Some results on the integral geometry of unions of independent families

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ABSTRACT. A notion of independence for families of varieties is presented, and some results of integral geometry are established in relation to their unions.

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1. Independent families

Let $F_t$ and $F_m$ be two families of varieties, $p$ and $s$ dimensional, placed in a space $X_n$ and depending on parameters $A_1, A_2, \ldots, A_t$ and $B_1, B_2, \ldots, B_m$ respectively:

$$F_t : F^\lambda(x_1, x_2, \ldots, x_n, A_1, A_2, \ldots, A_t) = 0, \quad \lambda = 1, 2, \ldots, n - p,$$

and

$$F_m : G^\mu(x_1, x_2, \ldots, x_n, B_1, B_2, \ldots, B_m) = 0, \quad \mu = 1, 2, \ldots, n - s.$$

We denote such families by $F_t(A)$ and $F_m(B)$.

Definition 1.1. The families $F_t(A)$ and $F_m(B)$ are said to be independent if there exists no relation $\varphi(A, B) = 0$ between their parameter sets $\{A\}$ and $\{B\}$. 
Let $F_q$ be the family of systems of two independent families $F_t(A)$ and $F_m(B)$. The above definition means that the parameters $A_1, A_2, \ldots, A_t, B_1, B_2, \ldots, B_m$ are all essential in $F_q(A, B)$. Further, if the maximal group of invariance of $F_q$ is the intersection of the maximal groups of invariance of $F_t$ and $F_m$, then $F_q$ is also called the independent union of $F_t$ and $F_m$.

Hence, every group of invariance of an independent union of independent families is a group of invariance of both of them.

**Theorem 1.2.** Let $G_1, G_2$ and $G_3$ be the maximal groups of invariance of the families of varieties $F_q, F_t$ and $F_m$, respectively, where $F_q = F_t + F_m$ is the family of systems of $F_t$ and $F_m$. If $\tau \in G_1$, then $\tau$, or at least $\tau^2$, belongs to $G_2 \cap G_3$.

**Proof.** An element of $F_q$ is a pair $(V, W)$ where $V \in F_t, W \in F_m$. If $\tau \in G_1$ then $\tau(V, W) = (\tau(V), \tau(W)) = (V_1, W_1)$ with $V_1 \in F_t$ and $W_1 \in F_m$. Three cases may arise:

1. $\tau(V) = V_1$ and $\tau(W) = W_1$. This implies $\tau \in G_2$ and $\tau \in G_3$, so $\tau \in G_2 \cap G_3$.
2. $\tau(V) = W_1$, $\tau(W) = V_1$, and $\tau(V_1) \in F_t$. This implies $\tau(W_1) \in F_m$. Consequently $\tau$ changes the variety $V_1 \in F_t$ into another variety of $F_t$, and therefore $\tau \in G_2$. Analogously $\tau$ changes the variety $W_1 \in F_m$ into a variety of $F_m$, and so $\tau \in G_3$. Hence $\tau \in G_2 \cap G_3$.
3. $\tau(V) = W_1$, $\tau(W) = V_1$, and $\tau(V_1) \in F_m$. This implies $\tau(W_1) \in F_t$. Consequently $\tau^2$ changes the variety $W \in F_m$ into another variety of $F_m$, so that $\tau^2 \in G_3$. Analogously, $\tau^2$ changes the variety $V \in F_t$ into a variety of $F_t$, so $\tau^2 \in G_2$. Hence $\tau^2 \in G_2 \cap G_3$. $\square$

**Remark 1.3.** Clearly the maximal group of invariance $G_1$ of $F_q = F_t + F_m$ always contains the intersection $G_2 \cap G_3$ of the maximal groups of invariance of $F_t$ and $F_m$. Therefore, in the cases 1 and 2 of the above theorem, we have $G_1 = G_2 \cap G_3$, so that, if $F_t$ and $F_m$ are independent, $F_q$ is their independent union.

**2. Systems of independent families**

Let $F_q(A, B) = F_t(A) + F_m(B), (q = t + m)$ be the family of systems of two independent families. Let us suppose that $F_t$ and $F_m$ are both measurable with respect to the maximal group of invariance $G_r$ of $F_q(A, B), r \geq q$, with respective densities

$$d\psi_t = \Phi_1(A_1, A_2, \ldots, A_t)dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t, \quad (1)$$

and

$$d\psi_m = \Phi_2(B_1, B_2, \ldots, B_m)dB_1 \wedge dB_2 \wedge \cdots \wedge dB_m. \quad (2)$$
The group $H_r$ associated to $G_r$ in the parameter space $X_q$ is measurable if and only if there exists a single non-trivial solution $\Phi = \Phi(C_1, C_2, \ldots, C_q)$ of the Deltheil system

$$\sum_{i=1}^{q} \frac{\partial (\xi_h^i \Phi)}{\partial C_i} = 0, \quad h = 1, 2, \ldots, r.$$  \hspace{1cm} (3)

Here $\xi_h^i$ are the coefficients of the infinitesimal transformations of the group $H_r$ and $C_1, C_2, \ldots, C_q$, the essential parameters of $F_q$ ([2], [7]).

Since $F_t$ and $F_m$ are independent families, the Deltheil system (3) may be written in the form

$$\sum_{k=1}^{t} \frac{\partial (\xi_h^k \Phi)}{\partial A_k} + \sum_{j=1}^{m} \frac{\partial (\zeta_h^{j+t} \Phi)}{\partial B_j} = 0, \quad h = 1, 2, \ldots, r.$$ \hspace{1cm} (4)

The coefficients of the infinitesimal transformations ([7]) are

$$\xi_h^i = \left( \frac{\partial C_i'}{\partial \alpha_h} \right)_0,$$

where $\{C_1', C_2', \ldots, C_q'\}$ is a group isomorphic to $G_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the parameters of $G_r$. The independence of $F_t$ and $F_m$ also implies that

$$\{C_1', C_2', \ldots, C_q'\} = \{A_1, A_2, \ldots, A_t, B_1', B_2', \ldots, B_m'\},$$

where $\{A_1, A_2, \ldots, A_t\}$ is the contribution of $F_t$ and $\{B_1', B_2', \ldots, B_m'\}$ that of $F_m$. That is, $\{A_1', A_2', \ldots, A_t'\}$ and $\{B_1', B_2', \ldots, B_m'\}$ jointly determine a group isomorphic to $G_r$ and are associated to the families $F_t$ and $F_m$, respectively. Consequently, we have

$$\xi_h^k = \left( \frac{\partial A_k'}{\partial \alpha_h} \right)_0, \quad k = 1, 2, \ldots, t,$$

$$\zeta_h^{j+t} = \left( \frac{\partial B_j'}{\partial \alpha_h} \right)_0, \quad j = 1, 2, \ldots, m.$$

Since $F_t$ is measurable with respect to $G_r$ then by (1) we obtain

$$\sum_{k=1}^{t} \frac{\partial (\xi_h^k \Phi_1(A_1, A_2, \ldots, A_t))}{\partial A_k} = 0, \quad h = 1, 2, \ldots, r,$$ \hspace{1cm} (5)

and since also $F_m$ is measurable with respect to $G_r$, it follows from (2) that

$$\sum_{j=1}^{m} \frac{\partial (\zeta_h^{j+t} \Phi_2(B_1, B_2, \ldots, B_m))}{\partial B_j} = 0, \quad h = 1, 2, \ldots, r.$$ \hspace{1cm} (6)
Now take $\Phi = \Phi_1 \Phi_2$ and replace in (3). We get
\[
\sum_{i=1}^{q} \frac{\partial (\xi_i \Phi)}{\partial C_i} = \sum_{k=1}^{t} \frac{\partial (\xi^k \Phi)}{\partial A_k} + \sum_{j=1}^{m} \frac{\partial (\xi^{j+t} \Phi)}{\partial B_j}
\]
\[
= \Phi_2 (B_1, B_2, \ldots, B_m) \sum_{k=1}^{t} \frac{\partial (\xi^k \Phi_1 (A_1, A_2, \ldots, A_t))}{\partial A_k}
\]
\[
+ \Phi_1 (A_1, A_2, \ldots, A_t) \sum_{j=1}^{m} \frac{\partial (\xi^{j+t} \Phi_2 (B_1, B_2, \ldots, B_m))}{\partial B_j}
\]
\[
= 0.
\]

The above argument shows that the Deltheil system associated to the group $H_r$, which is isomorphic to the maximal group of invariance $G_r$ of the family of systems of two independent families, both measurable with respect to $G_r$, always has at least a non-trivial solution. Furthermore, (7) ensures the following result.

**Theorem 2.1.** If $F_q = F_t + F_m$ is the family of systems of two independent families of varieties $F_t, F_m$ ($q = t + m$), both measurable with respect to the maximal group of invariance $G_r$ of $F_q$ ($r \geq q$), the densities being $d\psi_t$ and $d\psi_m$, then $F_q$ assumes, with respect to $G_r$, the density $d\psi_q = d\psi_t \wedge d\psi_m$.

**Corollary 2.2.** If the group $H_r$ isomorphic to $G_r$ in the parameter space $X_q$ of $F_q$ is transitive, then $F_q$ is measurable, and its density is $d\psi_q = d\psi_t \wedge d\psi_m$.

**Remark 2.3.** The assumption that $F_t$ and $F_m$ are both measurable with respect to the maximal group of invariance $G_r$ of $F_q$ implies that $G_r$ is a group of invariance of both these families. Consequently, by Theorem 1.2, $F_q$ is the independent union of $F_t$ and $F_m$.

More in general, the theorem holds whenever $F_t$ and $F_m$ have a density with respect to a same group of invariance.

**Example 2.4.** In the projective space $\mathbb{P}_3$, let $F_9$ be the family of non-degenerate quadrics having elliptic points ($\Delta < 0$, where $\Delta$ is the determinant of the quadrics), and let $F_6$ be the family of pairs plane-point, with the point out of the plane:
\[
F_9 : x^2 + (A^2 + C)y^2 + (B + D)z^2 + 2Axy + 2Bxz + 2(AB + E)yz + 2Lx + 2My + 2Nz + P = 0,
\]
\[
F_6 : \left\{ \begin{array}{l}
A_1 x + A_2 y + A_3 z + 1 = 0, \\
x = \alpha_1, \\
y = \alpha_2, \\
z = \alpha_3, \\
(\sum_{i=1}^{3} A_i \alpha_i + 1 \neq 0). \end{array} \right.
\]
These families are measurable ([3], [5]), with respective densities
\[ d\psi_9 = |\Delta|^{-\frac{5}{2}} dA \wedge dB \wedge dC \wedge dD \wedge dE \wedge dL \wedge dM \wedge dN \wedge dP, \]
and
\[ d\psi_6 = (A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1)^{-4} dA_1 \wedge dA_2 \wedge dA_3 \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3. \]

The maximal group of invariance of the family of systems \( \mathcal{F}_{15} = \mathcal{F}_9 + \mathcal{F}_6 \) is the projective group \( G_{15} \), with respect to which both \( \mathcal{F}_9 \) and \( \mathcal{F}_6 \) are measurable. Then \( \mathcal{F}_{15} \) is the independent union of \( \mathcal{F}_9 \) and \( \mathcal{F}_6 \), and so, by Theorem 2.1, it assumes the density \( d\psi_{15} = d\psi_9 \wedge d\psi_6 \) with respect to \( G_{15} \). Here \( r = q = 15 \), so that if the determinant \( \Delta = |\xi_i^1| \neq 0 \), then Corollary 2.2 holds. Note that \( \Delta \) is a polynomial; hence, it is non-trivial as soon as we can find a point \( Q \in X_{15} \) whose coordinates specify a variety \( V \in \mathcal{F}_{15} \) such that \( \Delta(Q) \neq 0 \). If we take the point
\[ Q = (A, B, C, D, E, L, M, N, P, A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3) \]
\[ = (1, 1, 2, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1) \]
then \( \Delta = -2 \) and \( A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1 = 4 \). Therefore, the variety \( V \) associated to \( Q \) belongs to \( \mathcal{F}_{15} \). Moreover, we have \( \Delta(Q) = -3072 \neq 0 \). Hence, we can apply Corollary 2.2, to deduce that \( \mathcal{F}_{15} \) is measurable, with density
\[ d\psi_{15} = |\Delta|^{-\frac{5}{2}} \left( \sum_{i=1}^{3} (A_i\alpha_i + 1) \right)^{-4} dA \wedge \cdots \wedge dP \wedge dA_1 \wedge \cdots \wedge d\alpha_3. \]

In the same way we can handle the independent union \( \overline{\mathcal{F}}_{15} = \overline{\mathcal{F}}_9 + \mathcal{F}_6 \) of the measurable families [3]
\[
\overline{\mathcal{F}}_9 : x^2 + (A^2 + C)y^2 + (B + D)z^2 + 2Axy + 2Bxz + 2(AB + E)yz + 2Lx + 2My + 2Nz + P = 0
\]
consisting of non-degenerate quadrics having hyperbolic points \( (C(DC - E^2) \leq 0 \) and \( \Delta > 0 \), where \( \Delta \) is the determinant of the quadrics) and the previous \( \mathcal{F}_6 \). Here we can take the point
\[ Q = (A, B, C, D, E, L, M, N, P, A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3) \]
\[ = (1, 1, 2, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1). \]

With this choice we have \( \Delta = 1 \) and \( A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1 = 4 \). Therefore the variety \( V \) associated to \( Q \) belongs to \( \overline{\mathcal{F}}_{15} \). Moreover \( \Delta(Q) = -384 \neq 0 \) and hence, by [3] and Corollary 2.2, \( \overline{\mathcal{F}}_{15} \) is measurable, with density
\[ d\overline{\psi}_{15} = |\Delta|^{-\frac{5}{2}} \left( \sum_{i=1}^{3} (A_i\alpha_i + 1) \right)^{-4} dA \wedge \cdots \wedge dP \wedge dA_1 \wedge \cdots \wedge d\alpha_3. \]
Remark 2.5. If \( H_r \) is not transitive, the family \( \mathcal{F}_q \) is not measurable, even though \( r \geq q \) and the Deltheil system associated to \( H_r \) have a solution, as this solution is not unique. Here we have an unusual kind of non-measurability, different from that pointed out by the Stoka's second condition [8], since in this case the family \( \mathcal{F}_q \) has different measures with respect to the same group, namely, the maximal group of invariance.

3. Iterated unions on a family of varieties

A special kind of independent union of families of varieties can be obtained by replacing the parameters \( A_1, A_2, \ldots, A_t \) of a family \( \mathcal{F}_t \) by new parameters \( B_1, B_2, \ldots, B_t \) having no relation with the former, and by taking \( \mathcal{F}_q(A, B) = \mathcal{F}_t(A) + \mathcal{F}_t(B) \). In this case we write \( \mathcal{F}_q = 2\mathcal{F}_t \). We can easily see that the groups of invariance of \( \mathcal{F}_q \) and \( \mathcal{F}_t \) are the same. Hence \( \mathcal{F}_q \) actually is an independent union. By iterating this construction we can build up the family \( \mathcal{F}_q(A^1, A^2, \ldots, A^m) = m\mathcal{F}_t \) that is the independent union of \( m \) copies of the family \( \mathcal{F}_t \).

Let \( \mathcal{F}_q = 2\mathcal{F}_t \) and assume \( \mathcal{F}_t \) to be measurable with respect to the maximal group of invariance \( G_r, r \geq 2t \) (so that \( \mathcal{F}_t \) is measurable), with density

\[
\psi_t = \phi_t(A_1, A_2, \ldots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.
\]

Further assume that the coefficients of the infinitesimal transformations \( \xi^k_h \), \( k = 1, 2, \ldots, t; h = 1, 2, \ldots, r \), are polynomials of degree \( d \leq 1 \), that is

\[
\xi^k_h = f^k_h(A_1, A_2, \ldots, A_t) = \sum_{j=1}^{t} \lambda^k_{hj} A_j + \mu^k_h, \quad \lambda^k_{hj}, \mu^k_h \in \mathbb{R}. \tag{8}
\]

The Deltheil system associated to \( \mathcal{F}_q \) is

\[
\sum_{i=1}^{q} \frac{\partial (\xi^k_h \Phi) \Phi}{\partial C_i} = \sum_{k=1}^{t} \frac{\partial (\xi^k_h \Phi) \Phi}{\partial A_k} + \sum_{k=1}^{t} \frac{\partial (\eta^k_h \Phi) \Phi}{\partial B_k}, \quad h = 1, 2, \ldots, r, \tag{9}
\]

where \( \eta^k_h = f^k_h(B_1, B_2, \ldots, B_t) \).

Now let \( \Phi = [\Phi_t(A_1, A_2, \ldots, A_t)]^\alpha [\Phi_t(B_1, B_2, \ldots, B_t)]^{2-\alpha} \). Replacing in (9) we get

\[
\sum_{k=1}^{t} \frac{\partial [\xi^k_h \Phi_t(A)^\alpha \Phi_t(B)^{2-\alpha}]}{\partial A_k} + \sum_{k=1}^{t} \frac{\partial [\eta^k_h \Phi_t(A)^{\alpha} \Phi_t(B)^{2-\alpha}]}{\partial B_k} = \]
\[
= \sum_{k=1}^{t} \left[ \frac{\partial \xi^k}{\partial A_k} \Phi_t(B)^{2-\alpha} + \xi^k \alpha \Phi_t(B)^{1-\alpha} \frac{\partial \Phi_t(A)}{\partial A_k} \Phi_t(B)^{2-\alpha} + \eta^k (2 - \alpha) \Phi_t(B)^{1-\alpha} \frac{\partial \Phi_t(A)}{\partial B_k} \right]
\]

Since \( \mathcal{F}_t \) is measurable, we have

\[
\sum_{k=1}^{t} \left[ f^k(A) \frac{\partial \Phi_t(A)}{\partial A_k} + \frac{\partial f^k(A)}{\partial A_k} \Phi_t(A) \right] = 0,
\]

and

\[
\sum_{k=1}^{t} \left[ f^k(B) \frac{\partial \Phi_t(B)}{\partial B_k} + \frac{\partial f^k(B)}{\partial B_k} \Phi_t(B) \right] = 0.
\]

By (11) and (12) we have

\[
\sum_{k=1}^{t} \frac{\partial f^k(A)}{\partial A_k} = - \sum_{k=1}^{t} f^k(A) [\Phi_t(A)]^{-1} \frac{\partial \Phi_t(A)}{\partial A_k},
\]

and

\[
\sum_{k=1}^{t} \frac{\partial f^k(B)}{\partial B_k} = - \sum_{k=1}^{t} f^k(B) [\Phi_t(B)]^{-1} \frac{\partial \Phi_t(B)}{\partial B_k}.
\]

Moreover, from (8) we obtain

\[
\frac{\partial f^k(A)}{\partial A_k} = \frac{\partial f^k(B)}{\partial B_k} = \chi^k_{hk}.
\]

Therefore, by replacing (13), (14) and (15) in (10), we get

\[
\Phi_t(A)^\alpha \Phi_t(B)^{2-\alpha} \sum_{k=1}^{t} \left[ \frac{\partial f^k(A)}{\partial A_k} - \alpha \frac{\partial f^k(A)}{\partial A_k} + \frac{\partial f^k(B)}{\partial B_k} - (2 - \alpha) \frac{\partial f^k(B)}{\partial B_k} \right]
\]

\[
= (\alpha - 1) \Phi_t(A)^\alpha \Phi_t(B)^{2-\alpha} \sum_{k=1}^{t} \left[ \frac{\partial f^k(B)}{\partial B_k} - \frac{\partial f^k(A)}{\partial A_k} \right] = 0.
\]

Consequently, the following result holds.
Theorem 3.1. Let \( \mathcal{F}_t = \mathcal{F}_t(A_1, A_2, \ldots, A_t) \) be a family of varieties which is measurable with respect to its maximal group of invariance \( G_r \), with density

\[
\Phi_t(A_1, A_2, \ldots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.
\]

If the coefficients of the infinitesimal transformations are polynomials of degree \( d \leq 1 \), then the independent union \( \mathcal{F}_q = 2\mathcal{F}_t \), \( r \geq q = 2t \), is a non-trivially non-measurable family, and it assumes at least the densities

\[
[\Phi_t(A_1, A_2, \ldots, A_t)]^\alpha [\Phi_t(B_1, B_2, \ldots, B_t)]^{2-\alpha} dA_1 \wedge \cdots \wedge dA_t \wedge dB_1 \wedge \cdots \wedge dB_t,
\]

for every \( \alpha \in \mathbb{R} \).

Remark 3.2. A family \( \mathcal{F}_q \) is said to be a trivially non-measurable family of varieties, if its maximal group of invariance \( G_r \) depends on \( r < q \) parameters. In this case the group \( H_r \), isomorphic to \( G_r \), is not transitive ([1], [2]).

By an analogous argument we can also state the following theorem.

Theorem 3.3. Let \( \mathcal{F}_t = \mathcal{F}_t(A_1, A_2, \ldots, A_t) \) be a family of varieties which is measurable with respect to its maximal group of invariance \( G_r \), with density

\[
\Phi_t(A_1, A_2, \ldots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.
\]

If the coefficients of the infinitesimal transformations are polynomials of degree \( d \leq 1 \), then the independent union \( \mathcal{F}_q = m\mathcal{F}_t \), \( r \geq q = mt \), is, for every integer \( m \neq 1 \), a non-trivially non-measurable family, and it assumes at least the densities

\[
[\Phi_t(A^1)]^{\alpha_1} [\Phi_t(A^2)]^{\alpha_2} \cdots [\Phi_t(A^m)]^{\alpha_m} \wedge^t dA^1 \wedge^t dA^2 \wedge^t \cdots \wedge^t dA^m,
\]

where \( A^i = (A^i_1, A^i_2, \ldots, A^i_t) \), \( i = 1, 2, \ldots, m \), is the set of parameters of the \( i \)-th copy of \( \mathcal{F}_t \), \( \wedge^t dA^i = dA^i_1 \wedge dA^i_2 \wedge \cdots \wedge dA^i_t \), and \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are real numbers such that \( \alpha_1 + \alpha_2 + \cdots + \alpha_m = m \).

Example 3.4. From [11] we know that the family \( \mathcal{F}_{n+1} \) of hyperspheres of the \( n \)-dimensional projective space \( \mathbb{P}_n \)

\[
x_1^2 + x_2^2 + \cdots + x_n^2 - 2u_1x_1 - 2u_2x_2 - \cdots - 2u_nx_n + v = 0,
\]

is measurable with respect to its maximal group of invariance, namely the group of similarities \( G_{n(n+1)/2+1}(\rho, \alpha, \theta) \).

Here \( \rho, \alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) and

\[
\theta = \{\theta_{12}, \ldots, \theta_{1n}, \theta_{23}, \ldots, \theta_{2n}, \ldots, \theta_{(n-1)n}\}
\]
are, respectively, the homothety, translation and rotation parameters. The density is
\[ d\psi'_{n+1} = R^{-(n+1)}du_1 \wedge du_2 \wedge \cdots \wedge du_n \wedge dR, \]
where
\[ R = \left[ \left( \sum_{j=1}^{n} u_j^2 \right) - v \right]^{-\frac{1}{2}} \]
is the radius of the hyperspheres. The coefficients of the infinitesimal transformations are ([11])
\[
\begin{align*}
\left( \frac{\partial u'_j}{\partial \rho} \right)_0 &= -u_j, & j &= 1, 2, \ldots, n, \\
\left( \frac{\partial u'_j}{\partial \alpha_i} \right)_0 &= \begin{cases} 0, & \text{if } j \neq i, \\
-1, & \text{if } j = i, & j, i = 1, 2, \ldots, n,
\end{cases} \\
\left( \frac{\partial u'_j}{\partial \theta_{hk}} \right)_0 &= \begin{cases} 0, & \text{if } j \neq h, k, \\
u_k, & \text{if } j = k, \\
-u_h, & \text{if } j = h, & j = 1, 2, \ldots, n, & h < k = 1, 2, \ldots, n,
\end{cases} \\
\left( \frac{\partial v'}{\partial \rho} \right)_0 &= -2v, \\
\left( \frac{\partial v'}{\partial \alpha_i} \right)_0 &= -2u_i, & i &= 1, 2, \ldots, n, \\
\left( \frac{\partial v'}{\partial \theta_{hk}} \right)_0 &= 0, & h < k &= 1, 2, \ldots, n.
\end{align*}
\]
Note that all of them are polynomials of degree \( d \leq 1 \).

Let \( m, n \) be two integers such that
\[ m(n + 1) \leq \frac{n(n + 1)}{2} + 1. \]
For fixed \( m \) this holds for any \( n \geq n_0 \), where \( n_0 \) is the least integer greater than
\[ \frac{2m + 1 + \sqrt{4m^2 + 4m - 7}}{2}, \]
so that \( n \geq 2m \). By Theorem 3.3, the independent union \( \mathcal{F}_{m(n+1)} = m\mathcal{F}_{n+1} \),
\[
(x_1 - u_{11})^2 + (x_2 - u_{12})^2 + \cdots + (x_n - u_{1n})^2 = R_1^2, \\
(x_1 - u_{21})^2 + (x_2 - u_{22})^2 + \cdots + (x_n - u_{2n})^2 = R_2^2, \\
\vdots \\
(x_1 - u_{m1})^2 + (x_2 - u_{m2})^2 + \cdots + (x_n - u_{mn})^2 = R_m^2,
\]
is, for every integer \( m \geq 2 \), a non-trivially non-measurable family of varieties of the projective space \( \mathbb{P}_{2m} \). This family assumes, at least, the densities

\[
[R_1^{a_1} R_2^{a_2} \cdots R_m^{a_m}]^{-(n+1)} \, du_{11} \wedge \cdots \wedge du_{mn} \wedge dR_1 \wedge \cdots \wedge dR_m, \quad \sum_{s=1}^{m} a_s = m.
\]

If \( a_1 = a_2 = \cdots = a_m = 1 \), this yields the product measure.

References


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