_n(q^{4k_2};q^2)_j(q^{4k_1+4k_2+2j-2};q^2)_j}}.$$

Proof. Observe that $C^{k_1,k_2,k}_{n_1,n_2,n}$ is independent of s, $x=\mu(z)$ and $y=\mu(w)$. Specialise $w=q^{2k_2}s$ and $z=q^{2k_1}/w=q^{2k_1-2k_2}/s$, then the Al-Salam and

Chihara polynomials in the summand on the left hand side of Lemma 7.3.1 can be evaluated explicitly, since the ${}_3\varphi_2$-series reduces to 1. For this choice the Askey-Wilson polynomial on the right hand side can also be evaluated explicitly, and we obtain the generating function for the Clebsch-Gordan coefficients

$$\sum_{n_1+n_2=n+j} C^{k_1,k_2,k}_{n_1,n_2,n}\, q^{2n_1(k_2-k_1)-2n_2k_2} s^{n_1-n_2} \frac{\sqrt{(q^{4k_1};q^2)_{n_1}(q^{4k_2};q^2)_{n_2}}}{\sqrt{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}}} =$$

$$\frac{q^{-2jk_2-2n(k_1+k_2+j)} s^{n-j} (q^{4k_1}, q^{4k_2}, q^{4k_2}s^2; q^2)_j}{\sqrt{(q^2, q^{4k_1}, q^{4k_2}, q^{4k_1+4k_2+2j-2}; q^2)_j}} \sqrt{\frac{(q^{4k_1+4k_2+4j};q^2)_n}{(q^2;q^2)_n}}$$
$$\times\ {}_3\varphi_2\left(\begin{matrix} q^{-2n}, q^{4k_2+2j}, q^{4k_1+2j}/s^2 \\ q^{4k_1+4k_2+4j},\ 0 \end{matrix}; q^2, q^2\right).$$

This determines $C^{k_1,k_2,k}_{n_1,n_2,n}$, but it takes some work to find the expression in terms of q-Hahn polynomials. First, take n_1 as the summation parameter in the sum and multiply both sides by s^{n+j} to find that both sides are polynomials of degree $n+j$ in s^2. Apply

$${}_2\varphi_1(q^{-n}, b; c; q, z) = \frac{(c/b;q)_n}{(c;q)_n}(bz/q)^n\ {}_3\varphi_2\left(\begin{matrix} q^{-n}, c^{-1}q^{1-n}, q/z \\ bq^{1-n}/c,\ 0 \end{matrix}; q, q\right),$$

which is just Exercise 3.6 with $a = q^{-n}$ and the series in the terminating ${}_2\varphi_2$-series inverted, and the q-binomial theorem to $(q^{4k_2}s^2;q^2)_j$ to find

$$s^{n+j}(q^{4k_2}s^2;q^2)_j\ {}_3\varphi_2\left(\begin{matrix} q^{-2n}, q^{4k_2+2j}, q^{4k_1+2j}/s^2 \\ q^{4k_1+4k_2+4j},\ 0 \end{matrix}; q^2, q^2\right) =$$

$$\frac{(q^{2-2n-4k_2-2j};q^2)_n}{(q^{2-2n-4k_1-4k_2-4j};q^2)_n}\, {}_1\varphi_0\left(\begin{matrix} q^{-2j} \\ - \end{matrix}; q^2, s^2q^{2j+4k_2}\right) {}_2\varphi_1\left(\begin{matrix} q^{-2n}, q^{4k_1+2j} \\ q^{2-2n-4k_2-2j} \end{matrix}; q^2, s^2q^{2-4k_1-2j}\right).$$

The product of the q-hypergeometric series is written as a polynomial in s^2 by

$$\sum_{n_1=0}^{n+j} s^{2n_1} \sum_r \frac{(q^{-2j};q^2)_{n_1-r}(q^{-2n}, q^{4k_1+2j};q^2)_r}{(q^2;q^2)_{n_1-r}(q^2, q^{2-2n-4k_2-2j};q^2)_r} q^{2(n_1-r)(j+2k_2)+2r(1-2k_1-j)} =$$

$$\sum_{n_1=0}^{n+j} s^{2n_1} \frac{(q^{-2j};q^2)_{n_1}}{(q^2;q^2)_{n_1}} q^{n_1(2j+4k_2)}\, {}_3\varphi_2\left(\begin{matrix} q^{-2n}, q^{4k_1+2j}, q^{-2n_1} \\ q^{2-2n-4k_2-2j}, q^{2+2j-2n_1} \end{matrix}; q^2, q^{2-4k_1-4k_2-2j}\right).$$

This gives an explicit expression for the Clebsch-Gordan coefficients in terms of a terminating ${}_3\varphi_2$-series. To put it into the required form in terms of q-Hahn polynomials, we need to apply some transformations for ${}_3\varphi_2$-series, namely [27, (III.13), (III.11)]. Then the ${}_3\varphi_2$-series can be rewritten as

$$\frac{(q^{2-2n_1-4k_1};q^2)_{n_1}}{(q^{2+2j-2n_1};q^2)_{n_1}}\ {}_3\varphi_2\left(\begin{matrix} q^{-2n_1}, q^{4k_1+2j}, q^{2-2j-4k_2} \\ q^{2-2j-4k_2-2n}, q^{4k_1} \end{matrix}; q^2, q^2\right)$$

by [27, (III.13)]. And by [27, (III.11)] this ${}_3\varphi_2$-series can be written as

$$\frac{(q^{-2j-2n};q^2)_{n_1}}{(q^{2-2j-4k_2-2n};q^2)_{n_1}} q^{2(1-2k_2)n_1}\, {}_3\varphi_2\left(\begin{matrix} q^{-2n_1}, q^{-2j}, q^{4k_1+4k_2+2j-2} \\ q^{4k_1}, q^{-2n-2j} \end{matrix}; q^2, q^2\right),$$

which is of the desired form. The constant follows by a straightforward calculation. ☑

Now all ingredients for the general convolution theorem for the Al-Salam and Chihara polynomials are known. Applying Lemma 7.3.2 in Lemma 7.3.1, and rewriting the result proves the following theorem.

Theorem 7.3.3. *With $x = \mu(z)$ and $y = \mu(w)$ and $n, j \in \mathbb{Z}_+$, $k_1, k_2 > 0$ we have*

$$\begin{aligned}\sum_{l=0}^{n+j} q^{2lk_1} \begin{bmatrix} n+j \\ l \end{bmatrix}_{q^2} & Q_j(q^{-2l}; q^{4k_1-2}, q^{4k_2-2}, n+j; q^2) \\ & \times s_l(x; q^{2k_1}w, q^{2k_1}/w|q^2)\, s_{n+j-l}(y; q^{2k_2}s, q^{2k_2}/s|q^2) = \\ \frac{q^{2n(k_1+k_2+2j)}}{(q^{4k_1};q^2)_j} & s_n(x; q^{2k_1+2k_2+2j}s, q^{2k_1+2k_2+2j}/s|q^2) \\ & \times p_j(y; q^{2k_1}z, q^{2k_1}/z, q^{2k_2}s, q^{2k_2}/s|q^2).\end{aligned}$$

Remark 7.3.4. (i) Theorem 7.3.3 is a connection coefficient formula for orthogonal polynomials in two variables, orthogonal for the same measure, cf. Remarks 7.2.4(i) and 7.2.6(i). The connection coefficients being given by the q-Hahn polynomials. Since the Clebsch-Gordan coefficients form a unitary matrix, we also have $e^{k_1}_{n_1} \otimes e^{k_2}_{n_2} = \sum_{n,k} C^{k_1,k_2,k}_{n_1,n_2,n}\, e^k_n$, and from this we can obtain the inverse connection coefficient problem. This also follows from the orthogonality relations for the dual q-Hahn polynomials, cf. Exercise 7.4.

(ii) The case $j = 0$ gives a simple convolution property for the Al-Salam and Chihara polynomials, since the q-Hahn and the Askey-Wilson polynomial reduce to 1. The case $n = 0$ is also of interest, since then the q-Hahn polynomial can be evaluated and the Al-Salam and Chihara polynomial on the right hand side reduces to 1. In both cases we have a free parameter in the sum.

(iii) Formally, in the representation space $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$ we have two bases of (generalised) eigenvectors for the action of $Y_s A$, namely $v^k(x)$ as in Remark 7.2.2 and $w(x; y)$ as in Remark 7.2.4(ii). They are connected by Clebsch-Gordan coefficients, which are now expressible as Askey-Wilson polynomials;

$$w(x;y) = \sum_{j=0}^{\infty} \frac{p_j(\mu(y); q^{2k_1}x, q^{2k_1}/x, q^{2k_2}s, q^{2k_2}/s|q^2)}{\sqrt{(q^2, q^{4k_1}, q^{4k_2}, q^{4k_1+4k_2+2j-2}; q^2)_j}}\, v^{k_1+k_2+j}(x).$$

This follows immediately from Lemma 7.3.1 and the orthogonality relations for the Clebsch-Gordan coefficients, cf. Remark 7.3.4(i), using the explicit expressions for $v^k(x)$ and $w(x;y)$ as in Remarks 7.2.2 and 7.2.4(ii).

The orthogonality relations for the Askey-Wilson polynomials then imply

$$v^{k_1+k_2+j}(x) = \frac{1}{h_j}\int w(x;y)\frac{p_j(\mu(y);q^{2k_1}x,q^{2k_1}/x,q^{2k_2}s,q^{2k_2}/s|q^2)}{\sqrt{(q^2,q^{4k_1},q^{4k_2},q^{4k_1+4k_2+2j-2};q^2)_j}}$$
$$\times\, dm(\mu(y);q^{2k_1}x,q^{2k_1}/x,q^{2k_2}s,q^{2k_2}/s|q^2),$$
$$h_j = \frac{1-q^{2j-2+4k_1+4k_2}}{1-q^{4j-2+4k_1+4k_2}}\frac{(q^2,q^{4k_1},q^{4k_2},q^{2k_1+2k_2}x^{\pm1}s^{\pm1};q^2)_j}{(q^{4k_1+4k_2};q^2)_j},$$

where we have to take all possible choices of signs at the $\pm$. This can also be proved using Lemma 7.3.1.

Notes and references. The results are obtained in joint work with Van der Jeugt [53] elaborating an idea of Granovskii and Zhedanov [28]. For the case of the Lie algebra $\mathfrak{su}(1,1)$ see [53] and Van der Jeugt [100]. A similar approach can be used in the case $U_q(\mathfrak{su}(2))$ to lead to a formula for q-Krawtchouk polynomials, see [53] for details. The method can also be extended to three-fold tensor products, and then we use the q-Racah coefficients, see [53]. Some related results on overlap coefficients for quantum algebras can be found in Klimyk and Kachurik [41].

Lemma 7.1.1 is well-known, see e.g. Kalnins, Manocha and Miller [35], where also the explicit expression for the Clebsch-Gordan coefficients is derived. Exercises 7.1-2 are also taken from [35]. The generating function for the Askey-Wilson polynomials, see Remark 7.2.6(ii), is given by Ismail and Wilson [30]. Taking $c=d=0$ also gives a generating function for the Al-Salam and Chihara polynomials, the other generating function needed is

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}t^n}{(q,ab;q)_n}s_n(\mu(x);a,b\mid q) = (-t/a;q)_\infty\,{}_2\varphi_1\left(\begin{matrix}ax,a/x\\ ab\end{matrix};q,-t/a\right).$$

The case $j=0$ of Theorem 7.3.3 was the reason for Al-Salam and Chihara [3] to introduce the Al-Salam and Chihara polynomials as the most general set of orthogonal polynomials still satisfying a convolution property, see also Al-Salam [2, §8]. Al-Salam and Chihara obtained the three-term recurrence relation, and later Askey and Ismail [6] determined the orthogonality measure.

The complete representation theory of the quantised universal enveloping algebra $U_q(\mathfrak{su}(1,1))$ can be found in Burban and Klimyk [14]. The representation theory is similar to that of $\mathfrak{su}(1,1)$, but there is an extra series of representations, the so-called strange series. The dual Hopf $*$-algebra $A_q(SU(1,1))$ is defined in Theorem 2.6.1, and Masuda et al. [69] give explicit expressions for the matrix elements in terms of ${}_2\varphi_1$-series of argument $-q^{-1}\beta\gamma$, which are q-analogues of the Jacobi functions. There are no corresponding orthogonality relations for the spherical elements, but the result has to be stated in terms of transform pairs. The transform pair is a q-analogue of the Mehler-Fock transform, see Vaksman and Korogodskiĭ [95] and Kakehi, Masuda and Ueno [34], Kakehi [33] for analytic proofs.

Exercises.

1. An explicit model for the positive discrete series π_k can be obtained as follows. First prove that the operators $A = q^k T_{q^2}$,
$$B = \frac{M}{q-q^{-1}}\left(q^{-2k}T_q^{-1} - q^{2k}T_q\right), \qquad C = T_q^{-1}D_{q^2},$$
acting on the formal power series give a representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$. Here M is the multiplication operator, $Mf(z) = zf(z)$, T_a, $a \neq 0$, is the shift operator, $T_a f(z) = f(az)$, and D_q is the q-derivative, $D_q f(z) = (f(z) - f(qz))/((1-q)z)$, see Exercise 3.5. Then show that taking $f_n(z) = z^n$ as an orthogonal basis gives a unitary representation of $U_q(\mathfrak{su}(1,1))$. Prove that
$$\| f_n \|^2 = q^{n(2k-1)} \frac{(q^2;q^2)_n}{(q^{4k};q^2)_n} \| f_0 \|^2 .$$
So the representation space is the space of formal power series $\sum_{n=0}^{\infty} c_n z^n$ such that $\sum_{n=0}^{\infty} |c_n|^2 \| f_n \|^2 < \infty$, which are the functions analytic on the disc with radius $q^{1/2-k}$.
2. By Exercise 7.1 $f_{n_1,n_2}(z,w) = z^{n_1}w^{n_2}$ forms an orthogonal basis for the representation space of the tensor product of the model for $\pi_{k_1}\otimes\pi_{k_2}$. Show that all eigenvectors of $\Delta(A)$ in the kernel of $\Delta(C)$ are given by
$$p_{j,0}(z,w) = z^j (q^{2-k_1-k_2-3s}w/z;q^2)_j, \qquad s \in \mathbb{Z}_+$$
And $\Delta(A)p_{j,0} = q^{k_1+k_2+j}p_{j,0}$. Now define inductively
$$p_{j,n} = \frac{q-q^{-1}}{(q^{-2k_1-2k_2-2j-n} - q^{2k_1+2k_2+2j+n})}\Delta(B)p_{j,n-1}$$
and prove that the span of $p_{j,n}$, $n \in \mathbb{Z}_+$, realises an irreducible unitary representation of $U_q(\mathfrak{su}(1,1))$ equivalent to the discrete series representation $\pi_{k_1+k_2+j}$. Finish the proof of Lemma 7.1.1 by proving that $p_{n,j}$, $n,j \in \mathbb{Z}_+$ form an orthogonal basis.
3. Derive the recurrence relation mentioned in Remark 7.1.3 and use Lemma 7.3.2 to obtain a contiguous relation for ${}_3\varphi_2$-series.
4. Use the unitarity of the Clebsch-Gordan coefficients, i.e.
$$\sum_{n_1=0}^{n+j} C^{k_1,k_2,k_1+k_2+j}_{n_1,n+j-n_1,n} C^{k_1,k_2,k_1+k_2+i}_{n_1,n+j-n_1,n+j-i} = \delta_{ij}$$

with Lemma 7.3.2 to derive the orthogonality relations for the q-Hahn polynomials $Q_n(q^{-x}) = Q_n(q^{-x}; a, b, N; q)$; for $n, m \in \{0, 1, \ldots, N\}$,

$$\sum_{x=0}^{N} Q_m(q^{-x}) Q_n(q^{-x}) \frac{(aq;q)_x (bq;q)_{N-x}}{(q;q)_x (q;q)_{N-x}} (aq)^{-x} = \delta_{nm} h_n.$$

What are the dual orthogonality relations?

8. More examples

In the last section we shortly discuss some more known examples between quantum groups and q-special functions, which are similarly using the duality of Hopf $*$-algebras. We also consider the quantum algebra approach, by considering it for a special case. Finally, we give some open problems. The references are given separately for each subsection, and there more details can be found.

§8.1. The quantum group of plane motions and q-Bessel functions. In Exercise 2.4 the Hopf $*$-algebra $U_q(\mathfrak{m}(2))$ is obtained by a contraction procedure from $U_q(\mathfrak{su}(2))$. We use the same letters A, B, C and D for the generators, then the comultiplication, counit, antipode and $*$-operator are unchanged, i.e. given by (2.1.2) and Theorem 2.3.4(ii). The relations among the generators change;

$$AB = qBA, \quad AC = q^{-1}CA, \quad AD = 1 = DA, \quad BC = CB.$$

We transpose the contraction procedure to the dual Hopf $*$-algebra by demanding that the duality on the level of generators, cf. (2.4.1), is not changed. Denoting the resulting generators of the Hopf $*$-algebra $A_q(M(2))$ by the same letters α, β, γ and δ, we see that the action of the counit, antipode and $*$-operator remains, cf. (2.4.4) and Theorem 2.6.1(ii). The commutation relations and the action of the comultiplication change;

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta, \quad \alpha\delta = \delta\alpha = 1.$$

Then $A_q(M(2))$ is Hopf algebra. The comultiplication Δ, the counit ε and the antipode S given on the generators by

$$\begin{gathered}
\Delta(\alpha) = \alpha \otimes \alpha, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \\
\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \delta \otimes \delta, \\
\varepsilon(\alpha) = \varepsilon(\delta) = 1, \qquad \varepsilon(\beta) = \varepsilon(\gamma) = 0, \\
S(\alpha) = \delta, \quad S(\beta) = -q^{-1}\beta, \quad S(\gamma) = -q\gamma, \quad S(\delta) = \alpha.
\end{gathered}$$

Lemma 8.1.1. *For $p, l \in \mathbb{Z}$, $n, m, r, s \in \mathbb{Z}_+$ we have* $\langle A^p B^r C^s, \alpha^l \beta^m \gamma^n \rangle =$

$$\delta_{rm}\delta_{sn} q^{(pl+l(m+n)+p(m-n))/2} q^{-m(m-1)/2} q^{-n(n-1)/2} \frac{(q^2;q^2)_m (q^2;q^2)_n}{(1-q^2)^{m+n}}.$$

As a consequence, we see that the extension of $A_q(SU(2)) \subset \big(U_q(\mathfrak{m}(2))\big)^*$, the linear dual of $U_q(\mathfrak{m}(2))$, given by

$$\sum_{l\in\mathbb{Z}} \sum_{n,m=0}^{\infty} c_{lmn} \alpha^l \beta^m \gamma^n$$

with c_{lmn} non-zero for only finitely many l, is still well-defined.

Let us now consider a special unitary representation t^R of $U_q(\mathfrak{m}(2))$ acting in $\ell^2(\mathbb{Z})$ equipped with an orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$ by

$$t^R(A)\, e_n = q^n e_n, \qquad t^R(B)\, e_n = Re_{n+1},$$
$$t^R(C)\, e_n = Re_{n-1}, \quad t^R(D)\, e_n = q^{-n} e_n,$$

where $R > 0$. Note that the operators for A and D are unbounded. Denote the corresponding matrix elements by t^R_{nm}.

Theorem 8.1.2. *The matrix elements t^R_{ij} are contained in the extension of $A_q(M(2))$. For $i \geq j$*

$$t^R_{ij} = \Big(\frac{R(1-q^2)}{q^{j+1/2}}\Big)^{i-j} \frac{\alpha^{i+j}\beta^{i-j}}{(q^2;q^2)_{i-j}}\, {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{2(i-j+1)} \end{matrix} ; q^2, -(1-q^2)R^2 q^{-2j}\beta\gamma \right)$$

and for $i \leq j$

$$t^R_{ij} = \Big(\frac{R(1-q^2)}{q^{i+1/2}}\Big)^{j-i} \frac{\alpha^{i+j}\gamma^{j-i}}{(q^2;q^2)_{j-i}}\, {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{2(j-i+1)} \end{matrix} ; q^2, -(1-q^2)R^2 q^{-2i}\beta\gamma \right).$$

Proof. Use Lemma 8.1.1 and the definition of the representation t^R to see that both sides agree on $A^p B^r C^s$. ☑

The ${}_1\varphi_1$-series in Theorem 8.1.2 are known as Hahn-Exton q-Bessel functions, which are defined by

$$J_\nu(z;q) = z^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} ; q, qz^2 \right).$$

We can now use this interpretation of the Hahn-Exton q-Bessel function as matrix elements of irreducible unitary representations of $U_q(\mathfrak{m}(2))$ to obtain identities for these q-analogues of the Bessel function.

Theorem 8.1.3. *The Hahn-Exton q-Bessel function satisfies the Hansen-Lommel type orthogonality relations;*

$$\sum_{i=-\infty}^{\infty} q^i J_{k+i}(z;q)\, J_{j+i}(z;q) = \delta_{kj} q^{-k}, \qquad |z| < q^{-1/2},\ k,j \in \mathbb{Z},$$

the Hankel type orthogonality relations;

$$\sum_{p=-\infty}^{\infty} q^{2p} J_n(q^{k+p};q^2)\, J_n(q^{l+p};q^2) = C q^{-2k} \delta_{kl}, \qquad k,l \in \mathbb{Z},$$

for some non-zero constant C, and the Graf type addition formula; for $n,y,x,z \in \mathbb{Z}$, $R > 0$

$$(-q)^n J_{x-n}(q^{z-y};q^2) J_n(Rq^{n+z};q^2) = \sum_{k=-\infty}^{\infty} q^{2k} J_k(Rq^{y+x};q^2) J_{k-n}(Rq^y;q^2) J_x(q^{z+k-y};q^2),$$

for $|R^2 q^{2x+2y+2}| < 1$.

Remark 8.1.4. The Hahn-Exton q-Bessel function of negative integer order is defined by using

$$(q^{1-n};q)_\infty \sum_{m=0}^{\infty} \frac{c_m}{(q^{1-n};q)_\infty} = \sum_{m=n}^{\infty} c_m (q^{1+m-n};q)_\infty$$

for $n \in \mathbb{Z}_+$. This leads to $J_{-n}(z;q) = (-1)^n q^{n/2} J_n(zq^{n/2};q)$, which is valid for $n \in \mathbb{Z}$.

Sketch of Proof. The first relation is a consequence of the unitarity of the representation t^R. The second relation uses the fact that there exists a Haar functional on $A_q(M(2))$ as well as on its extension, which is only defined on a suitable subspace, and for which Schur type orthogonality relations can be derived. The Hankel type orthogonality relations then follow from the Schur orthogonality relations. The constant C is involved, since there is no proper normalisation for the Haar functional. However, $C = 1$, see Exercise 8.3. Finally, the addition formula follows by representing the identity $\Delta(t^R_{0n}) = \sum_{k\in\mathbb{Z}} t^R_{0k} \otimes t^R_{kn}$ in $A_q(M(2)) \otimes A_q(M(2))$ using a suitable representation of $A_q(M(2))$ in $\ell^2(\mathbb{Z})$ given by $\alpha\, e_n = e_{n-1}$, $\gamma\, e_n = q^n e_n$. ☑

Remark 8.1.5. Since the contraction procedure is a limit, we can show that some of the results of Theorem 8.1.3 can be obtained from a suitable limit in the corresponding result for the little q-Jacobi polynomials, which occur as matrix elements on the quantum $SU(2)$ group, cf. Corollary 5.3.1.

Notes and references. The interpretation of the Hahn-Exton q-Bessel functions on the quantum group of plane motions is due to Vaksman and Korogodskiĭ [94]. The presentation is taken from [46]. For the C^*-algebra approach to the quantum group of plane motions, see Woronowicz [105], and Pal [82] for the interpretation of the Hahn-Exton q-Bessel functions in this context. The addition formula in Theorem 8.1.3 is also derived by Kalnins, Miller and Mukherjee [39] using only the quantum algebra. See Koornwinder and Swarttouw [63] for more information on the Hahn-Exton q-Bessel function, as well as for the limit transition from the little q-Jacobi polynomials. For the quantum group of plane motions we can also speak of the analogue of generalised matrix elements, see [49], and we can then obtain the analogue of the addition formula (6.1.1) for this situation. Here the Jackson q-Bessel functions [29] play the role of transition coefficients.

§8.2. The quantum algebra approach. Let us now briefly consider the quantum algebra approach. The quantum algebra approach is based on the observation that for X in the Lie algebra $\mathfrak{g}$ the exponential mapping $\exp tX$ gives a function on the corresponding group G. The representation theory of $U_q\mathfrak{g}$ is usually similar to the representation theory of $\mathfrak{g}$. Classically we can obtain elements of the corresponding group G by exponentiating Lie algebra elements. In the quantum algebra approach the action of $\exp_q(\alpha_1X_1)\ldots\exp_q(\alpha_nX_n)$ is calculated in a representation of $U_q\mathfrak{g}$. Here $\exp_q$ can be one of the q-analogues of the exponential function, see e.g. Corollary 3.2.2, α_i are scalars and X_i are generators of $U_q\mathfrak{g}$. For a suitable basis $\{f_m\}$ of the representation space we get

$$\exp_q(\alpha_1X_1)\ldots\exp_q(\alpha_nX_n)\,f_m=\sum_k U_{m,k}(\alpha_1,\ldots,\alpha_n)\,f_k.$$

The matrix coefficients $U_{m,k}(\alpha_1,\ldots,\alpha_n)$ can be calculated in terms of special functions in $\alpha_1,\ldots,\alpha_n$. Let us discuss shortly the example of $U_q(\mathfrak{m}(2))$. Let t^R be the representation of $U_q(\mathfrak{m}(2))$ as in the previous section. Using the notation of §8.1 and Corollary 3.2.2 we get e.g.

$$t^R\big(e_q(bB)E_q(cC)\big)\,e_m=\sum_{k=-\infty}^{\infty}U_{m,k}(b,c)\,e_k. \tag{8.2.1}$$

Then it is straightforward to calculate the matrix elements in terms of q-hypergeometric series;

$$U_{m,k}(b,c)=(Rb)^{k-m}\frac{(q^{k-m};q)_\infty}{(q;q)_\infty}\,{}_1\varphi_1(0;q^{k-m+1};q,-bcR^2)$$

which are the Hahn-Exton q-Bessel functions. For $k<m$ we apply Remark 8.1.5. Having this interpretation, a number of properties like orthogonality

relations and addition formulas can be derived. Explicit models for the representation space in terms of function spaces are usually needed. As a very simple example we derive the Hansen-Lommel orthogonality relations of Theorem 8.1.3. Observe that $e_q(z)E_q(-z)=1$, and hence

$$
\begin{aligned}
e_l &= t^R\big(e_q(bB)E_q(cC)e_q(-cC)E_q(-bB)\big)\, e_l \\
&= \sum_{k=-\infty}^{\infty}\Big(\sum_{p=-\infty}^{\infty} T_{l,p}(-b,-c)\, U_{p,k}(b,c)\Big) e_k,
\end{aligned}
$$

where the matrix coefficient $T_{l,p}(b,c)$ is defined by

$$
t^R\big(E_q(bB)e_q(cC)\big)\, e_m = \sum_{k=-\infty}^{\infty} T_{m,k}(b,c)\, e_k.
$$

Since interchanging B and C and A and D is a symmetry, say J, of $U_q(\mathfrak{m}(2))$ for which $t^R_{nm}(J(X)) = t^R_{mn}(X)$ we find that $T_{m,k}(b,c) = U_{k,m}(c,b)$. Hence,

$$
\sum_{p=-\infty}^{\infty} U_{p,l}(-c,-b)\, U_{p,k}(b,c) = \delta_{kl}
$$

and this is equivalent to the Hansen-Lommel orthogonality relations of Theorem 8.1.3.

Notes and references. The method sketched here is motivated by the classical relation between Lie algebras and special functions as described in Miller's book [71]. The example is taken from Kalnins, Miller and Mukherjee [39]. There exists a huge amount of papers on this approach, in particular Kalnins, Miller et al. [35], [37], [38], [39] and Floreanini and Vinet [23], [24] and references given there, see also the references in [51].

Using the model of t^R in the space of Laurent series with the actions of A, B and C given by T_q, the shift operator defined in Exercise 7.1, RM_z and $RM_{1/z}$, where M_f is multiplication by f. Then the basis e_m corresponds to z^m, $m \in \mathbb{Z}$. In this model (8.2.1) is the generating function for the Hahn-Exton q-Bessel function, and a large number of results can already be obtained by working only with the generating function, see Koornwinder and Swarttouw [63] and Koelink [44] for more general generating functions leading to q-Bessel functions. The quantum algebra becomes important when dealing with different models, and in particular when using the tensor product of representations, see Kalnins, Miller and Mukherjee [39] for a derivation of the addition formula of Theorem 8.1.3 by using the Clebsch-Gordan decomposition of $t^R \otimes t^S$.

§8.3. Quantum spheres and spherical functions. In sections 4 and 5 we discussed the analogue of functions on $K \setminus SU(2)/H$ for K, H one-parameter subgroups, which are precisely the elements $b^l_{00}(\tau,\sigma)$. We can also consider the elements $b^l_{n0}(\tau,\sigma)$ as the analogues of the functions on the sphere $S^2 = SU(2)/H$. These q-analogues of the sphere are parametrised by σ and were first introduced by Podleś [83], see Dijkhuizen and Koornwinder [18], Noumi and Mimachi [79].

This can be generalised to give q-analogues of $U(n)/U(n-1)$, the sphere in $\mathbb{C}^n$. There is a quantised universal enveloping algebra $U_q(\mathfrak{gl}(n,\mathbb{C}))$, which is in duality as Hopf algebras with $A_q(GL(n,\mathbb{C}))$. These Hopf algebras can be made into Hopf $*$-algebras, and we obtain a Hopf $*$-algebra $A_q(U(n))$, cf. [25], [43], [81]. The algebra structure of $A_q(U(n))$ is as follows. We have n^2 generators t_{ij} satisfying the relations

$$\begin{aligned} t_{ij}t_{il} &= qt_{il}t_{ij}, \quad j<l;\\ t_{ij}t_{kj} &= qt_{kj}t_{ij}, \quad i<k;\\ t_{ij}t_{kl} &= t_{kl}t_{ij}, \quad i>k,\ j<l;\\ t_{ij}t_{kl}-t_{kl}t_{ij} &= (q-q^{-1})t_{il}t_{kj}, \quad i<k,\ j<l. \end{aligned}$$

Then the t_{ij} generate a bi-algebra, which is an analogue of the semigroup of $n\times n$-matrices. In order to obtain a Hopf algebra we have to localise along the quantum determinant, which is a central element, see [25], [43], [81] for detailed relations. The important thing to notice is that we have a surjective (Hopf $*$-)algebra homomorphism $\pi: A_q(U(n)) \to A_q(U(n-1))$, which is the identity on t_{ij}, $1\le i,j<n$ and $\pi(t_{ni}) = \delta_{ni} = \pi(t_{in})$. On the dual level we have a natural inbedding

$$U_q(\mathfrak{gl}(n-1,\mathbb{C})) \oplus U_q(\mathfrak{gl}(1,\mathbb{C})) \hookrightarrow U_q(\mathfrak{gl}(n,\mathbb{C})),$$

and we can talk of $U_q(\mathfrak{gl}(n-1,\mathbb{C}))\oplus U_q(\mathfrak{gl}(1,\mathbb{C}))$-invariant vectors in irreducible representations of $U_q(\mathfrak{gl}(n,\mathbb{C}))$. The space of such invariant vectors is at most one-dimensional, and the corresponding representations can be labeled by two integers l and m. Let $\psi_{l,m}$ be the matrix element with respect to the invariant vector of the dual algebra $A_q(U(n))$, then this means precisely

$$(id\otimes\pi)\Delta(\psi_{l,m}) = \psi_{l,m}\otimes 1, \qquad (\pi\otimes id)\Delta(\psi_{l,m}) = 1\otimes\psi_{l,m}.$$

So we can view $\psi_{l,m}$ as an element of the deformed algebra of $U(n-1)$-biinvariant functions on $U(n)$. Using the Schur orthogonality for the Haar functional it is possible to derive a very explicit expression for the zonal spherical elements $\psi_{l,m}$;

$$\psi_{l,m} = \begin{cases} t_{nn}^{l-m}\, p_m^{(n-2,l-m)}(1-t_{nn}t_{nn}^*;q^2), & \text{if } l\ge m,\\ p_l^{(n-2,m-l)}(1-t_{nn}t_{nn}^*;q^2)(t_{nn}^*)^{m-l}, & \text{if } m\ge l, \end{cases}$$

where $p_l^{(n-2,m-l)}$ are little q-Jacobi polynomials as in §5, see Noumi, Yamada and Mimachi [81]. For $n=2$ this corresponds to Corollary 5.3.1.

Floris [25] obtains an abstract addition formula, i.e. in non-commuting variables, for the little q-Jacobi polynomials $p_l^{(n-2,m-l)}(\cdot;q^2)$ by calculating $\Delta(\psi_{l,m})$ explicitly modulo $U_q(\mathfrak{gl}(n-2,\mathbb{C}))$-invariance in each factor of the tensor product. Using the representation theory of the Hopf $*$-algebra $A_q(U(n))$, cf. [43], Floris and Koelink [26] derive an explicit addition and product formula for the little q-Jacobi polynomials $p_l^{(\alpha,m-l)}(\cdot;q^2)$, which contains as a special case the addition formula for the little q-Legendre polynomial $p_l^{(0,0)}(\cdot;q^2)$ discussed in §6.

From the $n=2$ case discussed in §§4-5, we may suspect that we can also have (τ,σ)-spherical elements in this case. This is indeed the case, as shown by Dijkhuizen and Noumi [19]; they obtain an interpretation of $p_m^{(n-2,0)}(\cdot;q^\tau,q^\sigma|q^2)$ as spherical functions on the quantum analogue of $U(n)/U(n-1)$. See also Dijkhuizen and Koornwinder [18] for a general discussion of quantum homogeneous spaces.

Instead of considering an analogue of the sphere in $\mathbb{C}^n$, Sugitani [90] considers the analogue of the sphere $SO(n)/SO(n-1)$ in $\mathbb{R}^n$. The zonal spherical functions can be expressed using big q-Jacobi polynomials and continuous q-ultraspherical polynomials, i.e. $p_n^{(\alpha,\alpha)}(\cdot;1,1|q)$ with the notation for the Askey-Wilson polynomials in §5.

§8.4. Multi-variable orthogonal polynomials as spherical functions.

The multi-variable orthogonal polynomials of importance are the Macdonald polynomials [65], [67], which are associated with root systems. The polynomials depend on n variables and are invariant under the Weyl group, see Macdonald [66, Ch. VI] for the case of symmetric functions, i.e. for the root system of type A. Koornwinder [59] has obtained multivariable analogues of the Askey-Wilson polynomials by considering the non-reduced root system BC_n. The orthogonality measures for these polynomials are absolutely continuous. For the special case $n=1$ we obtain the Askey-Wilson polynomials, whereas the case $n=1$, i.e. for root system A_1, of the Macdonald polynomials gives the continuous q-ultraspherical polynomials.

Noumi [73] shows that for appropriate analogues of $GL(n)/SO(n)$ and $GL(2n)/Sp(n)$, where $Sp(n)$ is the symplectic group, the Macdonald polynomials for root system A_{n-1} and a specific choice of the free parameter t arise as the zonal spherical functions. Noumi, Dijkhuizen and Sugitani [74] have announced that they can parametrise the quantum homogeneous spaces continuously, similarly as for the quantum $U(n)/U(n-1)$ space, and the corresponding zonal spherical functions are expressible in terms of Koornwinder's [59] multivariable Askey-Wilson polynomials. It is expected that multivariable orthogonal polynomials with (partly) discrete orthogonality measure introduced by Stokman [87], [88] can be obtained as limit cases from the general setting, see Stokman and Koornwinder [89] for the analytic proof of this limit transition.

§8.5. Some open problems.

Problem 8.5.1. The quantum $SU(2)$ group and the quantum $SU(1,1)$ group have been treated in some detail. The other real form, the quantum $SL(2,\mathbb{R})$ group, cf. Theorems 2.3.4 and 2.6.1, presents us with some difficulties, although its representation theory is known, cf. Schmüdgen [85], [86]. Is it possible to associate special functions of q-hypergeometric type for $|q| = 1$ to this quantum group? In particular, is there a relation with sieved orthogonal polynomials?

Problem 8.5.2. Is it possible to give a proof of Theorem 4.2.4 along the lines of the proof of Lemma 4.2.1? Or, can we derive a simple recurrence relation for $h(p_n(\rho_{\tau,\sigma}))$ for some suitable choosen polynomial p_n, and next identify it with the corresponding Askey-Wilson integral?

Problem 8.5.3. Derive an addition formula for Askey-Wilson polynomials by first deriving an abstract addition formula using the interpretation as zonal spherical function on the quantum analogue of $U(n)/U(n-1)$, cf. §8.3. Then use the representation theory of $A_q(U(n))$ to derive an addition formula in commuting variables. Classically, i.e. for $q = 1$ and working in the group case, an addition formula for the Jacobi polynomials $R_n^{(\alpha,0)}$ (notation of Exercise 6.1) is derived in this way and from this an addition formula for general Jacobi polynomials $R_n^{(\alpha,\beta)}$ can be obtained by differentiation, cf. [55]. For which cases of σ and τ can we obtain a general addition formula, i.e. an addition formula for $p_n^{(\alpha,\beta)}(\cdot, s, t|q)$ for all α, β? The Rahman-Verma [84] addition formula suggests that it might be possible for continuous q-Jacobi polynomials, i.e. $s = t = 1$.

Exercises.

1. Prove Lemma 8.1.1. Use Exercise 2.5, or apply the contraction procedure to Theorem 2.5.2.
2. Use the q-gamma function as in Exercise 3.9 to see that $J_\nu((1-q)z;q)$ tends to $J_\nu(2z)$ as $q \uparrow 1$. The Bessel function is defined by $J_\nu(z) = \sum_{k=0}^{\infty}(-1)^k z^{\nu+2k}/(k!\,\Gamma(\nu+k+1))$.
3. Prove $(w;q)_\infty\, {}_1\varphi_1(0;w;q,z) = (z;q)_\infty\, {}_1\varphi_1(0;z;q,w)$ and use this symmetry to derive the Hankel type orthogonality relations of Theorem 8.1.3 from the Hansen-Lommel type orthogonality relations. Show that $C = 1$.
4. Define matrix coefficients by
$$t^R\big(E_q(bB)E_q(cC)\big)\, e_m = \sum_{k=-\infty}^{\infty} U_{m,k}(b,c)\, e_k$$
and
$$t^R\big(e_q(bB)e_q(cC)\big)\, e_m = \sum_{k=-\infty}^{\infty} T_{m,k}(b,c)\, e_k.$$

Calculate the matrix coefficients explicitly, and derive the corresponding Hansen-Lommel (bi-)orthogonality relations. The q-Bessel functions are known as Jackson's q-Bessel functions, see Ismail [29].

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