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A note on $\mathcal{H}_3$-algebras

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ABSTRACT. In this note we shall generalize some results of the three-valued Lukasiewicz algebras to three-valued symmetrical Heyting algebras. We prove that a three-valued Heyting algebra with axe is a product of a Boolean algebra by a three-valued symmetrical Heyting symmetric algebra with center. We also introduce the involutive rough sets and we prove a representation theorem for three-valued symmetrical Heyting algebras by means of rough sets.

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1. Introduction

The class of three-valued symmetrical Heyting algebras ($\mathcal{H}_3$-algebras) are a generalization of three-valued Lukasiewicz algebras ($\mathcal{L}_3$-algebras). These algebras were mainly studied by Luisa Iturrioz in [5] and [7] (see also [9] p. 156). In [10] L. Monteiro proves that the variety of $\mathcal{L}_3$-algebras can be characterized as $\mathcal{H}_3$-algebras such that they verify the Kleene's axiom: $x \wedge \sim x \leq y \vee \sim y$. In this note we obtain some results on the variety of $\mathcal{H}_3$-algebras that generalize analogous results on $\mathcal{L}_3$-algebras. We shall introduce the centered $\mathcal{H}_3$-algebras and $\mathcal{H}_3$-algebras with axe. We prove that all $\mathcal{H}_3$-algebras with axe is the product of a Boolean algebra by a centered $\mathcal{H}_3$-algebras. There is an analogous result for $\mathcal{L}_3$-algebras (see [10]).

In [3] (see also [4]) S. Comer proves that every $\mathcal{L}_3$-algebra (or regular double Stone algebra) can be represented as an algebra of rough subsets of an approximation space. In the present paper we shall introduce the involutive rough sets and we shall show that an $\mathcal{H}_3$-algebra is isomorphic to a subalgebra of rough subsets of an involutive approximation space. We conjecture that a similar representation result can be proved for other classes of algebras with weaker operations than the pseudo complementation and dual pseudo complementation, as for example for some class of double MS-algebras (see [2]). If we analyze the proof of Theorem 18 below we see that this depends on the regularity axiom and of a subalgebra that allows, in a certain sense, builds up the original algebra. Our next task is to investigate this topic.

We assume that the reader is familiarized with the theory of Heyting algebras and the De Morgan algebras as it is given, for instance, in [1], [9] or in [13].

2. Preliminaries

Definition 1. *An $\mathcal{H}_3$-algebra is an algebra $(L, \vee, \wedge, \Rightarrow, \sim, 0, 1)$ such that: $(L, \vee, \wedge, \Rightarrow, 0, 1)$ is a Heyting algebra, $(L, \vee, \wedge, \sim, 0, 1)$ is a the De Morgan algebra and for all $x, y, z \in L$ the following axiom is verified:*

$$\textbf{\textit{T}}.\quad ((x \Rightarrow z) \Rightarrow y) \Rightarrow (((y \Rightarrow x) \Rightarrow y) \Rightarrow y) = 1.$$

Let L be an $\mathcal{H}_3$-algebra. Recall that $x^* = x \Rightarrow 0$ and $x^+ = \sim (\sim x)^*$ are the pseudo complement and the dual pseudo complement of x, respectively.

In [7] L. Iturrioz characterized the $\mathcal{H}_3$-algebras as algebras $(L, \vee, \wedge, ^*, \sim, 0, 1)$ of type (2,2,1,1,0,0) such that $(L, \vee, \wedge, ^*, 0, 1)$ is a bounded pseudo complemented distributive lattice and $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra and the following conditions are held for all $x, y \in L$,

$\mathbf{H}_1$. $(x \wedge y)^* = x^* \vee y^*$.

$\mathbf{H}_2$. If $x^* = y^*$ y $(\sim x)^* = (\sim y)^*$ then $x = y$ (regularity axiom).

We note that $\mathbf{H}_2$ may be replaced by the following identity:

$\mathbf{H}'_2$. $(x \wedge x^+) \wedge (y \vee y^*) = x \wedge x^+$, for any $x, y \in L$.

Let L be an $\mathcal{H}_3$-algebra. By $\mathbf{H}_1$ L, is a *Stone algebra* and from the duality given by $\sim$, L is also a *double Stone algebra*.

For an $\mathcal{H}_3$-algebra L, we define the set $D(L) = \{x \in L : x^* = 0\}$ of dense elements. This set is a filter of L. Dually, the set $D(L)^+ = \{x \in L : x^+ = 1\}$ is an ideal of L.

For an $\mathcal{H}_3$-algebra L let us write $R(L) = \{x \in L : x^{**} = x\}$ to denote the center of L. It is known that $R(L)$ is a Boolean algebra and that it is a subalgebra of L. It is easy to prove that

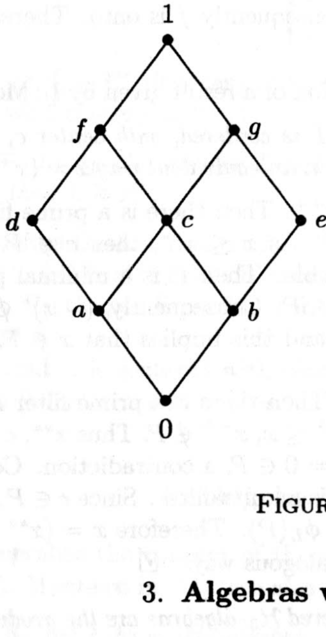
$$R(L) = \{x \in L : x^*\} = \{x \in L : x^* = x^+\} = \{x \in L : x = x^{++}\}.$$

The set of all prime filters of L is denoted by $X(L)$. As L is also a De Morgan algebra, then we can define the application $\phi_L : X(L) \to X(L)$ given by $\phi_L(P) = X(L) \setminus \sim P = (\sim P)^c$, where $\sim P = \{\sim x : x \in P\}$. It is known that if $P \in X(L)$ then $\phi_L(P) \in X(L)$ and $\phi_L(\phi_L(P)) = P$. The filter (ideal) generated by a set $H \subseteq L$ is denoted by $F(X)$ $(I(X))$. Let us recall that in any Stone algebra L, if P is maximal prime filter (i.e. an ultrafilter), then $D(L) \subseteq P$.

In [7] (see also [9] p.209) was proved that an $\mathcal{H}_3$-algebra L may be characterized by means of the set $X(L)$ as follows:

Theorem 2. *A symmetrical Heyting algebra L is an $\mathcal{H}_3$-algebra iff for all $P \in X(L)$, P is an ultrafilter or there is an unique ultrafilter Q such that $P \subset Q$.*

Hence, for each $P \in X(L)$, P is an ultrafilter iff $\phi_L(P)$ is a minimal prime filter. Let us recall that the variety of $\mathcal{H}_3$-algebras is semisimple and generated by the $\mathcal{H}_3$-algebra S_9 (see Figure 1) and its subalgebras are $S_2 = \{0, 1\}$, $S_3 = \{0, a, 1\}$ and $S_4 = \{0, d, e, 1\}$ (see [5], [7] and [9]).



x	$\sim x$
a	f
b	g
c	c
d	d
e	e
f	a
g	b

FIGURE 1

3. Algebras with center

Definition 3. *An $\mathcal{H}_3$-algebra L is centered if there is an element c, called the center of L, such that $c^{**} = 1$ and $c^{++} = 0$.*

We note that if c is the center of L, $\sim c = c$.

Lemma 4. *Let L be an $\mathcal{H}_3$-algebra.*

1. *L is centered, where c denotes the center, iff $D(L) \cap D(L)^+ = \{c\}$. Consequently c is unique .*
2. *If L is centered, then $D(L) = F(c)$, $D(L)^+ = I(c)$ and $D(L) = \sim D(L)^+$.*
3. *If L is centered, then $R(L) \cong D(L) \cong D(L)^+$.*

Proof. 1. We will suppose that L is centered. Let $z \in D(L) \cap D(L)^+$, then $z^{++} = 0$ and $z^{**} = 1$. By $\mathbf{H}_2$, $z = c$. The converse is immediate.

2. We prove that $D(L) = F(c)$. By *1.* and as $D(L)$ is a filter, $F(c) \subseteq D(L)$. Suppose that there is $z \in D(L)$ such that $z \notin F(c)$. Then there is a prime filter P such that $c \in P$ y $z \notin P$. If P is an ultrafilter, $D(L) \subseteq P$ and this implies that $z \in P$, which is a contradiction. Suppose that P is minimal. As $\sim c = c$, $c \notin \phi_L(P)$, which is also a contradiction, since $\phi_L(P)$ is an ultrafilter. Therefore $F(c) = D(L)$. The other equalities are easy to check.

3. It is enough to prove that $I(c) \cong R(L)$. Let the application $f : I(c) \longrightarrow R(L)$ be given by $f(x) = x^{**}$. It is clear that f is a lattice homomorphism. We check that is one-one. Let $f(x) = f(y)$. Then $x^{**} = y^{**}$ and as $0 = x^{++} = y^{++}$, then by condition $\mathbf{H}_2$, $x = y$. Therefore f is one-one. Let $z \in R(L)$ and let $z \wedge c \in I(c)$. Clearly $f(z \wedge c) = z$. Consequently f is onto. Therefore f is an isomorphism. ☑

The following result is a generalization of a result given by L. Monteiro [11].

Theorem 5. *Let L be $\mathcal{H}_3$-algebra. If L is centered, with center c, then for all element $x \in L$, $x = (x^{**} \wedge c) \vee x^{++}$, or in an equivalent way $x = (x^{++} \vee c) \wedge x^{**}$.*

Proof. Suppose that $x \nleq (x^{**} \wedge c) \vee x^{++}$. Then there is a prime filter P such that $x \in P$ and $(x^{**} \wedge c) \vee x^{++} \notin P$. As $x \leq x^{**}$, then $c \notin P$. If P is an ultrafilter, $F(c) \subseteq P$, which is impossible. Then P is a minimal prime filter. As $x^+ \vee x^{++} = 1 \in P$, $x^+ = \sim (\sim x)^* \in P$. Consequently, $(\sim x)^* \notin \phi_L(P)$. As $\phi_L(P)$ is an ultrafilter, $\sim x \in \phi_L(P)$ and this implies that $x \notin P$, which is a contradiction.

Suppose that $(x^{**} \wedge c) \vee x^{++} \nleq x$. Then there is a prime filter P such that $x \notin P$ y $(x^{**} \wedge c) \vee x^{++} \in P$. Since $x^{++} \leq x$, $x^{++} \notin P$. Thus $x^{**}, c \in P$. If P is an ultrafilter, $x^* \in P$. Then $x^* \wedge x^{**} = 0 \in P$, a contradiction. Consequently P is a minimal prime filter and $\phi_L(P)$ is an ultrafilter. Since $c \in P$, $c \notin \phi_L(P)$, but this is impossible because $F(c) \subseteq \phi_L(P)$. Therefore $x = (x^{**} \wedge c) \vee x^{++}$. The other equality is derived in an analogous way. ☑

Theorem 6. *The only finite and centered $\mathcal{H}_3$-algebras are the products $S_3^k \times S_9^r$, with $k, r \in w$.*

Proof. It is easy to see that $S_3^k \times S_9^r$ is centered. Let L be a finite and centered $\mathcal{H}_3$-algebra. As L is semisimple then it is a product of simple algebras,

$L = \prod_{i=1}^{n} L_i$. We prove that $L_i \cong S_3$ or $L_i \cong S_9$, for $i = 1, .., n$. Suppose that there is a j such that $L_j \cong S_2$ or $L_j \cong S_4$. If Π_j is the projection onto L_j, then $\Pi_j(L) = L_j$. If c is the center of L, then by the above Lemma, $\Pi_j(c)$ is the center of L_j, which is a contradiction. Therefore $L_i \cong S_3$ or $L_i \cong S_9$. ☑

We end this section with a representation theorem. An analogous result was given in [9] (p. 199) for three-valued Lukasiewicz algebras.

Let L be an $\mathcal{H}_3$-algebra and let $R(L)$ be the center of L. We define the set

$$R^* = \{(x, y) \in R(L) \times R(L) : x \le y\}$$

and the operations:

$$(a, b) \wedge (c, d) = (a \wedge c, b \wedge d),$$
$$(a, b) \vee (c, d) = (a \vee c, b \vee d),$$
$$\sim (a, b) = (\sim b, \sim a),$$
$$(a, b)^* = (b^*, b^*),$$
$$\mathbf{0} = (0, 0),$$
$$\mathbf{1} = (1, 1),$$

for all $(a, b), (c, d) \in R^*$. It is easy to check that $\langle R^*, \vee, \wedge, ^*, \sim, 0, 1\rangle$ is a centered $\mathcal{H}_3$-algebra with center $c = (0, 1)$.

Theorem 7. *Every $\mathcal{H}_3$–algebra L is isomorphic to a subalgebra of R^*. If L is centered, then $L \cong R^*$.*

Proof. We define the function $\alpha : L \longrightarrow R^*$ given by $\alpha(x) = (x^{++}, x^{**})$. Since $x^{++**} = x^{++}$, $(x^{++}, x^{**}) \in R^*$. It is easy to see that α is a homomorphism of $\mathcal{H}_3$-algebras. By the condition $\mathbf{H}_2$ it follows that α is one-one. Then $\alpha(L)$ is a subalgebra of R^*.

Suppose that L is centered with center c. Let $(a, b) \in R^*$. Then it is clear that $x = a \vee (b \wedge c) \in L$ and $\alpha(x) = (a, b)$. Then α is onto and $L \cong R^*$. ☑

4. Algebras with axe

We shall generalize the concept of three-valued Lukasiewicz algebra with axe studied by L. Monteiro in [11] (see also [9]).

Definition 8. *Let L be an $\mathcal{H}_3$-algebra. We shall say that L has axe if there is an element $e \in L$, called the axe of L, such that the following conditions hold for all $x \in L$:*

1. $e^{++} = 0$.

2. $x^{**} \leq x^{++} \vee e^{**}$.

The following theorem gives a representation of the elements of an $\mathcal{H}_3$-algebra L with axe.

Theorem 9. *Let L be an $\mathcal{H}_3$-algebra with axe e. Then for all $x \in L$ it is verified that $x = x^{++} \vee (x^{**} \wedge e)$.*

Proof. Suppose that $x \nleq x^{++} \vee (x^{**} \wedge e)$. Then there is a prime filter P such that $x \in P$ and $x^{++} \vee (x^{**} \wedge e) \notin P$. As $x^{++} \leq x \leq x^{**}$, $x^{++} \notin P, e \notin P$ and $x^{**} \in P$. If P is an ultrafilter,

$$e^{*} \in P, \tag{1}$$

since $e \notin P$. By *2.* of Definition 8, $x^{**} \leq x^{++} \vee e^{**} \in P$. Then $e^{**} \in P$ and as P is an ultrafilter $e^{*} \notin P$, which is a contradiction by (1).

Now, if we suppose that P is a minimal prime filter, $\phi_L(P)$ is an ultrafilter. As $x^{++} \notin P$, then by the identity $x^{+} \vee x^{++} = 1$, $x^{+} = \sim (\sim x)^{*} \in P$. Thus, $\sim x \in \phi_L(P)$ which is equivalent to $x \notin P$, which is a contradiction. Therefore $x \leq x^{++} \vee (x^{**} \wedge e)$.

Suppose that $x^{++} \vee (x^{**} \wedge e) \nleq x$. Then there is a prime filter P such that $x^{++} \vee (x^{**} \wedge e) \in P$ and $x \notin P$. Then $x^{++} \notin P$ and e, $x^{**} \in P$. If P is an ultrafilter, $x^{*} \notin P$, but as $x \notin P$, $x^{*} \in P$, which is a contradiction. Suppose that $\phi_L(P)$ is an ultrafilter. As $e \in P$, $\sim e \notin \phi_L(P)$ and this implies that

$$(\sim e)^{*} \in \phi_L(P). \tag{2}$$

On the other hand, as $\sim (\sim e)^{*} = e^{+} = 1 \in P$

$$(\sim e)^{*} \notin \phi_L(P),$$

in contradiction with (2). Therefore, $x^{++} \vee (x^{**} \wedge e) \leq x$. ☑

Lemma 10. *Let L be an $\mathcal{H}_3$-algebra. If L has axe, then it is unique.*

Proof. It is immediate by regularity axiom. ☑

Lemma 11. *Let L be an $\mathcal{H}_3$ algebra and let $e \in L$ such that $e^{++} = 1$. Then $e \leq \sim e$.*

Proof. If we suppose that $e \nleq \sim e$, then there is a prime filter P such that $e \in P$ and $\sim e \notin P$. So $e \in \phi_L(P)$. If P is an ultrafilter, $(\sim e)^{*} \in P$. But as $e^{++} = 1$, $(\sim e)^{*} = 0 \in P$, which is a contradiction. If $\phi_L(P)$ is an ultrafilter, $(\sim e)^{*} = 0 \in \phi_L(P)$, since $\sim e \notin \phi_L(P)$, which is a contradiction. Therefore $e \leq \sim e$. ☑

Lemma 12. *Let L be an $\mathcal{H}_3$ algebra with axe e. Then the following properties hold:*

1. $\sim e \wedge e^{**} = e$.

2. $\sim e^{**} \vee e^{**} = 1$.
3. $\sim e^{**} = e^*$.
4. *for all* $x \in L$, *if* $x \leq \sim e^{**}$, *then* $x \in R(L)$.

Proof. 1. It is clear that $e \leq \sim e \wedge e^{**}$. If $\sim e \wedge e^{**} \not\leq e$ there is a prime filter P such that $\sim e \wedge e^{**} \in P$ and $e \notin P$. If P is an ultrafilter, $e^* \in P$ and $e^{**} \in P$, which is impossible. If $\phi_L(P)$ is an ultrafilter, as $(\sim e)^* = 0 \notin \phi_L(P)$, $\sim e \in \phi_L(P)$ and this implies that $e \in P$, which is a contradiction.

2. Suppose that $\sim e^{**} \vee e^{**} < 1$. Then there is a prime filter P such that $\sim e^{**} \notin P$ y $e^{**} \notin P$. By the identity $e^* \vee e^{**} = 1$, $e^* \in P$. By 1 we have the following identities:

$$\begin{aligned} 0 &= (\sim e)^* &&= (e \vee \sim e^{**})^* \\ &= e^* \wedge (\sim e^{**})^* &&= e^* \wedge (\sim e)^{+++} \\ &= e^* \wedge (\sim e)^{+} &&= e^* \wedge \sim e^*. \end{aligned}$$

Thus $\sim e^* \vee e^* = 1 \in \phi_L(P)$. As $\sim e^{**} \notin P$, $e^* \notin \phi_L(P)$. Then $e^* \in P$, which is a contradiction.

3. It is a consequence of 2.

4. Let $x = x \wedge e^{**}$. Then

$$\begin{aligned} x &= x \wedge \sim e^{**} \\ &= ((x^{++} \vee (x^{**} \wedge e)) \wedge \sim e^{**} && \text{(by Theorem 9)} \\ &= (x^{**} \wedge \sim e \wedge e^{**} \wedge \sim e^{**}) \vee (x^{++} \wedge \sim e^{**}) && \text{(by 1)} \\ &= 0 \vee (x^{++} \wedge \sim e^{**}) && \text{(by 2)} \\ &= x^{++} \wedge \sim e^{**}. \end{aligned}$$

By the above identity

$$x^{**} = (x^{++} \wedge \sim e^{**})^{**} = x^{++} \wedge (\sim e^{**})^{**} = x^{++} \wedge \sim e^{**} = x.$$

Therefore $x \in R(L)$. ☑

Now, we shall prove the main result of this section. We recall that if F is a filter, $\theta(F) = \{(x, y) : \text{there is } f \in F \text{ such that } x \wedge f = y \wedge f\}$ is a lattice congruence.

Theorem 13. *Let L be an $\mathcal{H}_3$ algebra with axe e. Then $L \cong L_1 \times L_2$, where L_2 is a Boolean algebra and L_1 is an $\mathcal{H}_3$ algebra with center.*

Proof. Let e be the axe of L. Let us take the filters $F(e^{**})$ and $F(\sim e^{**})$. We shall denote $\theta_1 = \theta(F(e^{**}))$ and $\theta_2 = \theta(F(\sim e^{**}))$. As e^{**} y $\sim e^{**} \in R(L)$. Then by known results of theory of distributive lattices (see [1]) $L \cong L/\theta_1 \times L/\theta_2$, where $L/\theta_1 \cong [0, e^{**}]$ and $L/\theta_2 \cong [0, \sim e^{**}]$. As $\sim e^* = e^{**}$, $F(e^{**})$ is a regular filter (or deductive system in the terminology of [9]). Thus $L_1 = L/\theta_1$ is an $\mathcal{H}_3$ algebra. The operations of De Morgan negation α, pseudo complemented $\neg$ and dual pseudo complemented λ are defined in $[0, e^{**}]$ by means of the identities $\alpha x = \sim x \wedge e^{**}$, $\neg x = x^{*} \wedge e^{**}$ and $\lambda x = x^{++} \wedge e^{**}$, respectively. We now show that L_1 is centered. The center is defined by the equivalence class of e, $|e|_{\theta_1}$, since $\neg\neg\, |e|_{\theta_1} = |e^{**}|_{\theta_1}$ and $\lambda\lambda\, |e|_{\theta_1} = |e^{++}|_{\theta_1} = |0|_{\theta_1}$.

We check that $L_2 = L/\theta_2$ is a Boolean algebra. By Lemma 12, for all $x \in [0, \sim e^{**}]$ we have that $x \in R(L)$.

It is easy to check that the application f given by:

$$\begin{aligned} f : L &\longrightarrow [0, e^{**}] \times [0. \sim e^{**}] \\ x &\longmapsto f(x) = (x \wedge e^{**}, x \wedge \sim e^{**}) \end{aligned}$$

is an isomorphism of $\mathcal{H}_3$ algebras. ☑

The proof the following result is analogous to the proof given in [9].

Corollary 14. *Let $\mathcal{L}(\alpha)$ be the free $\mathcal{H}_3$ algebra, with a finite set of generators G of cardinal α. Then $\mathcal{L}(\alpha) \cong \mathcal{B}(\alpha) \times \mathcal{C}(\alpha)$, where $\mathcal{B}(\alpha)$ is the free Boolean algebra and $\mathcal{C}(\alpha)$ is the centered free $\mathcal{H}_3$-algebra.*

5. Involutive rough sets

In this section we shall establish the mentioned connection between $\mathcal{H}_3$-algebras and a generalization of rough sets.

Definition 15. *The triple $X = \langle X, \Theta, \phi \rangle$ is an involutive approximation space (***IAS***) if Θ is an equivalence relation on X and the following conditions hold:*

1. *ϕ is an application $\phi : X \longrightarrow X$ such that $\phi^2(x) = x$,*
2. *For all $x, y \in X$, $(x, y) \in \Theta \Leftrightarrow (\phi(x), \phi(y)) \in \Theta$.*

Let X be an **IAS** and let $U \subseteq X$. Then we define two sets U_Θ and U^Θ as follows:

$$\begin{aligned} U_\Theta &= \{x \in X : \Theta(x) \subseteq U\}, \\ U^\Theta &= \{x \in X : \Theta(x) \cap U \neq \emptyset\}. \end{aligned}$$

A *rough subset* is a pair (U_Θ, U^Θ), where $U \subseteq X$. The collections of all rough subset of X is denoted by $Ro(X)$. Let $U \subseteq X$. Let us define the set $\sim U$ by $\sim U = X \setminus \phi(U)$. As ϕ is an involution, then it is easy to check that $\sim\sim U = U$.

On the set $Ro(X)$ we shall define the operations $\vee, \wedge, \sim, {}^*, 0$ and 1 by:

$$\begin{aligned}(U_\Theta, U^\Theta) \vee (V_\Theta, V^\Theta) &= (U_\Theta \cup V_\Theta, U^\Theta \cup V^\Theta),\\ (U_\Theta, U^\Theta) \wedge (V_\Theta, V^\Theta) &= (U_\Theta \cap V_\Theta, U^\Theta \cap V^\Theta),\\ \approx (U_\Theta, U^\Theta) &= (\sim U^\Theta, \sim U_\Theta),\\ (U_\Theta, U^\Theta)^* &= (X \setminus U^\Theta, X \setminus U^\Theta),\\ 0 &= (\emptyset, \emptyset),\\ 1 &= (X, X).\end{aligned}$$

In [12] it is shown that $\langle Ro(X), \vee, \wedge, {}^*, 0, 1\rangle$ is a Stone algebra and if we define an operation $^+$ by

$$(U_\Theta, U^\Theta)^+ = (X \setminus U_\Theta, X \setminus U_\Theta),$$

then $\langle Ro(X), \vee, \wedge, {}^*, {}^+, 0, 1\rangle$ is a regular double Stone algebra.

Lemma 16. *Let X be an* **IAS**. *Then for all $U \subseteq X$ we have*

1. $\sim U_\Theta = (\sim U)^\Theta$ *and* $\sim U^\Theta = (\sim U)_\Theta$.
2. $\approx\approx (U_\Theta, U^\Theta) = (U_\Theta, U^\Theta)$.
3. $(U_\Theta, U^\Theta)^+ = \approx (\approx (U_\Theta, U^\Theta))^*$.

Proof. We shall prove that $\sim U_\Theta = (\sim U)^\Theta$.

$$\begin{aligned} x \in \sim U_\Theta &\Leftrightarrow \phi(x) \notin U_\Theta\\ &\Leftrightarrow \text{there is } y \in X \text{ such that } (\phi(x), y) \in \Theta \text{ and } y \notin U\\ &\Leftrightarrow \text{there is } y \in X \text{ such that } (x, \phi(y)) \in \Theta \text{ and } \phi(y) \notin \phi(U)\\ &\Leftrightarrow \text{there is } y \in X \text{ such that } (x, \phi(y)) \in \Theta \text{ and } \phi(y) \in \sim U\\ &\Leftrightarrow \Theta(x) \cap \sim U \neq \emptyset\\ &\Leftrightarrow x \in (\sim U)^\Theta.\end{aligned}$$

The proof of $\sim U^\Theta = (\sim U)_\Theta$ is analogous.

For 2,

$$\approx\approx (U_\Theta, U^\Theta) = \approx (\sim U^\Theta, \sim U_\Theta) = (\sim\sim U_\Theta, U^\Theta) = (U_\Theta, U^\Theta).$$

The proof of 3 follows from 1 and the following equivalences

$$\begin{aligned}
\approx (\approx (U_\Theta, U^\Theta)^*) &= \approx (\sim U^\Theta, \sim U_\Theta)^* \\
&= \approx ((\sim U)_\Theta, (\sim U)^\Theta)^* \\
&= \approx (X \smallsetminus (\sim U)^\Theta, X \smallsetminus (\sim U)^\Theta) \\
&= \approx (X \smallsetminus \sim U_\Theta, X \smallsetminus \sim U_\Theta) \\
&= \approx (\phi(U_\Theta), \phi(U_\Theta)) \\
&= (X \smallsetminus \phi(\phi(U_\Theta)), X \smallsetminus \phi(\phi(U_\Theta))) \\
&= (X \smallsetminus U_\Theta, X \smallsetminus U_\Theta) \\
&= (U_\Theta, U^\Theta)^+. \quad ☑
\end{aligned}$$

Theorem 17. *The algebra* $\mathbf{Ro}(X) = \langle Ro(X), \vee, \wedge, \approx, ^*, 0, 1\rangle$ *is an* $\mathcal{H}_3$*-algebra.*

Proof. By Lemma 16 $\langle Ro(X), \vee, \wedge, \approx, 0, 1\rangle$ is a De Morgan algebra. Also by Lemma 16 and the duality given by the negation $\approx$, $\langle Ro(X), \vee, \wedge, ^*, ^+, 0, 1\rangle$ is a double Stone algebra. We shall just have to prove that the regularity axiom is verified or, what in other words, is the same as to prove the condition

$$(U_\Theta, U^\Theta) \wedge (U_\Theta, U^\Theta)^+ \subseteq (V_\Theta, V^\Theta) \vee (V_\Theta, V^\Theta)^*,$$

for any $U, V \subseteq X$. By the definitions of operations * and $^+$ we have that the above condition is equivalent to

$$(U_\Theta, U^\Theta) \wedge (X \smallsetminus U_\Theta, X \smallsetminus U_\Theta) \subseteq (V_\Theta, V^\Theta) \vee (X \smallsetminus V^\Theta, X \smallsetminus V^\Theta),$$

whose proof is immediate. ☑

Now we shall prove the main result of this section.

Theorem 18. *Every* $\mathcal{H}_3$*-algebra* L *is isomorphic to a subalgebra of rough subsets of an involutive approximation space.*

Proof. Let $X(L)$ be the set of all prime filters of L and let $R(L)$ be the Boolean algebra of the complemented elements of L. We define the relation $\Theta \subseteq X(L) \times X(L)$ by:

$$(P, Q) \in \Theta \Leftrightarrow P \cap R(L) = Q \cap R(L),$$

for all $P, Q \in X(L)$. It is immediate that Θ is an equivalence relation. We shall prove that $(P, Q) \in \Theta \Leftrightarrow (\phi_L(P), \phi_L(Q)) \in \Theta$. Suppose that $P \cap R(L) =$

$Q \cap R(L)$, then

$$\begin{aligned} a \in \phi_L(P) \cap R(L) &\Leftrightarrow\ \sim a \notin P \text{ and } a \in R(L) \\ &\Leftrightarrow (\sim a)^* \in P \text{ and } a \in R(L) \\ &\Leftrightarrow\ \sim a^* \in P \text{ and } a \in R(L) \\ &\Leftrightarrow\ \sim a^* \in Q \text{ and } a \in R(L) \\ &\Leftrightarrow\ a^* \notin \phi_L(Q) \text{ and } a \in R(L) \\ &\Leftrightarrow\ a \in \phi_L(Q) \cap R(L) \end{aligned}$$

The other direction is analogous. Therefore $\langle X(L), \Theta, \phi_L \rangle$ is an **IAS.** Let the application $\sigma : L \longrightarrow \mathcal{P}(X(L))$ be defined by

$$\sigma(a) = \{P \in X(L) : a \in P\}\,.$$

For $a \in L$ we define $\beta(a) = (\sigma(a^{++}), \sigma(a^{**}))$. We shall prove that

$$\begin{aligned} \sigma(a^{**}) &= \sigma(a)^{\Theta} \\ \sigma(a^{++}) &= \sigma(a)_{\Theta} \end{aligned}$$

for $a \in L$. For $\sigma(a^{**}) = \sigma(a)^{\Theta}$ we shall check $a^{**} \in P$ if and only if there is $Q \in X(L)$ such that $P \cap R(L) = Q \cap R(L)$ and $a \in Q$. Suppose that $a^{**} \in P$. Let us take the filter $F(P \cap R(L) \cup \{a\})$. This filter is proper, since if we suppose that $0 \in F(P \cap R(L) \cup \{a\})$ then there is $p \in P \cap R(L)$ such that $p \wedge a = 0$, which implies that $p \leq a^* \in P$, which is a contradiction. Therefore, there is an $Q \in X(L)$ such that $P \cap R(L) \subseteq Q \cap R(L)$ and $a \in Q$. Since $P \cap R(L)$ and $Q \cap R(L)$ are ultrafilters of the Boolean algebra $R(L)$, $P \cap R(L) = Q \cap R(L)$. For the other direction we suppose that there is $Q \in X(L)$ such that $P \cap R(L) = Q \cap R(L)$ and $a \in Q$. Then, since $a \leq a^{**}$, $a^{**} \in Q$.

For $\sigma(a^{++}) = \sigma(a)^{\Theta}$ we shall check $a^{++} \in P \Leftrightarrow$ for all $a \in X(L)$ such that $P \cap R(L) = Q \cap R(L)$, then $a \in Q$, which is immediate, since $a^{++} \leq a$.

Now we prove that $\beta(\sim a) = \approx \beta(a)$.

$$\begin{aligned} \beta(\sim a) &= (\sigma((\sim a)^{++}, \sigma((\sim a)^{**}) \\ &= (\sigma(\sim a^{**}), \sigma(\sim a^{++})) \\ &= (\sim \sigma(a^{**}), \sim \sigma(a^{++})) \\ &= \approx (\sigma(a^{++}), \sigma(a^{**})) = \approx \beta(a) \end{aligned}$$

Therefore we have proved that for each $a \in L$, $\beta(a) \in \mathbf{Ro}(\langle X(L), \Theta, \phi_L \rangle)$. The regularity axiom for L implies that β is one-one. It is easy to check that β preserves * and $^+$ (see [3]). This completes the proof of the theorem. ☑

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References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[2] T. S. Blyth and J. C. Varlet, *Ockham Algebras*, Oxford University Press, 1994.

[3] S. D. Comer, *Perfect extensions of regular double Stone algebras*, Algebra Universalis, **34** (1995), 96–109.

[4] S. D. Comer, *On connections between information systems, rough sets and algebraic logic*, Algebraic Methods in Logic and in Computer Science, Banach Center Pub., vol. 28, 117–124, 1993.

[5] L. Iturrioz, *Sur une classe particulière d'algebres de Moisil*. C. R Acad. Sc. Paris, **267** Sèrie A (1968), 585–588.

[6] L. Iturrioz, *Lukasiewicz and symmetrical Heyting algebras*. Zeitschr. f. Math. Logik und Grundlagen d. Math. **23** (1977), 300–317.

[7] L. Iturrioz, *Algèbres de Heyting trivalentes involutives*, Notas de Lógica Matemática, 31, Univ. Nac. del Sur. Bahía Blanca, Argentina, 1974.

[8] T. B. Iwinski, *Algebraic approach to rough sets*. Bull. Polish. Acad. Sci. Math. **35** (1987), 673–683.

[9] A. Monteiro, *Sur les algèbres de Heyting simétriques*. Portugaliae Math. **39** Fasc. 1-4 (1980), 1–237.

[10] L. Monteiro, *Sur les algèbres de Lukasiewicz injectives*. Proceedings of the Japan Academy **41** 7 (1965), 578–581.

[11] L. Monteiro, *Algebras de Lukasiewicz trivalentes monádicas*. Notas de Lógica Matemática, 32, Univ. Nac. del Sur, Bahía Blanca, Argentina, 1974.

[12] J. Pomykala and J. A. Pomykala, *The Stone algebra of rough sets*. Bull. Polish. Acad. Sci. Math. **36** (1988), 495–508.

[13] H. Rasiowa, *An algebraic aproach to non-classic logic*, North-Holland, Amsterdam, 1974.

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