

Regularizing effects for the linearized Kadomtsev-Petviashvili (KP) equation

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ABSTRACT. In this article we obtain estimates of Strichartz type, of maximal type, and of local type for the linearized KPI and KPII equations. These estimates show the gain of regularity of the solutions of these equations and are obtained by a detailed analysis of the oscillatory integrals which give the solutions explicitly.

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0. Introduction

We say that an evolution equation presents regularizing effects if, for $t > 0$, the solution u of the Cauchy problem for this equation is, in some sense, better than the initial datum u_0 , for $t = 0$.

This gain of regularity, for equations which are reversible and conservative, is in general measured by mixed L^p-L^q space-time norms of the solution. These norms often involve fractional derivatives. For more details, see, for example, [CS] and [KPV1].

In this paper we study certain regularizing effects for the linearizations of the Kadomtsev-Petviashvili equation (KP):

$$(u_t + u_{xxx} + uu_x)_x \mp u_{yy} = 0. \quad (0.1)$$

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Equation (0.1) with “-” sign (respectively with “+” sign) is called KPI (respectively KP II). More precisely we will consider the problem

$$\left. \begin{aligned} (u_t + u_{xxx})_x \mp u_{yy} &= 0 \\ u(x, y, 0) &= u_0(x, y) \end{aligned} \right\} \tag{0.2}$$

The study of these effects for linearized equations has been successfully used to prove existence of solutions for nonlinear evolution equations with dispersive linear part and nonregular initial data. In particular, a wide variety of articles using this approach in the Schrödinger and Korteweg-de Vries (KdV) equations has been recently published.

In the case of the linearizations of the KP equation, these effects will be obtained by a detailed analysis of the oscillatory integrals which, by means of the Fourier Transform, give the explicit solution of problem (0.2).

The methods we use here were suggested to us by the work done by C. Kenig, G. Ponce, and L. Vega [KPV1, KPV2], for the KdV equation.

Let $\{U_1(t)\}$ and $\{U_2(t)\}$ be the groups describing the solutions of (0.2) with “-” and “+” signs respectively, namely

$$\begin{aligned} [U_j(t)u_0](x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i[t\varphi_j(\zeta) + x\xi + y\eta]} \widehat{u}_0(\zeta) \, d\zeta \\ &= C[I_j(\cdot, \cdot, t) * u_0](x, y) \quad j = 1, 2, \end{aligned} \tag{0.3}$$

where $\zeta = (\xi, \eta)$, $\varphi_1(\zeta) = \xi^3 + \frac{\eta^2}{\xi}$, $\varphi_2(\zeta) = \xi^3 - \frac{\eta^2}{\xi}$, and $I_j(\cdot, \cdot, t)$ is defined by the oscillatory integrals:

$$I_j(x, y, t) = \int_{\mathbb{R}^2} e^{i[t\varphi_j(\zeta) + x\xi + y\eta]} \, d\zeta \quad j = 1, 2. \tag{0.4}$$

In [S], J.C. Saut proved that there is $C > 0$ with $|I_j(x, y, t)| \leq \frac{C}{|t|}$, and from this inequality he establishes certain $L^p - L^q$ (time-space) effects of Strichartz type for the groups U_j . Similar results have been obtained by A. Faminskii and L. Vega ([F]).

In this article we obtain estimates of Strichartz type, of maximal type, and of local type for the linearized KPI and KP II equations.

In order to formulate and prove our results we need to introduce some definitions and notations. The form of the phase functions φ_j in (0.3) suggests to consider the following regions in the ζ -plane:

$$\begin{aligned} \Omega &:= \{ \zeta \in \mathbb{R}^2 \mid |\eta| < \sqrt{2}\xi^2 \}, \\ \Gamma &:= \{ \zeta \in \mathbb{R}^2 \mid |\eta| > \frac{\sqrt{2}}{2}\xi^2 \}; \end{aligned}$$

and a partition of the unity $\{\alpha, \beta\}$ of $\mathbb{R}^2 - \{0\}$ subordinated to Ω and Γ . The functions α and β may be chosen in such a way that, in the regions where they are not zero,

$$|\alpha_\xi| = |\beta_\xi| \sim \eta^{-\frac{1}{2}}, \quad (0.5)$$

$$|\alpha_{\xi\xi}| = |\beta_{\xi\xi}| \sim \eta^{-1}, \quad (0.6)$$

$$|\alpha_\eta| = |\beta_\eta| \sim \xi^{-2}, \quad (0.7)$$

$$|\alpha_{\eta\eta}| = |\beta_{\eta\eta}| \sim \xi^{-4}. \quad (0.8)$$

We will need to write $U_j(t)$ as a sum $W_j(t) + V_j(t)$ where $W_j(t)$ and $V_j(t)$ are defined by means of the Fourier Transform as:

$$[W_j(t)u_0]^\wedge(\zeta) := [U_j(t)u_0]^\wedge(\zeta)\alpha(\zeta), \quad (0.9)$$

$$[V_j(t)u_0]^\wedge(\zeta) := [U_j(t)u_0]^\wedge(\zeta)\beta(\zeta). \quad (0.10)$$

For $s \in \mathbb{R}$, D_x^s and D_y^s will be operators defined by

$$(D_x^s u)^\wedge(\zeta) := |\xi|^s \hat{u}(\zeta),$$

$$(D_y^s u)^\wedge(\zeta) := |\eta|^s \hat{u}(\zeta).$$

On the other side, $\| \cdot \|_{L_x^q L_y^p}$, $\| \cdot \|_{L_t^q L_{xy}^p}$, $\| \cdot \|_{L_T^q L_{xy}^p}$ will denote the norms of the following spaces $L^q(\mathbb{R}_x; L^p(\mathbb{R}_{xy}^2))$, $L^q(\mathbb{R}_t; L^p(\mathbb{R}_{xy}^2))$, $L^q(\mathbb{R}_x; L^p([-T, T] \times \mathbb{R}_y))$, respectively.

Our results in this work are gathered in the following theorems:

Theorem 1. *There exists a constant C such that*

$$\|D_x U_1(\cdot)u_0\|_{L_t^4 L_{xy}^\infty} \leq C(\|D_x^{\frac{7}{4}}u_0\|_{L_{xy}^2} + \|D_y^{\frac{7}{8}}u_0\|_{L_{xy}^2}), \quad (0.11)$$

$$\|D_x U_2(\cdot)u_0\|_{L_t^6 L_{xy}^\infty} \leq C(\|D_x^2u_0\|_{L_{xy}^2} + \|D_y u_0\|_{L_{xy}^2}). \quad (0.12)$$

Theorem 2.

(i) *If $s > \frac{7}{4}$, then there is a constant C_s such that*

$$\|W_1(\cdot)u_0\|_{L_T^2 L_{xy}^\infty} \leq C_s(1+T)^{\frac{1}{2}}\|D_x^s u_0\|_{L_{xy}^2}. \quad (0.13)$$

(ii) *If $s > \frac{5}{4}$, then there is a constant C_s such that*

$$\|V_1(\cdot)u_0\|_{L_y^2 L_{Tx}^\infty} \leq C_s(1+T)^{\frac{1}{2}}\|(D_x^{-1}D_y)^s u_0\|_{L_{xy}^2}. \quad (0.14)$$

(iii) *If $s > \frac{7}{4}$, then there is a constant C_s such that*

$$\|U_2(\cdot)u_0\|_{L_T^2 L_{xy}^\infty} \leq C_s T^{\frac{s-1}{3}}\|[D_x^s + (D_x^{-1}D_y)^s]u_0\|_{L_{xy}^2}. \quad (0.15)$$

Theorem 3. *There exists a constant C such that*

$$\|D_x W_1(\cdot) u_0\|_{L_x^\infty L_y^2} \leq C \|u_0\|_{L_x^2}, \quad (0.16)$$

$$\|(D_x^{-1} D_y)^{\frac{1}{2}} U_j(\cdot) u_0\|_{L_y^\infty L_x^2} \leq C \|u_0\|_{L_x^2} \quad j = 1, 2, \quad (0.17)$$

$$\|(D_x + D_x^{-1} D_y) U_2(\cdot) u_0\|_{L_x^\infty L_y^2} \leq C \|u_0\|_{L_x^2}, \quad (0.18)$$

$$\|D_x^{\frac{1}{2}} V_j(\cdot) u_0\|_{L_y^\infty L_x^2} \leq C \|u_0\|_{L_x^2} \quad j = 1, 2. \quad (0.19)$$

1. Strichartz type estimates (Proof of Theorem 1)

We consider the oscillatory integrals for the KPI equation

$$A_1(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} |\xi|^{-\frac{3}{2}} \alpha(\zeta) d\zeta,$$

$$B_1(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} \left| \frac{\eta}{\xi} \right|^{-\frac{3}{2}} \beta(\zeta) d\zeta,$$

and

$$A_2(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\varphi_2(\zeta) + x\xi + y\eta]} |\xi|^{-2} \alpha(\zeta) d\zeta$$

$$B_2(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\varphi_2(\zeta) + x\xi + y\eta]} \left| \frac{\eta}{\xi} \right|^{-2} \beta(\zeta) d\zeta$$

for the KP-II equation.

By the symmetry of the problem we carry out our estimates integrating only in the first quadrant of the ζ plane.

We will often use the following version of Van der Corput's lemma: If $\psi \in C_0^1(\mathbb{R})$ and $\Phi \in C^2(\mathbb{R})$ with $|\Phi''(\xi)| \geq 1$ in the support of ψ , then

$$\left| \int_a^b e^{i\lambda\Phi(\xi)} \psi(\xi) d\xi \right| \leq C |\lambda|^{-\frac{1}{2}} \{ \|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \},$$

where C is a constant independent of a, b, λ, Φ , and ψ (see [St], pg. 311).

In particular, if $\psi \in C_0^1(\mathbb{R})$ and $\phi \in C^2(\mathbb{R})$ with $|\phi''(\xi)| \geq |\lambda|$ in the support of ψ , then

$$\left| \int_a^b e^{i\phi(\xi)} \psi(\xi) d\xi \right| \leq C |\lambda|^{-\frac{1}{2}} \{ \|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \}.$$

To estimate A_1 we consider the regions (see Figure 1)

$$\begin{aligned} \Omega_1(t) &:= \left\{ \zeta \in \Omega \mid \left| \frac{\partial \varphi_1}{\partial \xi} \right| < \frac{12}{|t|^{\frac{2}{3}}} \right\} \\ \Omega_2(x, t) &:= \left\{ \zeta \in \Omega \mid \left| \frac{\partial \varphi_1}{\partial \xi} \right| > \frac{3}{|t|^{\frac{2}{3}}} \wedge \left| t \frac{\partial \varphi_1}{\partial \xi} + x \right| < \frac{1}{2} |x| \right\} \\ \Omega_3(x, t) &:= \left\{ \zeta \in \Omega \mid \left| \frac{\partial \varphi_1}{\partial \xi} \right| > \frac{3}{|t|^{\frac{2}{3}}} \wedge \left| t \frac{\partial \varphi_1}{\partial \xi} + x \right| > \frac{1}{3} |x| \right\}. \end{aligned}$$

Let $\{\psi_k\}_{k=1}^3$ be a collection of functions in $C^\infty(\mathbb{R}^2)$ such that their restrictions to Ω form a partition of the unity of Ω subordinated to $\{\Omega_k\}_{k=1}^3$. Then

$$A_1(x, y, t) = \sum_{k=1}^3 \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} |\xi|^{-\frac{3}{2}} \alpha(\zeta) \psi_k(\zeta) d\zeta = I + II + III.$$

It is easily seen that

$$|I| \leq C \int_0^{c|t|^{-\frac{1}{3}}} \int_0^{\sqrt{2}\xi^2} \xi^{-\frac{3}{2}} d\eta d\xi = \frac{C}{|t|^{\frac{1}{2}}}.$$

To estimate II we consider two cases:

- (i) If $|x| < 2|t|^{\frac{1}{3}}$, then $\Omega_2(x, t) = \emptyset$ and thus $II = 0$.
- (ii) If $|x| \geq 2|t|^{\frac{1}{3}}$, we apply Van der Corput's lemma with respect to ξ . For this we observe that in Ω_2 :

$$\left| \frac{\partial^2}{\partial \xi^2} (t\varphi_1(\zeta) + x\xi) \right| = |t(6\xi + 2\frac{\eta^2}{\xi^3})| \geq 6|t||\xi| \geq C|t| \left| \frac{x}{t} \right|^{\frac{1}{2}} = C|x t|^{\frac{1}{2}}.$$

In this way,

$$\begin{aligned} |II| &\leq \int_0^{c|\frac{x}{t}|} \left| \int_{a_1(\eta)}^{a_2(\eta)} e^{i[t\varphi_1(\zeta) + x\xi]} |\xi|^{-\frac{3}{2}} \alpha(\zeta) \psi_2(\zeta) d\xi \right| d\eta \\ &\leq C \int_0^{c|\frac{x}{t}|} |x t|^{-\frac{1}{4}} \| |\xi|^{-\frac{3}{2}} \alpha(\zeta) \psi_2(\xi, \eta) \|_{L^\infty(a_1, a_2)} d\eta \\ &\leq C \int_0^{c|\frac{x}{t}|} |x t|^{-\frac{1}{4}} \left| \frac{x}{t} \right|^{-\frac{3}{4}} d\eta = \frac{C}{|t|^{\frac{1}{2}}}. \end{aligned}$$

For III , after integrating by parts with respect to ξ , we have:

$$|III| = \left| \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} \frac{\partial}{\partial \xi} \left[\frac{|\xi|^{-\frac{3}{2}} \alpha(\zeta) \psi_3(\zeta)}{(t \frac{\partial \varphi_1}{\partial \xi} + x)} \right] d\zeta \right|.$$

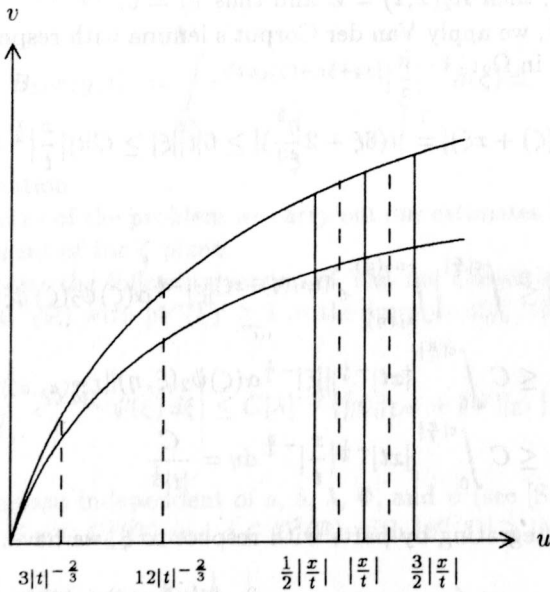
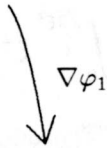
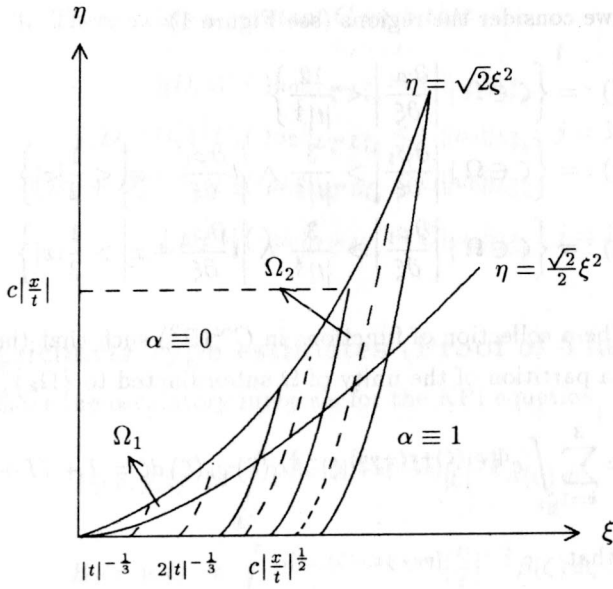


FIGURE 1.

The derivative in the integral above gives rise to terms of the following types:

$$\frac{\alpha\psi_3\xi^{-\frac{3}{2}}}{\left(t\frac{\partial\varphi_1}{\partial\xi}+x\right)}, \quad \frac{\alpha_\xi\psi_3\xi^{-\frac{3}{2}}}{\left(t\frac{\partial\varphi_1}{\partial\xi}+x\right)}, \quad \frac{\alpha\psi_3\xi^{-\frac{5}{2}}}{\left(t\frac{\partial\varphi_1}{\partial\xi}+x\right)}, \quad \frac{\alpha\psi_3\xi^{-\frac{3}{2}}t\frac{\partial^2\varphi_1}{\partial\xi^2}}{\left(t\frac{\partial\varphi_1}{\partial\xi}+x\right)^2},$$

which respectively originate 4 integrals III_j , $j = 0, 1, 2, 3$, with

$$|III| \leq |III_0| + \cdots + |III_3|.$$

As in *II* we consider the cases (i) and (ii). We restrict ourselves to (i) since (ii) is similar.

In $\Omega_3(x, t)$: $|t\frac{\partial\varphi_1}{\partial\xi} + x| > C|t\frac{\partial\varphi_1}{\partial\xi}| \geq C|t|\xi^2$. If $|x| < 2|t|^{\frac{1}{3}}$, then $\Omega_2(x, t) = \emptyset$, and thus ψ_3 grows from 0 to 1 in a horizontal interval whose length is of order $|t|^{-\frac{1}{3}}$. Therefore ψ_3 may be constructed in such a way that $|\psi_{3_\xi}| \sim |t|^{\frac{1}{3}}$. We also observe that $\text{Area}\{\zeta \in \Omega \mid \psi_{3_\xi} \neq 0\} \sim |t|^{-1}$. Hence

$$|III_0| \leq C \int_{\{\zeta \mid \psi_{3_\xi}(\zeta) \neq 0\}} \frac{|\xi|^{-\frac{3}{2}}|t|^{\frac{1}{3}}}{|t|\left|\frac{\partial\varphi_1}{\partial\xi}\right|} d\zeta.$$

If $\psi_{3_\xi}(\zeta) \neq 0$, then $|\xi| \sim |t|^{-\frac{1}{3}}$ and $\left|\frac{\partial\varphi_1}{\partial\xi}\right| \sim |t|^{-\frac{2}{3}}$, and thus:

$$|III_0| \leq \frac{C}{|t|^{\frac{1}{2}}}.$$

On the other side:

$$\begin{aligned} |III_1| &\leq C \int_{|c| |t|^{-\frac{2}{3}}}^{\infty} \int_{c_1 \eta^{\frac{1}{2}}}^{c_2 \eta^{\frac{1}{2}}} \frac{\eta^{-\frac{1}{2}} \xi^{-\frac{3}{2}}}{\left|t\frac{\partial\varphi_1}{\partial\xi}\right|} d\xi d\eta \leq \frac{C}{|t|} \int_{|c| |t|^{-\frac{2}{3}}}^{\infty} \eta^{-\frac{1}{2}} \int_{c_1 \eta^{\frac{1}{2}}}^{c_2 \eta^{\frac{1}{2}}} \frac{\xi^{-\frac{3}{2}}}{\xi^2} d\xi d\eta \\ &\leq \frac{C}{|t|^{\frac{1}{2}}}. \end{aligned}$$

$$\begin{aligned} |III_2| &\leq C \int_{\Omega_3(x, t)} \frac{|\xi|^{-\frac{5}{2}}}{\left|t\frac{\partial\varphi_1}{\partial\xi}\right|} d\zeta \leq C \int_{\Omega_3} \frac{|\xi|^{-\frac{5}{2}}}{|t|\xi^2} d\zeta \leq \frac{C}{|t|} \int_{|c| |t|^{-\frac{1}{3}}}^{\infty} \int_0^{\sqrt{2}\xi^2} \frac{1}{\xi^{\frac{3}{2}}} d\eta d\xi \\ &= \frac{C}{|t|^{\frac{1}{2}}}. \end{aligned}$$

A similar estimate follows for III_3 if we take into account that in Ω , $\left|\frac{\partial^2\varphi_1}{\partial\xi^2}\right| = \left|6\xi + 2\frac{\eta^2}{\xi^3}\right| < 10|\xi|$. Hence, we conclude that

$$|A_1(x, y, t)| \leq \frac{C}{|t|^{\frac{1}{2}}}. \quad (1.1)$$

To estimate B_1 we consider the regions (see Figure 2)

$$\begin{aligned}\Gamma_1^1(t) &:= \left\{ \zeta \in \Gamma \mid \left| \frac{\partial \varphi_1}{\partial \eta} \right| < \frac{2}{|t|^{\frac{2}{3}}} \right\}, \\ \Gamma_2^1(y, t) &:= \left\{ \zeta \in \Gamma \mid \left| \frac{\partial \varphi_1}{\partial \eta} \right| > \frac{1}{|t|^{\frac{2}{3}}} \wedge \left| t \frac{\partial \varphi_1}{\partial \eta} + y \right| < \frac{1}{2}|y| \right\}, \\ \Gamma_3^1(y, t) &:= \left\{ \zeta \in \Gamma \mid \left| \frac{\partial \varphi_1}{\partial \eta} \right| > \frac{1}{|t|^{\frac{2}{3}}} \wedge \left| t \frac{\partial \varphi_1}{\partial \eta} + y \right| > \frac{1}{3}|y| \right\}.\end{aligned}$$

and functions $\gamma_1, \gamma_2, \gamma_3 \in C^\infty(\mathbb{R}^2 - \{0\})$ whose restrictions to Γ form a partition of the unity of Γ subordinated to $\{\Gamma_k^1\}_{k=1}^3$. Then,

$$B_1(x, y, t) = \sum_{k=1}^3 \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} \left| \frac{\eta}{\xi} \right|^{-\frac{3}{2}} \beta(\zeta) \gamma_k(\zeta) d\zeta = I^* + II^* + III^*.$$

For I^* we have

$$\begin{aligned}|I^*| &= \left| \int_{\Gamma_1^1} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} \left| \frac{\eta}{\xi} \right|^{-\frac{3}{2}} \beta(\zeta) \gamma_1(\zeta) d\zeta \right| \leq \int_0^{\frac{\sqrt{2}}{2}|t|^{-\frac{1}{3}}} \int_{\frac{\sqrt{2}}{2}\xi^2}^{\xi|t|^{-\frac{1}{3}}} \left| \frac{\eta}{\xi} \right|^{-\frac{3}{2}} d\eta d\xi \\ &\leq \frac{C}{|t|^{\frac{1}{2}}}.\end{aligned}$$

To estimate II^* we see first that $\Gamma_2^1(y, t) = \emptyset$ when $|y| < \frac{2}{3}|t|^{\frac{2}{3}}$. If $|y| \geq \frac{2}{3}|t|^{\frac{2}{3}}$, we apply Van der Corput's lemma with respect to the variable η . For it we observe that

$$\left| \frac{\partial^2}{\partial \eta^2} [t\varphi_1(\zeta) + y\eta] \right| = 2 \left| \frac{t}{\xi} \right|.$$

Besides, in $\Gamma_2^1(y, t)$, $\left| \frac{\eta}{\xi} \right| \sim \left| \frac{y}{t} \right|$. Thus, if for fixed ξ and $(\eta, \xi) \in \Gamma_2^1(y, t)$, η ranges between two numbers $a_1(y, t, \xi)$ and $a_2(y, t, \xi)$, then:

$$\begin{aligned}|II^*| &= \left| \int_0^{c|\frac{y}{t}|} e^{ix\xi} \int_{a_1}^{a_2} e^{i[t\varphi_1(\zeta) + y\eta]} \left| \frac{\eta}{\xi} \right|^{-\frac{3}{2}} \beta(\zeta) \gamma_2(\zeta) d\eta d\xi \right| \\ &\leq C \int_0^{c|\frac{y}{t}|} \left| \frac{t}{\xi} \right|^{-\frac{1}{2}} \left| \frac{y}{t} \right|^{-\frac{3}{2}} d\xi \leq \frac{C}{|t|^{\frac{1}{2}}}.\end{aligned}$$

The integral III^* is estimated by integration by parts with respect to η :

$$|III^*| = \left| \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} \frac{\partial}{\partial \eta} \left[\frac{\left| \frac{\eta}{\xi} \right|^{-\frac{3}{2}} \beta(\zeta) \gamma_3(\zeta)}{\left(t \frac{\partial \varphi_1}{\partial \eta} + y \right)} \right] d\zeta \right|.$$

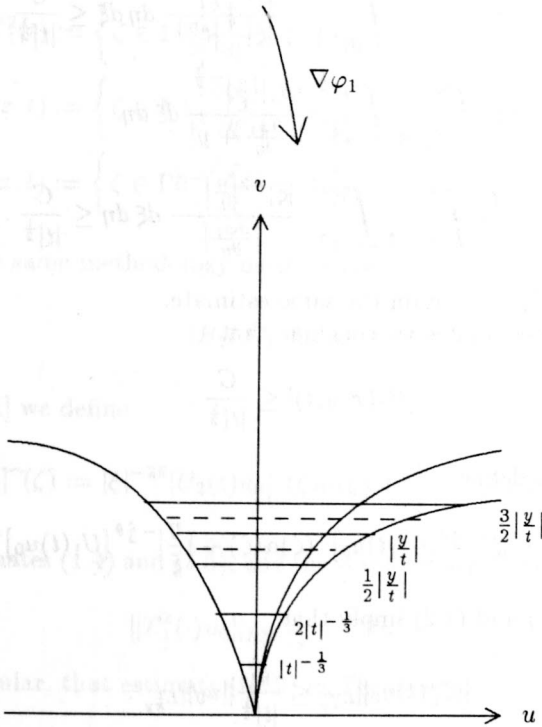
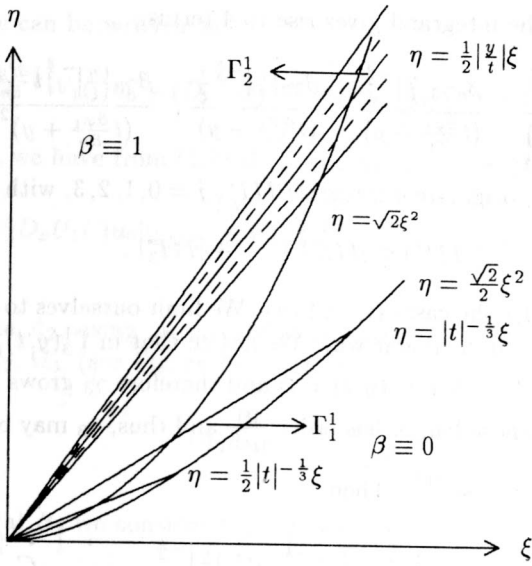


FIGURE 2.

The derivative in the integrand gives rise to 4 terms:

$$\frac{\beta\gamma_3\eta\left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}}}{\left(t\frac{\partial\varphi_1}{\partial\eta}+y\right)}, \quad \frac{\beta_\eta\gamma_3\left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}}}{\left(t\frac{\partial\varphi_1}{\partial\eta}+y\right)}, \quad \frac{\beta\gamma_3\left|\frac{\eta}{\xi}\right|^{-\frac{5}{2}}\frac{1}{\xi}}{\left(t\frac{\partial\varphi_1}{\partial\eta}+y\right)}, \quad \frac{\beta\gamma_3\left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}}t\frac{\partial^2\varphi_1}{\partial\eta^2}}{\left(t\frac{\partial\varphi_1}{\partial\eta}+y\right)^2},$$

which, respectively, originate 4 integrals III_j^* , $j = 0, 1, 2, 3$, with

$$|III^*| \leq |III_0^*| + \dots + |III_3^*|.$$

As in II^* we consider the cases (i) and (ii). We limit ourselves to (i) since both cases can be treated in a similar way. We notice that in $\Gamma_3^1(y, t)$, $|t\frac{\partial\varphi_1}{\partial\eta}+y| \geq C|t\frac{\partial\varphi_1}{\partial\eta}|$. If $|y| < \frac{2}{3}|t|^{\frac{2}{3}}$, then $\Gamma_2^1(y, t) = \emptyset$ and therefore γ_3 grows from 0 to 1 in a vertical interval whose length has order $\frac{|\xi|}{|t|^{\frac{1}{3}}}$ and thus, γ_3 may be constructed

in such a way that $\gamma_{3,\eta} \sim \frac{|t|^{\frac{1}{3}}}{|\xi|}$. Then

$$\begin{aligned} |III_0^*| &\leq C \int_0^{c|t|^{-\frac{1}{3}}} \int_{\xi|t|^{-\frac{1}{3}}}^{2\xi|t|^{-\frac{1}{3}}} \frac{|t|^{\frac{1}{3}}\left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}}}{|\xi|\left|\frac{\eta}{\xi}\right|} d\eta d\xi \leq \frac{C}{|t|^{\frac{1}{2}}}. \\ |III_1^*| &\leq C \int_{c_3|t|^{-\frac{2}{3}}}^\infty \int_{c_1\eta^{\frac{1}{2}}}^{c_2\eta^{\frac{1}{2}}} \frac{|\beta_\eta|\left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}}}{\left|t\frac{\partial\varphi_1}{\partial\eta}+y\right|} d\xi d\eta \\ &\leq \frac{C}{|t|} \int_{c_3|t|^{-\frac{2}{3}}}^\infty \int_{c_1\eta^{\frac{1}{2}}}^{c_2\eta^{\frac{1}{2}}} \frac{|\xi|^{-2}\left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}}}{\left|\frac{\partial\varphi_1}{\partial\eta}\right|} d\xi d\eta \leq \frac{C}{|t|^{\frac{1}{2}}}. \end{aligned}$$

For $|III_2^*|$ and $|III_3^*|$ we obtain the same estimate.

From the above estimates we conclude that

$$|B_1(x, y, t)| \leq \frac{C}{|t|^{\frac{1}{2}}}. \tag{1.2}$$

For $\theta \in [0, 1]$ let us define

$$[T_\theta^1(t)u_0]^\wedge(\zeta) := |\xi|^{-\frac{3}{2}\theta} [U_1(t)u_0]^\wedge(\zeta)\alpha(\zeta) + \left|\frac{\eta}{\xi}\right|^{-\frac{3}{2}\theta} [U_1(t)u_0]^\wedge(\zeta)\beta(\zeta).$$

The estimates (1.1) and (1.2) imply that

$$\|T_1^1(t)u_0\|_{L_{xy}^\infty} \leq \frac{C}{|t|^{\frac{1}{2}}} \|u_0\|_{L_{xy}^1}.$$

From a standard argument (see [KPV1]) it follows that

$$\|T_{\frac{1}{2}}^1(t)u_0\|_{L_t^1 L_{xy}^\infty} \leq C \|u_0\|_{L_{xy}^2}.$$

This estimate can be written as:

$$\|D_x^{-\frac{3}{4}}W_1(\cdot)u_0 + (D_x^{-1}D_y)^{-\frac{3}{4}}V_1(\cdot)u_0\|_{L_t^4L_{xy}^\infty} \leq C\|u_0\|_{L_{xy}^2}. \quad (1.3)$$

In particular, we have from (1.3) that estimate (0.11) in Theorem 1,

$$\|D_xU_1(\cdot)u_0\|_{L_t^4L_{xy}^\infty} \leq C(\|D_x^{\frac{7}{4}}u_0\|_{L_{xy}^2} + \|D_y^{\frac{7}{8}}u_0\|_{L_{xy}^2}),$$

takes place.

To estimate A_2 , using φ_2 instead of φ_1 , we again divide the region Ω into regions $\Omega_1, \Omega_2, \Omega_3$, (see Figure 3), to obtain

$$|A_2(x, y, t)| \leq \frac{C}{|t|^{\frac{1}{3}}}. \quad (1.4)$$

For the integral B_2 we consider the regions (see Figure 4)

$$\begin{aligned} \Gamma_1^2(t) &:= \left\{ \zeta \in \Gamma \mid \left| \frac{\partial\varphi_1}{\partial\xi} \right| < \frac{12}{|t|^{2/3}} \right\}, \\ \Gamma_2^2(x, t) &:= \left\{ \zeta \in \Gamma \mid \left| \frac{\partial\varphi_1}{\partial\xi} \right| > \frac{3}{|t|^{2/3}} \wedge \left| t \frac{\partial\varphi_1}{\partial\xi} + x \right| < \frac{1}{2}|x| \right\}, \\ \Gamma_3^2(x, t) &:= \left\{ \zeta \in \Gamma \mid \left| \frac{\partial\varphi_1}{\partial\xi} \right| < \frac{3}{|t|^{2/3}} \wedge \left| t \frac{\partial\varphi_1}{\partial\xi} + x \right| > \frac{1}{3}|x| \right\}. \end{aligned}$$

Following the same methodology used for the linearized KPI equation we conclude that

$$|B_2(x, y, t)| \leq \frac{C}{|t|^{\frac{1}{3}}}. \quad (1.5)$$

If for $\theta \in [0, 1]$ we define

$$[T_\theta^2(t)u_0]^\wedge(\zeta) := |\xi|^{-2\theta}[U_2(t)u_0]^\wedge(\zeta)\alpha(\zeta) + \left|\frac{\eta}{\xi}\right|^{-2\theta}[U_2(t)u_0]^\wedge(\zeta)\beta(\zeta),$$

then the estimates (1.4) and (1.5), and the standard argument mentioned above imply that

$$\|T_\theta^2(\cdot)u_0\|_{L_t^4L_{xy}^\infty} \leq C\|u_0\|_{L_{xy}^2},$$

and, in particular, that estimate (0.12) in Theorem 1,

$$\|D_xU_2(\cdot)u_0\|_{L_t^4L_{xy}^\infty} \leq C(\|D_x^2u_0\|_{L_{xy}^2} + \|D_yu_0\|_{L_{xy}^2}),$$

holds.

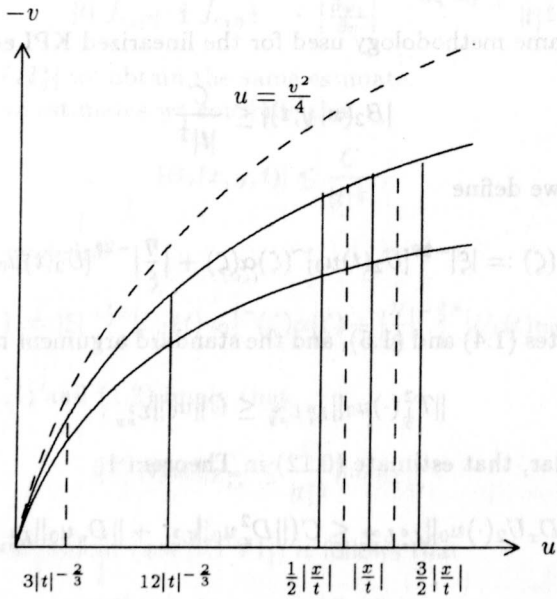
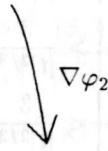
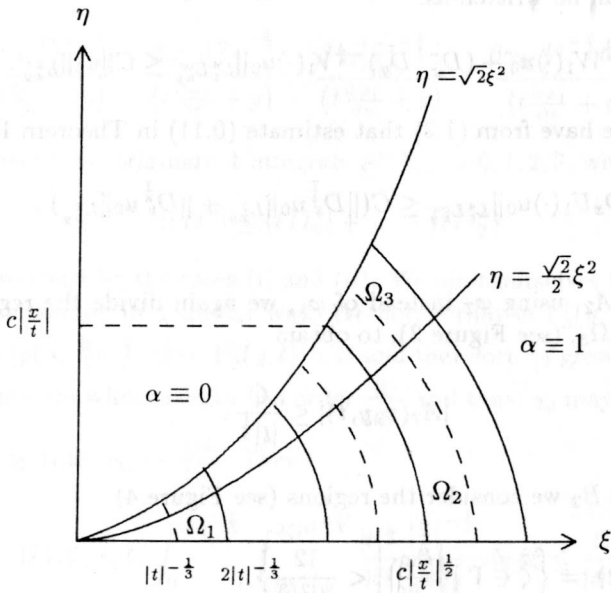


FIGURE 3.

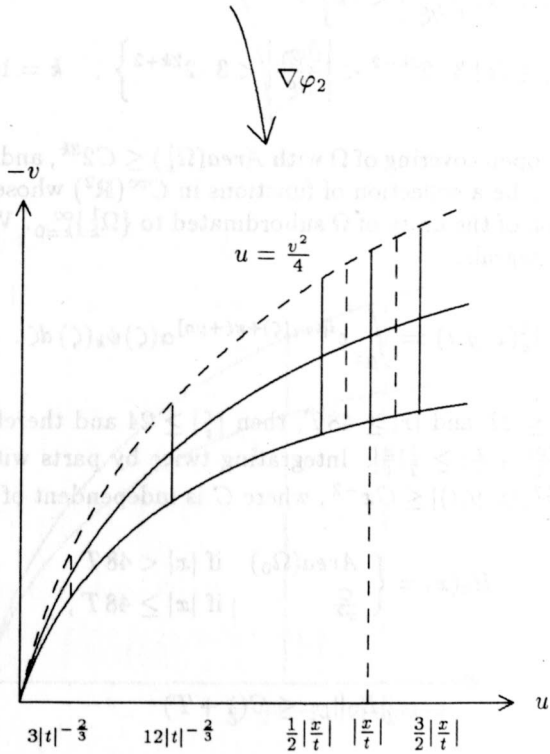
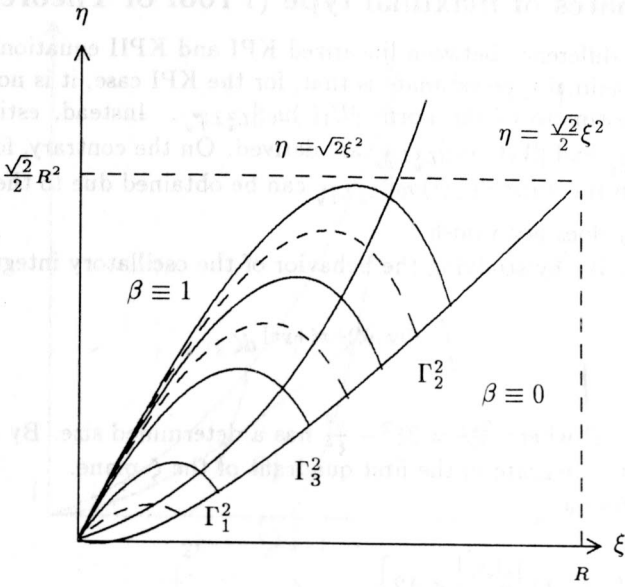


FIGURE 4.

2. Estimates of maximal type (Proof of Theorem 2)

The central difference between linearized KPI and KP II equations with respect to the maximal type estimate is that, for the KPI case, it is not possible to obtain an estimate of the norm $\|U_1(\cdot)u_0\|_{L_x^2 L_y^\infty}$. Instead, estimates for $\|W_1(\cdot)u_0\|_{L_x^2 L_y^\infty}$ and $\|V_1(\cdot)u_0\|_{L_y^2 L_x^\infty}$ are derived. On the contrary, for KP II, a full estimate on the norm $\|U_2(\cdot)u_0\|_{L_x^2 L_y^\infty}$ can be obtained due to the fact that $\frac{\partial \varphi_1}{\partial \xi} = 3\xi^2 + \frac{\eta^2}{\xi^2}$ does not vanish.

We estimate W_1 by studying the behavior of the oscillatory integral

$$\int e^{i[t\varphi_1(\zeta)+x\xi+y\eta]} d\zeta,$$

in subregions of Ω where $\frac{\partial \varphi_1}{\partial \xi} = 3\xi^2 - \frac{\eta^2}{\xi^2}$ has a determined size. By symmetry it is sufficient to integrate in the first quadrant of the ζ plane.

Let (see Figure 5)

$$\begin{aligned} \Omega_0^1 &:= \left\{ \zeta \in \Omega \mid \left| \frac{\partial \varphi_1}{\partial \xi} \right| < 12 \right\}, \\ \Omega_k^1 &:= \left\{ \zeta \in \Omega \mid 3 \cdot 2^{2k-2} < \left| \frac{\partial \varphi_1}{\partial \xi} \right| < 3 \cdot 2^{2k+2} \right\}, \quad k = 1, 2, \dots \end{aligned}$$

$\{\Omega_k^1\}_{k=0}^\infty$ is an open covering of Ω with $Area(\Omega_k^1) \leq C2^{3k}$, and C independent of k . Let $\{\psi_k\}_{k=0}^\infty$ be a collection of functions in $C^\infty(\mathbb{R}^2)$ whose restrictions to Ω form a partition of the unity of Ω subordinated to $\{\Omega_k^1\}_{k=0}^\infty$. We will estimate the oscillatory integrals:

$$I_k^1(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta)+x\xi+y\eta]} \alpha(\zeta) \psi_k(\zeta) d\zeta.$$

For $k = 0$, if $|t| \leq 2T$ and $|x| \geq 48T$, then $|\frac{x}{t}| \geq 24$ and therefore, for $\zeta \in \Omega_0^1$ we have that $|\frac{\partial \varphi_1}{\partial \xi} + \frac{x}{t}| \geq \frac{1}{2}|\frac{x}{t}|$. Integrating twice by parts with respect to ξ , we obtain that $|I_0^1(x, y, t)| \leq Cx^{-2}$, where C is independent of T . If we define

$$H_0(x) = \begin{cases} Area(\Omega_0) & \text{if } |x| < 48T \\ \frac{C}{x^2} & \text{if } |x| \geq 48T, \end{cases}$$

then

$$\|H_0\|_{L_x^1} \leq C(1+T) \tag{2.1}$$

and

$$|I_0^1(x, y, t)| \leq CH_0(x) \quad \text{for } |t| \leq 2T \text{ and } (x, y) \in \mathbb{R}^2. \tag{2.2}$$

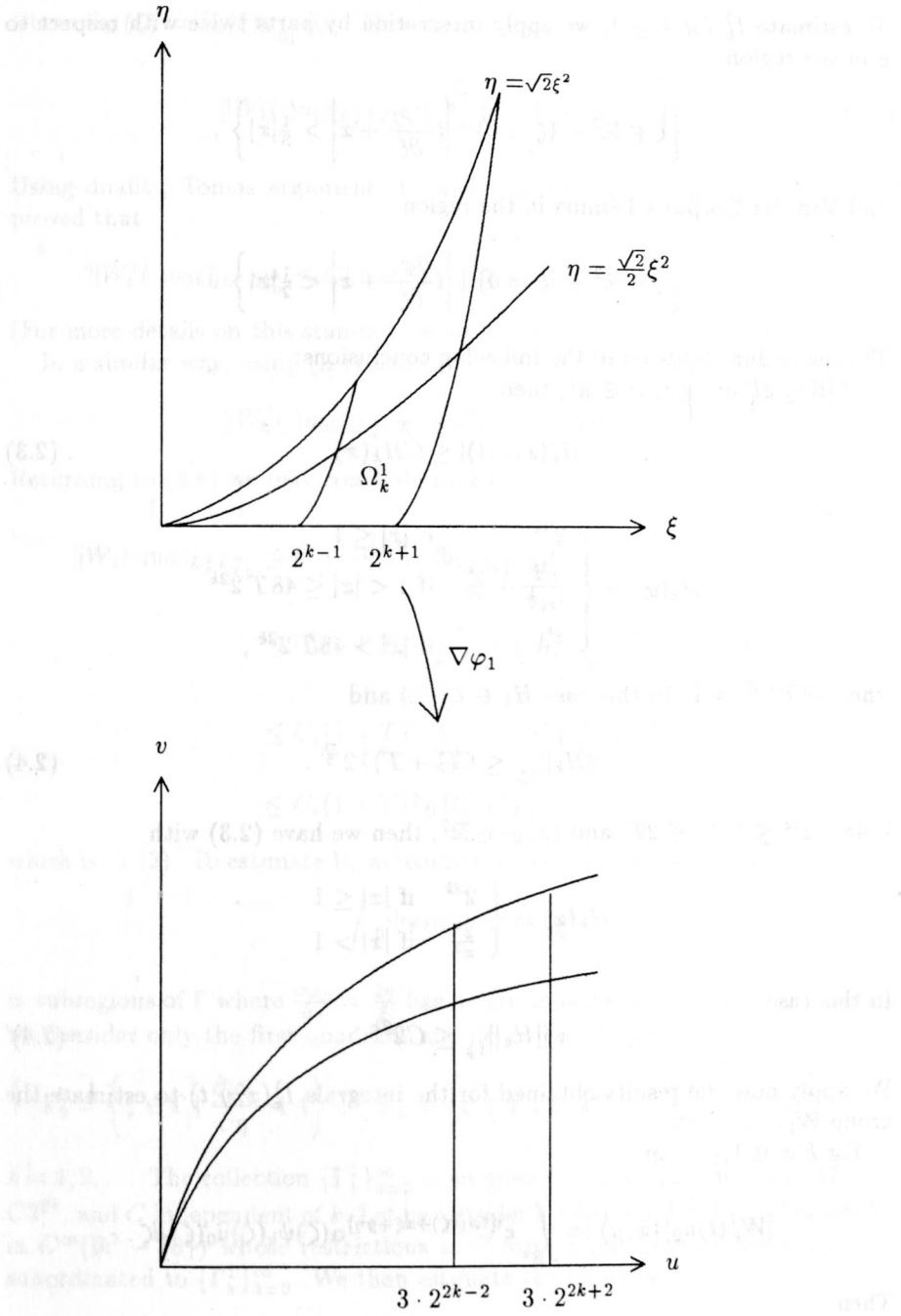


FIGURE 5.

To estimate I_k^1 for $k \geq 1$, we apply integration by parts twice with respect to ξ in the region

$$\left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \left| t \frac{\partial \varphi_1}{\partial \xi} + x \right| > \frac{1}{3} |x| \right\},$$

and Van der Corput's Lemma in the region

$$\left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \left| t \frac{\partial \varphi_1}{\partial \xi} + x \right| < \frac{1}{2} |x| \right\}.$$

This procedure leads us to the following conclusions:

If $|t| \leq 2T$ and $(x, y) \in \mathbb{R}^2$, then

$$|I_k^1(x, y, t)| \leq CH_k(x), \quad (2.3)$$

where

$$H_k(x) = \begin{cases} 2^{3k} & \text{if } |x| \leq 1 \\ \frac{2^{\frac{5k}{2}}}{|x|^{\frac{1}{2}}} + \frac{2^k}{x^2} & \text{if } 1 < |x| \leq 48T2^{2k} \\ \frac{2^k}{x^2} & \text{if } |x| > 48T2^{2k}, \end{cases}$$

when $48T2^{2k} > 1$. In this case $H_k \in L^1(\mathbb{R})$ and

$$\|H_k\|_{L^1_{\frac{1}{2}}} \leq C(1+T)^{\frac{1}{2}} 2^{\frac{7k}{2}}. \quad (2.4)$$

If $48T2^{2k} \leq 1$, $|t| \leq 2T$, and $(x, y) \in \mathbb{R}^2$, then we have (2.3) with

$$H_k(x) = \begin{cases} 2^{3k} & \text{if } |x| \leq 1 \\ \frac{2^k}{x^2} & \text{if } |x| > 1. \end{cases}$$

In this case

$$\|H_k\|_{L^1_{\frac{1}{2}}} \leq C2^{3k}. \quad (2.4)'$$

We apply now the results obtained for the integrals $I_k^1(x, y, t)$ to estimate the group W_1 .

For $k = 0, 1, \dots$, let

$$[W_k^1(t)u_0](x, y) := \int_{\mathbb{R}^2} e^{i[t\varphi_1(\zeta) + x\xi + y\eta]} \alpha(\zeta) \psi_k(\zeta) \hat{u}_0(\zeta) d\zeta.$$

Then

$$[W_1(t)u_0](x, y) = \sum_{k=0}^{\infty} [W_k^1(t)u_{0k}](x, y),$$

where $\widehat{u}_{0k}(\zeta) := \widehat{u}_0(\zeta)\chi_{\Omega_k^1}(\zeta)$. Therefore

$$\|W_1(\cdot)u_0\|_{L_x^2 L_y^\infty} \leq \sum_{k=0}^{\infty} \|W_k^1(\cdot)u_{0k}\|_{L_x^2 L_y^\infty}. \quad (2.5)$$

Using duality, Tomas argument [T], and, taking into account (2.4) it can be proved that

$$\|W_k^1(\cdot)u_{0k}\|_{L_x^2 L_y^\infty} \leq C(1+T)^{\frac{1}{2}} 2^{\frac{7k}{4}} \|u_{0k}\|_{L_{x,y}^2}, \quad k = 0, 1, 2, \dots$$

(For more details on this standard procedure see, for example, [KPV2]).

In a similar way, using (2.1) and (2.2), it may be seen that

$$\|W_0^1(\cdot)u_{00}\|_{L_x^2 L_y^\infty} \leq C(1+T)^{\frac{1}{2}} \|u_{00}\|_{L_{x,y}^2}.$$

Returning to (2.5) we may conclude that for $s > \frac{7}{4}$:

$$\begin{aligned} \|W_1(\cdot)u_0\|_{L_x^2 L_y^\infty} &\leq C \left[(1+T)^{\frac{1}{2}} \|\widehat{u}_{00}\|_{L_\zeta^2} + (1+T)^{\frac{1}{2}} \sum_{k=1}^{\infty} 2^{\frac{7k}{4}} \|\widehat{u}_{0k}\|_{L_\zeta^2} \right] \\ &\leq C_s (1+T)^{\frac{1}{2}} \sum_{k=0}^{\infty} 2^{\frac{7k}{4}} \left(\int_{\Omega_k^1} \left(\frac{\xi^2}{2^{2k}} \right)^s |\widehat{u}_0(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\leq C_s (1+T)^{\frac{1}{2}} \sum_{k=0}^{\infty} 2^{(\frac{7}{4}-s)k} \left(\int_{\mathbb{R}^2} |\xi|^{2s} |\widehat{u}_0(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\leq C_s (1+T)^{\frac{1}{2}} \|D_x^s u_0\|_{L_{x,y}^2}, \end{aligned}$$

which is (0.13). To estimate V_1 we study the behavior of the oscillatory integral

$$\int e^{i[t\varphi_1(\zeta)+x\xi+y\eta]} d\zeta$$

in subregions of Γ where $\frac{\partial\varphi_1}{\partial\eta} = \frac{2\eta}{\xi}$ has a determined size. Again, by symmetry, we consider only the first quadrant and define (see Figure 6)

$$\Gamma_0^1 = \left\{ \zeta \in \Gamma \mid \frac{\partial\varphi_1}{\partial\eta} < 4 \right\}, \quad \Gamma_k^1 = \left\{ \zeta \in \Gamma \mid 2 \cdot 2^{k-1} < \frac{\partial\varphi_1}{\partial\eta} < 2 \cdot 2^{k+1} \right\},$$

$k = 1, 2, \dots$. The collection $\{\Gamma_k^1\}_{k=0}^{\infty}$ is an open covering of Γ with $Area(\Gamma_k^1) \leq C2^{3k}$, and C independent of k . Let us consider a collection $\{\tilde{\psi}_k\}_{k=0}^{\infty}$ of functions in $C^\infty(\mathbb{R}^2 - \{0\})$ whose restrictions to Γ form a partition of the unity of Γ subordinated to $\{\Gamma_k^1\}_{k=0}^{\infty}$. We then estimate the integrals

$$J_k^1(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\phi_1(\zeta)+x\xi+y\eta]} \beta(\zeta) \tilde{\psi}_k(\zeta) d\zeta.$$

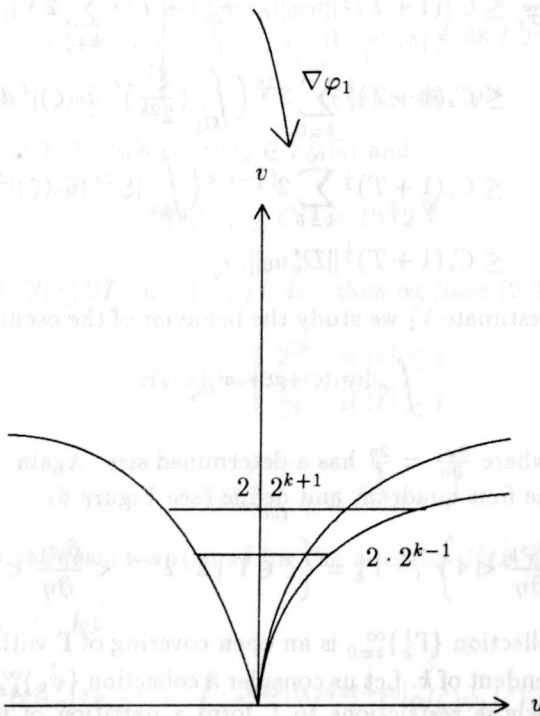
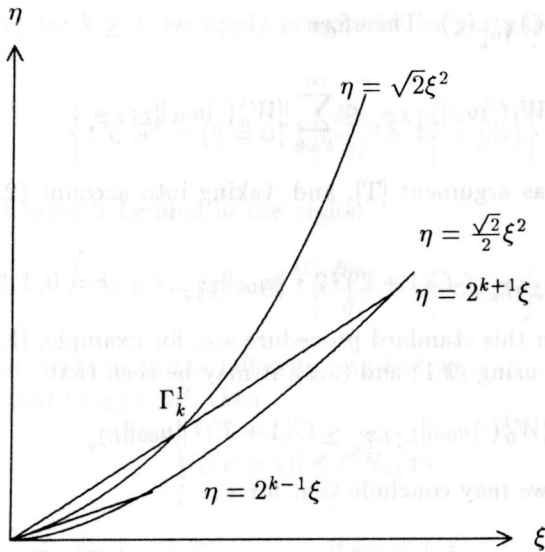


FIGURE 6.

For $k \geq 1$, if $|y| \leq 2^{-\frac{k}{2}}$, we estimate J_k^1 by the area of Γ_k^1 . If $|y| > 2^{-\frac{k}{2}}$, we write the integral defining J_k^1 as the sum of the integral in the region $\{|\xi| < \frac{2^{-k}}{|y|}\}$ which is bounded by $C2^{-k}y^{-2}$, and the integral in $\{|\xi| > \frac{2^{-k}}{|y|}\}$ which can be estimated by applying integration by parts twice with respect to η in the region

$$\left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \left| t \frac{\partial \varphi_1}{\partial \eta} + y \right| > \frac{1}{3}|y| \right\} \cup \{\xi = 0\},$$

and Van der Corput's Lemma with respect to η in the region

$$\left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \left| t \frac{\partial \varphi_1}{\partial \eta} + y \right| < \frac{1}{2}|y| \right\}.$$

With this procedure we conclude that if $|t| \leq 2T$, then

$$|J_k^1(x, y, t)| \leq C\tilde{H}_k(y), \tag{2.6}$$

where

$$\tilde{H}_k(y) = \begin{cases} 2^{3k} & \text{if } |y| \leq 2^{-\frac{k}{2}} \\ \frac{2^{-k}}{y^2} \log(\sqrt{2} \cdot 2^{2k}|y|) + \frac{2^{2k}}{|y|^{\frac{3}{2}}} & \text{if } 2^{-\frac{k}{2}} < |y| \leq 16T2^k \\ \frac{2^{-k}}{y^2} \log(\sqrt{2} \cdot 2^{2k}|y|) & \text{if } |y| > 16T2^k, \end{cases}$$

when $2^{-\frac{k}{2}} < 16T2^k$, and

$$\tilde{H}_k(y) = \begin{cases} 2^{3k} & \text{if } |y| \leq 2^{-\frac{k}{2}} \\ \frac{2^{-k}}{y^2} \log(\sqrt{2} \cdot 2^{2k}|y|) & \text{if } |y| > 2^{-\frac{k}{2}}, \end{cases}$$

if $2^{-\frac{k}{2}} \geq 16T2^k$.

We observe that

$$\|\tilde{H}_k\|_{L^1_y} \leq C(1+T)^{\frac{1}{2}}2^{\frac{5k}{2}}. \tag{2.7}$$

For $k = 0$, if $|y| \leq 1$, then $|J_0^1(x, y, t)| \leq C$. If $|y| > 1$, the integral defining J_0^1 can be divided as the sum of the integral in the region $\{|\xi| \leq \sqrt{2}|y|^{-2/3}\}$, which is bounded by $C|y|^{-4/3}$ - a bound for the area of this region-, and the integral in the region $\{|\xi| > \sqrt{2}|y|^{-2/3}\}$ which can be estimated by applying integration by parts twice with respect to η , to obtain finally that, for $|t| \leq 2T$,

$$|J_0^1(x, y, t)| \leq C\tilde{H}_0(y), \tag{2.8}$$

where

$$\tilde{H}_0(y) = \begin{cases} 1 & \text{if } |y| \leq 16T \\ \frac{T^{\frac{4}{3}}}{|y|^{\frac{4}{3}}} & \text{if } |y| > 16T, \end{cases}$$

when $T > \frac{1}{16}$, and

$$\tilde{H}_0(y) := \begin{cases} 1 & \text{if } |y| \leq 1 \\ \frac{1}{|y|^{\frac{3}{4}}} & \text{if } |y| > 1, \end{cases}$$

when $T \leq \frac{1}{16}$.

The function $\tilde{H}_0 \in L^1_y$, and

$$\|\tilde{H}_0\|_{L^1_y} \leq C(1+T). \tag{2.9}$$

We now apply the obtained results about the oscillatory integrals $J_k^1(x, y, t)$ to estimate the group V_1 .

By a procedure similar to that followed to estimate W_1 , and using (2.6)–(2.9), we have that

$$\|V_1(\cdot)u_0\|_{L^2_y L^{\infty}_{T_x}} \leq C \left[(1+T)^{\frac{1}{2}} \|\hat{u}_{00}^*\|_{L^2_\zeta} + (1+T)^{\frac{1}{4}} \sum_{k=1}^{\infty} 2^{\frac{3k}{4}} \|\hat{u}_{0k}^*\|_{L^2_\zeta} \right],$$

where $\hat{u}_{0k}^* = \hat{u}_0 \chi_{\Gamma_k^1}$. Therefore, if $s > 5/4$, then

$$\begin{aligned} \|V_1(\cdot)u_0\|_{L^2_y L^{\infty}_{T_x}} &\leq C(1+T)^{\frac{1}{2}} \sum_{k=0}^{\infty} 2^{(\frac{5}{4}-s)k} \left(\int_{\Gamma_k^1} 2^{2ks} |\hat{u}_0(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\leq C_s(1+T)^{\frac{1}{2}} \sum_{k=0}^{\infty} 2^{(\frac{5}{4}-s)k} \left(\int_{\Gamma_k^1} \left| \frac{\eta}{\xi} \right|^{2s} |\hat{u}_0(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\leq C_s(1+T)^{\frac{1}{2}} \|(D_x^{-1}D_y)^s u_0\|_{L^2_{x,y}}, \end{aligned}$$

and (0.14) is proved.

To study the group U_2 we consider the sets (see Figure 7)

$$\Omega_k^2 := \left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \frac{\partial \varphi_2}{\partial \xi} < 12 \right\}$$

and, for $k = 1, 2, \dots$,

$$\Omega_k^2 := \left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid 3 \cdot 2^{2k-2} < \frac{\partial \varphi_2}{\partial \xi} < 3 \cdot 2^{2k+2} \right\}.$$

The collection $\{\Omega_k^2\}_{k=0}^{\infty}$ is an open covering of $\mathbb{R}^2 - \{\xi = 0\}$ such that $Area(\Omega_k^2) \leq C 2^{3k}$. Let ψ_1, ψ_2, \dots be functions in $C^\infty(\mathbb{R}^2 - \{0\})$ whose restrictions to $\mathbb{R}^2 - \{\xi = 0\}$ form a partition of the unity of $\mathbb{R}^2 - \{\xi = 0\}$ subordinated to $\{\Omega_k^2\}_{k=0}^{\infty}$. We estimate the oscillatory integrals

$$I_k^2(x, y, t) := \int_{\mathbb{R}^2} e^{i[t\varphi_2(\zeta) + x\xi + y\eta]} \psi_k(\zeta) d\zeta.$$

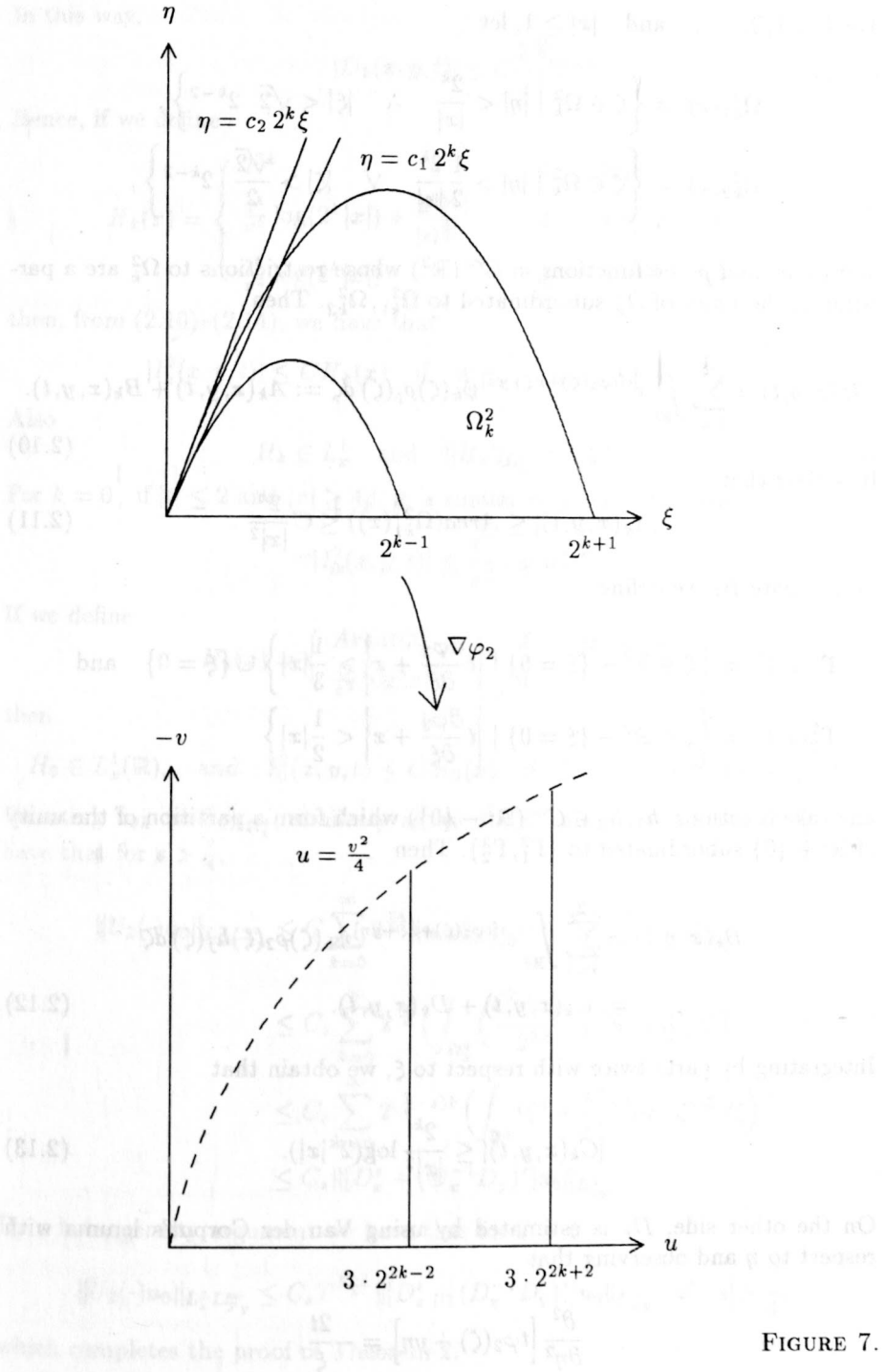


FIGURE 7.

For $k = 1, 2, \dots$, and $|x| \geq 1$, let

$$\begin{aligned}\Omega_{k1}^2(x) &:= \left\{ \zeta \in \Omega_k^2 \mid |\eta| < \frac{2^k}{|x|} \quad \wedge \quad |\xi| < \sqrt{2} \cdot 2^{k-2} \right\}, \\ \Omega_{k2}^2(x) &:= \left\{ \zeta \in \Omega_k^2 \mid |\eta| > \frac{1}{2} \frac{2^k}{|x|} \quad \vee \quad |\xi| > \frac{\sqrt{2}}{2} \cdot 2^{k-2} \right\},\end{aligned}$$

and let ρ_1 and ρ_2 be functions in $C^\infty(\mathbb{R}^2)$ whose restrictions to Ω_k^2 are a partition of the unity of Ω_k^2 subordinated to $\Omega_{k1}^2, \Omega_{k2}^2$. Then

$$I_k^2(x, y, t) = \sum_{j=1}^2 \int_{\mathbb{R}^2} e^{i[t\varphi_2(\zeta) + x\xi + y\eta]} \psi_k(\zeta) \rho_j(\zeta) d\zeta =: A_k(x, y, t) + B_k(x, y, t). \quad (2.10)$$

It is clear that

$$|A_k(x, y, t)| \leq \text{Area}(\Omega_{k1}^2(x)) \leq C \frac{2^k}{|x|^2}. \quad (2.11)$$

To estimate B_k we define

$$\begin{aligned}\Gamma_1^2(x, t) &:= \left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \left| t \frac{\partial \varphi_2}{\partial \xi} + x \right| > \frac{1}{3} |x| \right\} \cup \{\xi = 0\} \quad \text{and} \\ \Gamma_2^2(x, t) &:= \left\{ \zeta \in \mathbb{R}^2 - \{\xi = 0\} \mid \left| t \frac{\partial \varphi_2}{\partial \xi} + x \right| < \frac{1}{2} |x| \right\}\end{aligned}$$

and take functions $h_1, h_2 \in C^\infty(\mathbb{R}^2 - \{0\})$ which form a partition of the unity of $\mathbb{R}^2 - \{0\}$ subordinated to $\{\Gamma_1^2, \Gamma_2^2\}$. Then

$$\begin{aligned}B_k(x, y, t) &= \sum_{j=1}^2 \int_{\mathbb{R}^2} e^{i[t\varphi_2(\zeta) + x\xi + y\eta]} \psi_k(\zeta) \rho_2(\zeta) h_j(\zeta) d\zeta \\ &=: C_k(x, y, t) + D_k(x, y, t).\end{aligned} \quad (2.12)$$

Integrating by parts twice with respect to ξ , we obtain that

$$|C_k(x, y, t)| \leq \frac{2^k}{|x|^2} \log(2^k |x|). \quad (2.13)$$

On the other side, D_k is estimated by using Van der Corput's lemma with respect to η and observing that

$$\frac{\partial^2}{\partial \eta^2} [t\varphi_2(\zeta) + y\eta] = -\frac{2t}{\xi}.$$

In this way,

$$|D_k(x, y, t)| \leq C \frac{2^{3k}}{t^{\frac{1}{2}}}. \quad (2.14)$$

Hence, if we define

$$H_k(x) = \begin{cases} 2^{3k} & \text{if } |x| < 1 \\ \frac{2^k}{x^2} \log(2^k|x|) + \frac{2^{\frac{5k}{2}}}{|x|^{\frac{1}{2}}} & \text{if } 1 \leq |x| < 4 \cdot 2^{2k} \\ \frac{2^k}{x^2} \log(2^k|x|) & \text{if } 4 \cdot 2^{2k} \leq |x|, \end{cases}$$

then, from (2.10)–(2.14), we have that

$$|I_k^2(x, y, t)| \leq CH_k(x) \quad \text{if } |t| \leq 2 \quad \text{and } (x, y) \in \mathbb{R}^2. \quad (2.15)$$

Also

$$H_k \in L_x^1 \quad \text{and} \quad \|H_k\|_{L_x^1} \leq C 2^{\frac{7k}{2}}. \quad (2.16)$$

For $k = 0$, if $|t| \leq 2$ and $|x| \geq 48$, in a similar way, we obtain that

$$|I_0^2(x, y, t)| \leq \frac{C}{x^2} \log|x|.$$

If we define

$$H_0(x) = \begin{cases} \text{Area}(\Omega_0^2) & \text{if } |x| < 48 \\ \frac{C}{x^2} \log|x| & \text{if } |x| \geq 48, \end{cases}$$

then

$$H_0 \in L_x^1(\mathbb{R}), \quad \text{and} \quad I_0^2(x, y, t) \leq CH_0(x) \quad \text{if } |t| \leq 2 \quad \text{and } (x, y) \in \mathbb{R}^2.$$

Denoting $\widehat{u}_{0k} := \widehat{u}_0 \chi_{\Omega_k^2}$, similarly as we did to obtain (0.13) and (0.14), we have that for $s > \frac{7}{4}$,

$$\begin{aligned} \|U_2(\cdot)u_0\|_{L_x^2 L_T^\infty} &\leq C \sum_{k=0}^{\infty} 2^{\frac{7k}{4}} \|u_{0k}\|_{L_{xy}^2} = C \sum_{k=0}^{\infty} 2^{\frac{7k}{4}} \|\widehat{u}_{0k}\|_{L_\zeta^2} \\ &\leq C_s \sum_{k=0}^{\infty} 2^{\frac{7k}{4}} \left(\int_{\Omega_k^2} \left(\frac{\xi^2 + \frac{\eta^2}{\xi^2}}{2^{2k}} \right)^s |\widehat{u}_0(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\leq C_s \sum_{k=0}^{\infty} 2^{(\frac{7}{4}-s)k} \left(\int_{\mathbb{R}^2} (\xi^2 + \frac{\eta^2}{\xi^2})^s |\widehat{u}_0(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} \\ &\leq C_s \| [D_x^s + (D_x^{-1} D_y)^s] u_0 \|_{L_{xy}^2}. \end{aligned}$$

By a homogeneity argument we have that for $T > 0$:

$$\|U_2(\cdot)u_0\|_{L_x^2 L_T^\infty} \leq C_s T^{\frac{s-1}{3}} \| [D_x^s + (D_x^{-1} D_y)^s] u_0 \|_{L_{xy}^2} \quad \text{if } s > \frac{7}{4},$$

which completes the proof of Theorem 2.

3. Estimates of local type (Proof of Theorem 3)

To estimate (0.3) for $U_2(t)$, we divide \mathbb{R}^2 into the right and left halfplanes. We only integrate in the right half plane, the other estimate being similar. By the change of variable $\theta = \varphi_2(\zeta) = \xi^3 - \frac{\eta^2}{\xi}$, $\eta = \eta$, we obtain

$$[U_2(t)u_0](x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{it\theta} e^{i\xi(\theta, \eta)x} e^{i\eta y} \frac{\widehat{u}_0(\xi(\theta, \eta), \eta)}{(3\xi^2 + \frac{\eta^2}{\xi^2})} d\theta d\eta.$$

Applying Plancherel's theorem with respect to the variables t and y , it follows that for all x :

$$\begin{aligned} \|[U_2(\cdot)u_0](x, \cdot)\|_{L^2_{ty}} &= \left\| \frac{e^{i\xi(\cdot, \cdot)x} \widehat{u}_0(\xi(\cdot, \cdot), \cdot)}{3\xi^2 + \frac{\eta^2}{\xi^2}} \right\|_{L^2_{\theta\eta}} \\ &= \left(\int_{\mathbb{R}^2} \frac{|\widehat{u}_0(\xi, \eta)|^2}{(3\xi^2 + \frac{\eta^2}{\xi^2})^2} d\theta d\eta \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^2} \frac{|\widehat{u}_0(\xi, \eta)|^2}{3\xi^2 + \frac{\eta^2}{\xi^2}} d\xi d\eta \right)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\|(D_x + D_x^{-1}D_y)U_2(\cdot)u_0\|_{L^\infty_x L^2_{ty}} \leq C\|u_0\|_{L^2_{xy}},$$

proving (0.18). Similarly, by the change of variables $\theta = \varphi_1(\zeta) = \xi^3 + \frac{\eta^2}{\xi}$, $\eta = \eta$, and taking into account that in Ω , $3\xi^2 - \frac{\eta^2}{\xi^2} > \xi^2$, it can be proved that (0.16) holds.

Also, if we change variables in (0.3) by $\theta = \varphi_j(\zeta)$, $\xi = \xi$, and apply Plancherel's theorem with respect to t and x , we have (0.17):

$$\|(D_x^{-1}D_y)^{\frac{1}{2}}U_j(\cdot)u_0\|_{L^\infty_y L^2_{tx}} \leq C\|u_0\|_{L^2_{xy}}.$$

From this estimate, and taking into account that $|\frac{\eta}{\xi}| > \sqrt{2}|\xi|$ in Γ , (0.19) follows.

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