Revista Colombiana de Matemáticas
Volumen **31** *(1997), páginas 99–108*

Some results on the integral geometry of unions of independent families

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ABSTRACT. A notion of independence for families of varieties is presented, and some results of integral geometry are established in relation to their unions.

Key words and phrases. Family of varieties, maximal group of invariance of a family of varieties, systems of families of varieties, infinitesimal transformations, Deltheil systems, density of a measurable family of varieties.

1991 AMS. Subject Classification. Primary 60D05. Secondary 52A22.

1. Independent families

Let $\mathcal{F}_t$ and $\mathcal{F}_m$ be two families of varieties, p and s dimensional, placed in a space X_n and depending on parameters $A_1, A_2, \dots, A_t$ and $B_1, B_2, \dots, B_m$ respectively:

$$\mathcal{F}_t : F^{\lambda}(x_1, x_2, \dots, x_n, A_1, A_2, \dots, A_t) = 0, \quad \lambda = 1, 2, \dots, n-p,$$

and

$$\mathcal{F}_m : G^{\mu}(x_1, x_2, \dots, x_n, B_1, B_2, \dots, B_m) = 0, \quad \mu = 1, 2, \dots, n-s.$$

We denote such families by $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_m(\underline{B})$.

Definition 1.1. The families $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_m(\underline{B})$ are said to be independent if there exists no relation $\varphi(\underline{A}, \underline{B}) = 0$ between their parameter sets $\{\underline{A}\}$ and $\{\underline{B}\}$.

Let $\mathcal{F}_q$ be the family of systems of two independent families $\mathcal{F}_t(\underline{A})$ and $\mathcal{F}_m(\underline{B})$. The above definition means that the parameters $A_1, A_2, \ldots, A_t, B_1, B_2, \ldots, B_m$ are all essential in $\mathcal{F}_q(\underline{A}, \underline{B})$. Further, if the maximal group of invariance of $\mathcal{F}_q$ is the intersection of the maximal groups of invariance of $\mathcal{F}_t$ and $\mathcal{F}_m$, then $\mathcal{F}_q$ is also called the *independent union of* $\mathcal{F}_t$ *and* $\mathcal{F}_m$.

Hence, every group of invariance of an independent union of independent families is a group of invariance of both of them.

Theorem 1.2. *Let G_1, G_2 and G_3 be the maximal groups of invariance of the families of varieties $\mathcal{F}_q$, $\mathcal{F}_t$ and $\mathcal{F}_m$, respectively, where $\mathcal{F}_q = \mathcal{F}_t + \mathcal{F}_m$ is the family of systems of $\mathcal{F}_t$ and $\mathcal{F}_m$. If $\tau \in G_1$, then τ, or at least τ^2, belongs to $G_2 \cap G_3$.*

Proof. An element of $\mathcal{F}_q$ is a pair (V, W) where $V \in \mathcal{F}_t$, $W \in \mathcal{F}_m$. If $\tau \in G_1$ then $\tau(V, W) = (\tau(V), \tau(W)) = (V_1, W_1)$ with $V_1 \in \mathcal{F}_t$ and $W_1 \in \mathcal{F}_m$. Three cases may arise:

1. $\tau(V) = V_1$ and $\tau(W) = W_1$. This implies $\tau \in G_2$ and $\tau \in G_3$, so $\tau \in G_2 \cap G_3$.
2. $\tau(V) = W_1$, $\tau(W) = V_1$, and $\tau(V_1) \in \mathcal{F}_t$. This implies $\tau(W_1) \in \mathcal{F}_m$. Consequently τ changes the variety $V_1 \in \mathcal{F}_t$ into another variety of $\mathcal{F}_t$, and therefore $\tau \in G_2$. Analogously τ changes the variety $W_1 \in \mathcal{F}_m$ into a variety of $\mathcal{F}_m$, and so $\tau \in G_3$. Hence $\tau \in G_2 \cap G_3$.
3. $\tau(V) = W_1$, $\tau(W) = V_1$, and $\tau(V_1) \in \mathcal{F}_m$. This implies $\tau(W_1) \in \mathcal{F}_t$. Consequently τ^2 changes the variety $W \in \mathcal{F}_m$ into another variety of $\mathcal{F}_m$, so that $\tau^2 \in G_3$. Analogously, τ^2 changes the variety $V \in \mathcal{F}_t$ into a variety of $\mathcal{F}_t$, so $\tau^2 \in G_2$. Hence $\tau^2 \in G_2 \cap G_3$. ☑

Remark 1.3. Clearly the maximal group of invariance G_1 of $\mathcal{F}_q = \mathcal{F}_t + \mathcal{F}_m$ always contains the intersection $G_2 \cap G_3$ of the maximal groups of invariance of $\mathcal{F}_t$ and $\mathcal{F}_m$. Therefore, in the cases 1 and 2 of the above theorem, we have $G_1 = G_2 \cap G_3$, so that, if $\mathcal{F}_t$ and $\mathcal{F}_m$ are independent, $\mathcal{F}_q$ is their independent union.

2. Systems of independent families

Let $\mathcal{F}_q(\underline{A}, \underline{B}) = \mathcal{F}_t(\underline{A}) + \mathcal{F}_m(\underline{B})$, $(q = t + m)$ be the family of systems of two independent families. Let us suppose that $\mathcal{F}_t$ and $\mathcal{F}_m$ are both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_q(\underline{A}, \underline{B})$, $r \geq q$, with respective densities

$$d\psi_t = \Phi_1(A_1, A_2, \ldots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t, \tag{1}$$

and

$$d\psi_m = \Phi_2(B_1, B_2, \ldots, B_m) dB_1 \wedge dB_2 \wedge \cdots \wedge dB_m. \tag{2}$$

The group H_r associated to G_r in the parameter space X_q is measurable if and only if there exists a single non-trivial solution $\Phi = \Phi(C_1, C_2, \dots, C_q)$ of the Deltheil system

$$\sum_{i=1}^{q} \frac{\partial(\xi_h^i \Phi)}{\partial C_i} = 0, \quad h = 1, 2, \dots, r. \tag{3}$$

Here ξ_h^i are the coefficients of the infinitesimal transformations of the group H_r and $C_1, C_2, \dots, C_q$, the essential parameters of $\mathcal{F}_q$ ([2], [7]).

Since $\mathcal{F}_t$ and $\mathcal{F}_m$ are independent families, the Deltheil system (3) may be written in the form

$$\sum_{k=1}^{t} \frac{\partial(\xi_h^k \Phi)}{\partial A_k} + \sum_{j=1}^{m} \frac{\partial(\xi_h^{j+t} \Phi)}{\partial B_j} = 0, \quad h = 1, 2, \dots, r. \tag{4}$$

The coefficients of the infinitesimal transformations ([7]) are

$$\xi_h^i = \left(\frac{\partial C_i'}{\partial \alpha_h} \right) \Big|_0,$$

where $\{C_1', C_2', \dots, C_q'\}$ is a group isomorphic to G_r and $\alpha_1, \alpha_2, \dots, \alpha_r$ are the parameters of G_r. The independence of $\mathcal{F}_t$ and $\mathcal{F}_m$ also implies that

$$\{C_1', C_2', \dots, C_q'\} = \{A_1', A_2', \dots, A_t', B_1', B_2', \dots, B_m'\},$$

where $\{A_1', A_2', \dots, A_t'\}$ is the contribution of $\mathcal{F}_t$ and $\{B_1', B_2', \dots, B_m'\}$ that of $\mathcal{F}_m$. That is, $\{A_1', A_2', \dots, A_t'\}$ and $\{B_1', B_2', \dots, B_m'\}$ jointly determine a group isomorphic to G_r and are associated to the families $\mathcal{F}_t$ and $\mathcal{F}_m$, respectively. Consequently, we have

$$\xi_h^k = \left(\frac{\partial A_k'}{\partial \alpha_h} \right) \Big|_0, \quad k = 1, 2, \dots, t,$$

$$\xi_h^{j+t} = \left(\frac{\partial B_j'}{\partial \alpha_h} \right) \Big|_0, \quad j = 1, 2, \dots, m.$$

Since $\mathcal{F}_t$ is measurable with respect to G_r then by (1) we obtain

$$\sum_{k=1}^{t} \frac{\partial\left(\xi_h^k \Phi_1(A_1, A_2, \dots, A_t)\right)}{\partial A_k} = 0, \quad h = 1, 2, \dots, r, \tag{5}$$

and since also $\mathcal{F}_m$ is measurable with respect to G_r, it follows from (2) that

$$\sum_{j=1}^{m} \frac{\partial\left(\xi_h^{j+t} \Phi_2(B_1, B_2, \dots, B_m)\right)}{\partial B_j} = 0, \quad h = 1, 2, \dots, r. \tag{6}$$

Now take $\Phi = \Phi_1\Phi_2$ and replace in (3). We get

$$\begin{aligned}\sum_{i=1}^{q}\frac{\partial(\xi_h^i\Phi)}{\partial C_i} &= \sum_{k=1}^{t}\frac{\partial(\xi_h^k\Phi)}{\partial A_k} + \sum_{j=1}^{m}\frac{\partial(\xi_h^{j+t}\Phi)}{\partial B_j}\\ &= \Phi_2(B_1,B_2,\dots,B_m)\sum_{k=1}^{t}\frac{\partial(\xi_h^k\Phi_1(A_1,A_2,\dots,A_t))}{\partial A_k}\\ &\quad + \Phi_1(A_1,A_2,\dots,A_t)\sum_{j=1}^{m}\frac{\partial(\xi_h^{j+t}\Phi_2(B_1,B_2,\dots,B_m))}{\partial B_j} = 0.\end{aligned} \tag{7}$$

The above argument shows that the Deltheil system associated to the group H_r, which is isomorphic to the maximal group of invariance G_r of the family of systems of two independent families, both measurable with respect to G_r, always has at least a non-trivial solution. Furthermore, (7) ensures the following result.

Theorem 2.1. *If $\mathcal{F}_q = \mathcal{F}_t + \mathcal{F}_m$ is the family of systems of two independent families of varieties $\mathcal{F}_t$, $\mathcal{F}_m$ $(q = t + m)$, both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_q$ $(r \geq q)$, the densities being $d\psi_t$ and $d\psi_m$, then $\mathcal{F}_q$ assumes, with respect to G_r, the density $d\psi_q = d\psi_t \wedge d\psi_m$.*

Corollary 2.2. *If the group H_r isomorphic to G_r in the parameter space X_q of $\mathcal{F}_q$ is transitive, then $\mathcal{F}_q$ is measurable, and its density is $d\psi_q = d\psi_t \wedge d\psi_m$.*

Remark 2.3. The assumption that $\mathcal{F}_t$ and $\mathcal{F}_m$ are both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_q$ implies that G_r is a group of invariance of both these families. Consequently, by Theorem 1.2, $\mathcal{F}_q$ is the independent union of $\mathcal{F}_t$ and $\mathcal{F}_m$.

More in general, the theorem holds whenever $\mathcal{F}_t$ and $\mathcal{F}_m$ have a density with respect to a same group of invariance.

Example 2.4. In the projective space $\mathbb{P}_3$, let $\mathcal{F}_9$ be the family of non-degenerate quadrics having elliptic points ($\Delta < 0$, where Δ is the determinant of the quadrics), and let $\mathcal{F}_6$ be the family of pairs plane–point, with the point out of the plane:

$$\begin{aligned}\mathcal{F}_9 : x^2 + (A^2 + C)y^2 + (B + D)z^2 + 2Axy + 2Bxz\\ + 2(AB + E)yz + 2Lx + 2My + 2Nz + P = 0,\end{aligned}$$

$$\mathcal{F}_6 : \begin{cases} A_1x + A_2y + A_3z + 1 = 0,\\ x = \alpha_1,\\ y = \alpha_2,\\ z = \alpha_3, \qquad \left(\sum_{i=1}^{3} A_i\alpha_i + 1 \neq 0\right).\end{cases}$$

These families are measurable ([3], [5]), with respective densities

$$d\psi_9 = |\Delta|^{-\frac{5}{2}} dA \wedge dB \wedge dC \wedge dD \wedge dE \wedge dL \wedge dM \wedge dN \wedge dP,$$

and

$$d\psi_6 = (A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1)^{-4} dA_1 \wedge dA_2 \wedge dA_3 \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3.$$

The maximal group of invariance of the family of systems $\mathcal{F}_{15} = \mathcal{F}_9 + \mathcal{F}_6$ is the projective group G_{15}, with respect to which both $\mathcal{F}_9$ and $\mathcal{F}_6$ are measurable. Then $\mathcal{F}_{15}$ is the independent union of $\mathcal{F}_9$ and $\mathcal{F}_6$, and so, by Theorem 2.1, it assumes the density $d\psi_{15} = d\psi_9 \wedge d\psi_6$ with respect to G_{15}. Here $r = q = 15$, so that if the determinant $\overline{\Delta} = |\xi_h^i| \neq 0$, then Corollary 2.2 holds. Note that $\overline{\Delta}$ is a polynomial; hence, it is non-trivial as soon as we can find a point $Q \in X_{15}$ whose coordinates specify a variety $V \in \mathcal{F}_{15}$ such that $\overline{\Delta}(Q) \neq 0$. If we take the point

$$\begin{aligned} Q &= (A, B, C, D, E, L, M, N, P, A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3) \\ &= (1, 1, 2, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1) \end{aligned}$$

then $\Delta = -2$ and $A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1 = 4$. Therefore, the variety V associated to Q belongs to $\mathcal{F}_{15}$. Moreover, we have $\overline{\Delta}(Q) = -3072 \neq 0$. Hence, we can apply Corollary 2.2, to deduce that $\mathcal{F}_{15}$ is measurable, with density

$$d\psi_{15} = |\Delta|^{-\frac{5}{2}} \left(\sum_{i=1}^{3} (A_i\alpha_i + 1) \right)^{-4} dA \wedge \cdots \wedge dP \wedge dA_1 \wedge \cdots \wedge d\alpha_3.$$

In the same way we can handle the independent union $\overline{\mathcal{F}}_{15} = \overline{\mathcal{F}}_9 + \mathcal{F}_6$ of the measurable families [3]

$$\begin{aligned} \overline{\mathcal{F}}_9 : x^2 + (A^2 + C)y^2 + (B + D)z^2 + 2Axy + 2Bxz \\ + 2(AB + E)yz + 2Lx + 2My + 2Nz + P = 0 \end{aligned}$$

consisting of non-degenerate quadrics having hyperbolic points ($C(DC - E^2) \leq 0$ and $\Delta > 0$, where Δ is the determinant of the quadrics) and the previous $\mathcal{F}_6$. Here we can take the point

$$\begin{aligned} Q &= (A, B, C, D, E, L, M, N, P, A_1, A_2, A_3, \alpha_1, \alpha_2, \alpha_3) \\ &= (1, 1, 2, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1). \end{aligned}$$

With this choice we have $\Delta = 1$ and $A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 + 1 = 4$. Therefore the variety V associated to Q belongs to $\overline{\mathcal{F}}_{15}$. Moreover $\overline{\Delta}(Q) = -384 \neq 0$ and hence, by [3] and Corollary 2.2, $\overline{\mathcal{F}}_{15}$ is measurable, with density

$$d\overline{\psi}_{15} = |\Delta|^{-\frac{5}{2}} \left(\sum_{i=1}^{3} (A_i\alpha_i + 1) \right)^{-4} dA \wedge \cdots \wedge dP \wedge dA_1 \wedge \cdots \wedge d\alpha_3.$$

Remark 2.5. If H_r is not transitive, the family $\mathcal{F}_q$ is not measurable, even though $r \geq q$ and the Deltheil system associated to H_r have a solution, as this solution is not unique. Here we have an unusual kind of non-measurability, different from that pointed out by the Stoka's second condition [8], since in this case the family $\mathcal{F}_q$ has different measures with respect to the same group, namely, the maximal group of invariance.

3. Iterated unions on a family of varieties

A special kind of independent union of families of varieties can be obtained by replacing the parameters $A_1, A_2, \dots, A_t$ of a family $\mathcal{F}_t$ by new parameters $B_1, B_2, \dots, B_t$ having no relation with the former, and by taking $\mathcal{F}_q(\underline{A}, \underline{B}) = \mathcal{F}_t(\underline{A}) + \mathcal{F}_t(\underline{B})$. In this case we write $\mathcal{F}_q = 2\mathcal{F}_t$. We can easily see that the groups of invariance of $\mathcal{F}_q$ and $\mathcal{F}_t$ are the same. Hence $\mathcal{F}_q$ actually is an independent union. By iterating this construction we can build up the family $\mathcal{F}_q(\underline{A}^1, \underline{A}^2, \dots, \underline{A}^m) = m\mathcal{F}_t$ that is the independent union of m copies of the family $\mathcal{F}_t$.

Let $\mathcal{F}_q = 2\mathcal{F}_t$ and assume $\mathcal{F}_t$ to be measurable with respect to the maximal group of invariance G_r, $r \geq 2t$ (so that $\mathcal{F}_t$ is measurable), with density

$$\psi_t = \phi_t(A_1, A_2, \dots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.$$

Further assume that the coefficients of the infinitesimal transformations ξ_h^k, $k = 1, 2, \dots, t$; $h = 1, 2, \dots, r$, are polynomials of degree $d \leq 1$, that is

$$\xi_h^k = f_h^k(A_1, A_2, \dots, A_t) = \sum_{j=1}^{t} \lambda_{hj}^k A_j + \mu_h^k, \quad \lambda_{hj}^k,\ \mu_h^k \in \mathbb{R}. \tag{8}$$

The Deltheil system associated to $\mathcal{F}_q$ is

$$\sum_{i=1}^{q} \frac{\partial(\xi_h^i \Phi)}{\partial C_i} = \sum_{k=1}^{t} \frac{\partial(\xi_h^k \Phi)}{\partial A_k} + \sum_{k=1}^{t} \frac{\partial(\eta_h^k \Phi)}{\partial B_k}, \quad h = 1, 2, \dots, r, \tag{9}$$

where $\eta_h^k = f_h^k(B_1, B_2, \dots, B_t)$.

Now let $\Phi = [\Phi_t(A_1, A_2, \dots, A_t)]^\alpha [\Phi_t(B_1, B_2, \dots, B_t)]^{2-\alpha}$. Replacing in (9) we get

$$\sum_{k=1}^{t} \frac{\partial \left[\xi_h^k \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha}\right]}{\partial A_k} + \sum_{k=1}^{t} \frac{\partial \left[\eta_h^k \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha}\right]}{\partial B_k} =$$

$$\begin{aligned}
&= \sum_{k=1}^{t} \left[\frac{\partial \xi_h^k}{\partial A_k} \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} + \xi_h^k \alpha \Phi_t(\underline{A})^{\alpha-1} \frac{\partial \Phi_t(\underline{A})}{\partial A_k} \Phi_t(\underline{B})^{2-\alpha} + \right. \\
&\qquad \left. + \frac{\partial \eta_h^k}{\partial B_k} \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} + \eta_h^k (2-\alpha) \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{1-\alpha} \frac{\partial \Phi_t(\underline{B})}{\partial B_k} \right] \\
&= \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} \sum_{k=1}^{t} \left[\frac{\partial f_h^k(\underline{A})}{\partial A_k} + \alpha f_h^k(\underline{A}) \Phi_t(\underline{A})^{-1} \frac{\partial \Phi_t(\underline{A})}{\partial A_k} + \right. \\
&\qquad \left. + \frac{\partial f_h^k(\underline{B})}{\partial B_k} + (2-\alpha) f_h^k(\underline{B}) \Phi_t(\underline{B})^{-1} \frac{\partial \Phi_t(\underline{B})}{\partial B_k} \right]
\end{aligned} \tag{10}$$

Since $\mathcal{F}_t$ is measurable, we have

$$\sum_{k=1}^{t} \left[f_h^k(\underline{A}) \frac{\partial \Phi_t(\underline{A})}{\partial A_k} + \frac{\partial f_h^k(\underline{A})}{\partial A_k} \Phi_t(\underline{A}) \right] = 0, \tag{11}$$

and

$$\sum_{k=1}^{t} \left[f_h^k(\underline{B}) \frac{\partial \Phi_t(\underline{B})}{\partial B_k} + \frac{\partial f_h^k(\underline{B})}{\partial B_k} \Phi_t(\underline{B}) \right] = 0. \tag{12}$$

By (11) and (12) we have

$$\sum_{k=1}^{t} \frac{\partial f_h^k(\underline{A})}{\partial A_k} = -\sum_{k=1}^{t} f_h^k(\underline{A}) \left[\Phi_t(\underline{A})\right]^{-1} \frac{\partial \Phi_t(\underline{A})}{\partial A_k}, \tag{13}$$

and

$$\sum_{k=1}^{t} \frac{\partial f_h^k(\underline{B})}{\partial B_k} = -\sum_{k=1}^{t} f_h^k(\underline{B}) \left[\Phi_t(\underline{B})\right]^{-1} \frac{\partial \Phi_t(\underline{B})}{\partial B_k}. \tag{14}$$

Moreover, from (8) we obtain

$$\frac{\partial f_h^k(\underline{A})}{\partial A_k} = \frac{\partial f_h^k(\underline{B})}{\partial B_k} = \lambda_{hk}^k. \tag{15}$$

Therefore, by replacing (13), (14) and (15) in (10), we get

$$\begin{aligned}
\Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} \sum_{k=1}^{t} &\left[\frac{\partial f_h^k(\underline{A})}{\partial A_k} - \alpha \frac{\partial f_h^k(\underline{A})}{\partial A_k} + \frac{\partial f_h^k(\underline{B})}{\partial B_k} - (2-\alpha) \frac{\partial f_h^k(\underline{B})}{\partial B_k} \right] \\
&= (\alpha - 1) \Phi_t(\underline{A})^\alpha \Phi_t(\underline{B})^{2-\alpha} \sum_{k=1}^{t} \left[\frac{\partial f_h^k(\underline{B})}{\partial B_k} - \frac{\partial f_h^k(\underline{A})}{\partial A_k} \right] = 0.
\end{aligned}$$

Consequently, the following result holds.

Theorem 3.1. *Let $\mathcal{F}_t = \mathcal{F}_t(A_1, A_2, \dots, A_t)$ be a family of varieties which is measurable with respect to its maximal group of invariance G_r, with density*

$$\Phi_t(A_1, A_2, \dots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.$$

If the coefficients of the infinitesimal transformations are polynomials of degree $d \leq 1$, then the independent union $\mathcal{F}_q = 2\mathcal{F}_t$, $r \geq q = 2t$, is a non-trivially non-measurable family, and it assumes at least the densities

$$[\Phi_t(A_1, A_2, \dots, A_t)]^\alpha [\Phi_t(B_1, B_2, \dots, B_t)]^{2-\alpha} dA_1 \wedge \cdots \wedge dA_t \wedge dB_1 \wedge \cdots \wedge dB_t,$$

for every $\alpha \in \mathbb{R}$.

Remark 3.2. A family $\mathcal{F}_q$ is said to be a trivially non-measurable family of varieties, if its maximal group of invariance G_r depends on $r < q$ parameters. In this case the group H_r isomorphic to G_r is not transitive ([1], [2]).

By an analogous argument we can also state the following theorem.

Theorem 3.3. *Let $\mathcal{F}_t = \mathcal{F}_t(A_1, A_2, \dots, A_t)$ be a family of varieties which is measurable with respect to its maximal group of invariance G_r, with density*

$$\Phi_t(A_1, A_2, \dots, A_t) dA_1 \wedge dA_2 \wedge \cdots \wedge dA_t.$$

If the coefficients of the infinitesimal transformations are polynomials of degree $d \leq 1$, then the independent union $\mathcal{F}_q = m\mathcal{F}_t$, $r \geq q = mt$, is, for every integer $m \neq 1$, a non-trivially non-measurable family, and it assumes at least the densities

$$[\Phi_t(\underline{A}^1)]^{\alpha_1} [\Phi_t(\underline{A}^2)]^{\alpha_2} \cdots [\Phi_t(\underline{A}^m)]^{\alpha_m} \wedge^t d\underline{A}^1 \wedge^t d\underline{A}^2 \wedge^t \cdots \wedge^t d\underline{A}^m,$$

where $\underline{A}^i = (A_1^i, A_2^i, \dots, A_t^i)$, $i = 1, 2, \dots, m$, is the set of parameters of the i-th copy of $\mathcal{F}_t$, $\wedge^t d\underline{A}^i = dA_1^i \wedge dA_2^i \wedge \cdots \wedge dA_t^i$, and $\alpha_1, \alpha_2, \dots, \alpha_m$ are real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = m$.

Example 3.4. From [11] we know that the family $\mathcal{F}_{n+1}$ of hyperspheres of the n-dimensional projective space $\mathbb{P}_n$

$$x_1^2 + x_2^2 + \cdots + x_n^2 - 2u_1x_1 - 2u_2x_2 - \cdots - 2u_nx_n + v = 0,$$

is measurable with respect to its maximal group of invariance, namely the group of similarities $G_{\frac{n(n+1)}{2}+1}(\rho, \underline{\alpha}, \underline{\theta})$.

Here $\rho, \underline{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and

$$\underline{\theta} = \{\theta_{12}, \dots, \theta_{1n}, \theta_{23}, \dots, \theta_{2n}, \dots, \theta_{(n-1)n}\}$$

are, respectively, the homothety, translation and rotation parameters. The density is

$$d\psi_{n+1} = R^{-(n+1)} du_1 \wedge du_2 \wedge \cdots \wedge du_n \wedge dR,$$

where

$$R = \left[\left(\sum_{j=1}^{n} u_j^2 \right) - v \right]^{-\frac{1}{2}}$$

is the radius of the hyperspheres. The coefficients of the infinitesimal transformations are ([11])

$$\left(\frac{\partial u'_j}{\partial \rho} \right) \Big|_0 = -u_j, \quad j = 1, 2, \ldots, n,$$

$$\left(\frac{\partial u'_j}{\partial \alpha_i} \right) \Big|_0 = \begin{cases} 0, & \text{if } \ j \neq i, \\ -1, & \text{if } \ j = i, \quad j, i = 1, 2, \ldots, n, \end{cases}$$

$$\left(\frac{\partial u'_j}{\partial \theta_{hk}} \right) \Big|_0 = \begin{cases} 0, & \text{if } \ j \neq h, k, \\ u_k, & \text{if } \ j = k, \\ -u_h, & \text{if } \ j = h, \quad j = 1, 2, \ldots, n, \quad h < k = 1, 2, \ldots, n, \end{cases}$$

$$\left(\frac{\partial v'}{\partial \rho} \right) \Big|_0 = -2v,$$

$$\left(\frac{\partial v'}{\partial \alpha_i} \right) \Big|_0 = -2u_i, \quad i = 1, 2, \ldots, n,$$

$$\left(\frac{\partial v'}{\partial \theta_{hk}} \right) \Big|_0 = 0, \quad h < k = 1, 2, \ldots, n.$$

Note that all of them are polynomials of degree $d \leq 1$.

Let m, n be two integers such that

$$m(n+1) \leq \frac{n(n+1)}{2} + 1.$$

For fixed m this holds for any $n \geq n_0$, where n_0 is the least integer greater than

$$\frac{2m + 1 + \sqrt{4m^2 + 4m - 7}}{2},$$

so that $n \geq 2m$. By Theorem 3.3, the independent union $\mathcal{F}_{m(n+1)} = m\mathcal{F}_{n+1}$,

$$(x_1 - u_{11})^2 + (x_2 - u_{12})^2 + \cdots + (x_n - u_{1n}^2) = R_1^2,$$
$$(x_1 - u_{21})^2 + (x_2 - u_{22})^2 + \cdots + (x_n - u_{2n}^2) = R_2^2,$$
$$\vdots$$
$$(x_1 - u_{m1})^2 + (x_2 - u_{m2})^2 + \cdots + (x_n - u_{mn}^2) = R_m^2,$$

is, for every integer $m \geq 2$, a non-trivially non-measurable family of varieties of the projective space $\mathbb{P}_{2m}$. This family assumes, at least, the densities

$$[R_1^{a_1} R_2^{a_2} \cdots R_m^{a_m}]^{-(n+1)} du_{11} \wedge \cdots \wedge du_{mn} \wedge dR_1 \wedge \cdots \wedge dR_m, \qquad \sum_{s=1}^{m} a_s = m.$$

If $a_1 = a_2 = \cdots = a_m = 1$, this yields the product measure.

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(Recibido en mayo de 1996; revisado en noviembre de 1997)

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