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Flows and diffeomorphisms

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ABSTRACT. We discuss the question of when a given diffeomorphism on a bounded domain can be embedded in the flow of a smooth autonomous system of ordinary differential equations. This question is related to the existence of classical solutions of some nonlinear boundary value problems. We treat also the special case of diffeomorphisms defined on bounded intervals or on circles.

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1. Introduction

The autonomous differential equation $x' = f(x)$ associated with the complete vector field f on the bounded domain $\Omega \subset \mathbb{R}^n$ defines a flow Φ_f satisfying

- $\Phi_f(t,\cdot) : \Omega \longleftrightarrow \Omega$ is a diffeomorphism for each $t \in \mathbb{R}$,
- $\Phi_f(0,\cdot)$ is the identity map of Ω,
- $\Phi_f(t, \Phi_f(s,\cdot)) = \Phi_f(t+s,\cdot)$.

In this article we deal with the inverse problem, closely related to Jabotinsky and Abel equations (see [4]). More precisely, given a diffeomorphism $g : \Omega \leftrightarrow \Omega$, find a complete vector field f on Ω such that its time one map equals g, i.e.

$$\Phi_f(1,x) = g(x), \qquad x \in \Omega.$$

If such a field f exists the composition operation can be simplified. Indeed, $\Phi(1,\cdot) = g(\cdot)$, implies $g^k(\cdot) = \Phi(k,\cdot)$ for every integer k. Here g^k denotes the k-fold composition of g with itself. Additionally, on $\Omega \subset \mathbb{R}^2$ the existence of the field f prevents the diffeomorphism from being chaotic.

With $\Omega \subset \mathbb{R}^n$ seen as a universe, the field f as its governing laws, and $\Phi(t,\cdot)$ as its evolution on the time, then the problem we pose admits the following interpretation:

Given the state of evolution of the universe Ω at time $t = 1$,
is it possible to determine its governing laws?

Our goal is to establish the equivalence between the existence of the field f and the solvability of certain nonlinear boundary value problems. Additionally, we obtain an expression to recover the field f from g, on a neighborhood of an attracting equilibrium $\widetilde{x}$. For one dimensional domains we obtain stronger results: For Ω a bounded interval we show the uniqueness of the field f up to a scalar multiple (compare [4] and [8]) and relate the existence of f to the one dimensinal wave equation; for circles (compare [10]) we relate the existence and uniqueness of f to the rotation number of the diffeomorphism g.

2. General results

First notice that the flow determines the field, i.e. the map $f \to \Phi_f$ is one to one. Also, not every diffeomorphism stems from a flow. To see this, suppose Φ is the flow associated with the vector field f. Then $z(t) := \frac{\partial}{\partial x}\Phi(t,x)$ is a solution of the linear (nonautonomous) system of ordinary differential equations

$$z'(t) = Df\left(\Phi(t,x)\right) z(t), \tag{1}$$

where $Df(x)$ is the derivative of f at x. From well known results on the Wronskian we have

$$\det\left(\frac{\partial}{\partial x}\Phi(t,x)\right) = \exp\left(\int_0^t tr\left(Df\left(\Phi(\xi,x)\right)\right) d\xi\right). \tag{2}$$

So, diffeomorphisms with negative Jacobian determinant are not allowed.

The Hénon map

$$H_{a,b}(x,y) := \left(a - by - x^2, x\right), \quad b \neq 0,$$

is a diffeomorphism on $\mathbb{R}^2$ with Jacobian determinant b. It is interesting to note that numerical evidence indicates that the Hénon map possesses a strange attractor when $a = 1.4$, and $b = -0.3$. See [5] and [9] for more details.

In this paper we assume Ω is a domain with smooth boundary $\partial\Omega$. Denote the unit outer normal at $x \in \partial\Omega$ by $\nu(x)$. As it is well known the flow Φ_f leaves the domain Ω invariant iff $f(x)\cdot\nu(x)=0$ for all x in $\partial\Omega$. In this case the flow $\Phi_f(t,x)$ is defined for all $t\in\mathbb{R}$ and all $x\in\Omega$, i.e. the flow Φ_f is complete on Ω (see [1]).

The next lemma shows that to solve $\Phi(1,\cdot)=g(\cdot)$ is equivalent to solve the equation $\Phi(s,\cdot)=g(\cdot)$ for one $s\neq 0$.

Lemma 1. *Let $\lambda\neq 0$ be scalar. For a complete flow we have* $\Phi_f(t,x)=\Phi_{\lambda f}(\frac{t}{\lambda},x)$.

Proof. It suffices to remark that $t\to\Phi_{\lambda f}(\frac{t}{\lambda},x)$ satisfies the initial value problem $z'=f(z),\ z(0)=x$. ☑

Lemma 2. *If f is a vector field such that its flow $\Phi\equiv\Phi_f$ is complete on Ω then for all t in $\mathbb{R}$ and all x in Ω we have*

$$\frac{\partial\Phi}{\partial x}(t,x)\,f(x)=\frac{\partial\Phi}{\partial t}(t,x)=f\left(\Phi\left(t,x\right)\right).$$

Proof. Differentiating the equation $\Phi(t,\Phi(\xi,x))=\Phi_f(\xi+t,x)$ with respect to ξ gives

$$\frac{\partial\Phi}{\partial x}(t,\Phi(\xi,x))\,\frac{\partial\Phi}{\partial\xi}(\xi,x)=\frac{\partial\Phi}{\partial\xi}(\xi+t,x).$$

Now the result follows setting $\xi=0$ in the last equation and taking into account that $\frac{\partial\Phi}{\partial\xi}(\xi,x)=f(\Phi(\xi,x))$. ☑

We are now in a position to relate the flow to classical solutions of a specific boundary value problem.

Theorem 1. *The flow $\Phi\equiv\Phi_f$ satisfies $\Phi(1,\cdot)=g(\cdot)$ and leaves Ω invariant if and only if Φ is a classical solution of the boundary value problem*

$$\begin{cases}\dfrac{\partial}{\partial t}\left(\left(\dfrac{\partial\Phi}{\partial x}(t,x)\right)^{-1}\dfrac{\partial\Phi}{\partial t}(t,x)\right)=0, & x\in\Omega,\quad t\in\mathbb{R},\\ \Phi(0,x)=x,\quad \Phi(1,x)=g(x), & x\in\Omega,\\ \dfrac{\partial\Phi}{\partial t}(t,x)\cdot\nu(x)=0, & x\in\partial\Omega,\quad t\in\mathbb{R}.\end{cases}$$

Proof. The only if part follows from Lemma 2. To show if part suppose that $\Phi(t,x)$ satisfies the above boundary value problem. We define

$$f(x):=\left(\frac{\partial\Phi}{\partial x}(t,x)\right)^{-1}\frac{\partial\Phi}{\partial t}(t,x).$$

Then

$$\frac{\partial\Phi}{\partial x}(t,x)\,f(x)-\frac{\partial\Phi}{\partial t}(t,x)=0.$$

This equation has as its unique solution satisfying the initial condition $\Phi(0,x) = x$, the flow Φ_f (see [6]). ☑

Next, we develop an expression to recover the field f from the time one map $g : \Omega \longleftrightarrow \Omega$ of the flow Φ_f, under the assumptions that f and g are sufficiently regular, and $\widetilde{x}$ is an attracting fixed point of g with basin of attraction Ω. In this case we can write

$$g(x) = \widetilde{x} + A\,(x-\widetilde{x}) + R_g\,(x), \quad \text{where} \quad \lim_{x\to\widetilde{x}} \frac{R_g\,(x)}{|x-\widetilde{x}|} = 0, \tag{3}$$

$$f(x) = B\,(x-\widetilde{x}) + R_f\,(x), \quad \text{where} \quad |R_f\,(x)| \le \text{const}\;|x-\widetilde{x}|^{1+p}, \tag{4}$$

with $p > 0$. (1) implies the matrices A and B are related by $e^B = A$. We shall denote by r (respectively l) the maximum (respectively the minimum) of the module of the eigenvalues of A. It should be clear that $0 < l \le r < 1$.

Theorem 2. *Let g be the time one map of a flow Φ_f and suppose the estimates (3) and (4) hold. If $r^{p+1} < l$ then there exists a $\delta > 0$ such that*

$$f(x) = \lim_{n\to\infty} (D\,g^n)^{-1}\,(x)\,B\,(g^n(x) - \widetilde{x}), \quad |x| < \delta. \tag{5}$$

Proof. There is no loss of generality assuming $\widetilde{x} = 0$. From Lemma 2 we have the expression $Dg(x)f(x) = f(g(x))$. It follows by iteration that

$$f(x) = (D\,g^n)^{-1}\,(x)\,f\,(g^n(x)), \qquad n \in \mathbb{Z},$$

In view of (4) it remains to show that $\lim_{n\to\infty} (D\,g^n)^{-1}\,(x) R_f\,(g^n(x)) = 0$. To do so observe that $|R_f\,(g^n(x))| \le \text{const}\;|g^n(x)|^{p+1}$. Now let $\varepsilon > 0$. It is easily seen that there exists a $\delta > 0$ such that for any $|x| < \delta$ we have $|g^n(x)| \le (r+2\varepsilon)\,|x|^n$, and consequently $|R_f\,(g^n(x))|^{p+1} \le \text{const}\;(r+\varepsilon)^{n(p+1)}\,|x|$.

We may choose a scalar product norm on $\mathbb{R}^n$ such that the associated norm of A^{-1} satisfies $|A^{-1}| \le \frac{1}{l} + \frac{\varepsilon}{2}$. From this follows that $\left|(D\,g\,(x))^{-1}\right|$, for $|x| < \delta$ and δ small enough. Now

$$(D\,g^n)^{-1}\,(x) = (D\,g\,(x))^{-1} \cdots \left(D\,g\,(g^{n-1}(x))\right)^{-1}. \tag{6}$$

So for $|x| < \delta$ we have

$$\left|(D\,g^n)^{-1}\,(x) R_f\,(g^n(x))\right| \le \text{const}\left(\frac{1}{l} + \varepsilon\right)^n (r+\varepsilon)^{n(p+1)}$$

and the claim follows.

Note that f can be extended to the whole domain Ω employing the expression $Dg(x)f(x) = f(g(x))$. ☑

Let us suppose $g : \Omega \longleftrightarrow \Omega$ is a given diffeomorphism with an attracting fixed point $\widetilde{x}$ satisfying $r^{p+1} < l$, assume B is a matrix satisfying $e^B = Dg\,(\widetilde{x})$

and that (5) defines a smooth field. It is not at all clear whether g is the time one map of Φ_f. Nevertheless there are some useful facts.

First observe that a smooth field defined by (5) satisfies

$$\begin{aligned} f(g(x)) &= \lim_{n\to\infty} (Dg^n)^{-1} (g(x))\, B\left(g^{n+1}(x) - \widetilde{x}\right) \\ &= \lim_{n\to\infty} (Dg(x))\left(Dg^{n+1}\right)^{-1} (x)\, B\left(g^{n+1}(x) - \widetilde{x}\right), \end{aligned}$$

thus

$$Dg(x)\, f(x) = f(g(x)). \tag{7}$$

Next, we see that

$$\begin{aligned} &\frac{\partial}{\partial x_j} (Dg^n)^{-1} (x)\, B\left(g^n(x) - \widetilde{x}\right) = \\ &\qquad (Dg^n(x))^{-1} \frac{\partial}{\partial x_j} (Dg^n(x))\, (Dg^n(x))^{-1}\, B\left(g^n(x) - \widetilde{x}\right) \\ &\qquad\qquad + (Dg^n(x))^{-1}\, B \frac{\partial}{\partial x_j} g^n(x), \end{aligned}$$

replacing $x = \widetilde{x}$ we obtain

$$\begin{aligned} \frac{\partial}{\partial x_j} (Dg^n)^{-1} (x)\, B\left(g^n(x) - \widetilde{x}\right)\big|_{x=\widetilde{x}} &= (Dg^n(\widetilde{x}))^{-1}\, B \frac{\partial}{\partial x_j} g^n(\widetilde{x}) \\ &= B\,(Dg^n(\widetilde{x}))^{-1} \frac{\partial}{\partial x_j} g^n(\widetilde{x}) = B\,\mathbf{e}_j, \end{aligned}$$

where $\{\mathbf{e}_j\}$ is the standard basis of $\mathbb{R}^k$. From this we have

$$Df(\widetilde{x}) = B. \tag{8}$$

Equation (7) has another important consequence: $g(x(t))$ and $g^{-1}(x(t))$ are solutions of the differential equation $x' = f(x)$ provided $x(t)$ is solution as well. Hence we obtain

$$g^{-1}\left(\Phi_f(t, g(x))\right) = \Phi_f(t, x), \quad \text{or} \quad \Phi_f(t, g(x)) = g\left(\Phi_f(t, x)\right). \tag{9}$$

The last equation means that $G = \{g^n \mid n \in \mathbb{Z}\}$ is a group of symmetries for the flow Φ_f.

3. The one-dimensional case

We will now discuss the cases in which Ω is a semi bounded interval or a circle. As the methods are rather different, we treat them separately.

3.1. Intervals

In this section we suppose that $g : \overline{\Omega} \longleftrightarrow \overline{\Omega}$ is a C^2 diffeomorphism defined on a closed interval $\overline{\Omega}$.

Theorem 3. *If $\widetilde{x} \in \overline{\Omega}$ is an hyperbolic fixed point of g, then there exists at most one smooth field f on $\overline{\Omega}$ such that g is the time one map of Φ_f.*

Proof. Using a similar reasoning as in Theorem 2 one concludes that f has to be given by

$$f(x) = \lim_{n\to\infty} \frac{g^n(x) - \widetilde{x}}{D\,g^n(x)} \ln D\,g(\widetilde{x}), \quad x \in \overline{\Omega}, \tag{10}$$

provided $D\,g(\widetilde{x}) < 1$. The case $D\,g(\widetilde{x}) > 1$ yields essentially the same formula for f, however the limit has to be taken with $n \to -\infty$.

Suppose now that g is a C^3 diffeomorphism with a fixed point $\widetilde{x}$, and $0 < D\,g(\widetilde{x}) < 1$. Our goal is to show that expression (10) defines a smooth field f with time one map g. Let us write for $x \in \Omega$ and $x > \widetilde{x}$

$$g(x) = \widetilde{x} + (x - \widetilde{x})\,h(x), \qquad r = D\,g(\widetilde{x}).$$

Note that h is C^3 and satisfies $0 < h(\widetilde{x}) < 1$.

From (6) we know that $|D\,g^n(x)| \leq (r+\varepsilon)^n$, provided $|x - \widetilde{x}| < \delta$ and δ sufficiently small. Set

$$G_n(x) := \frac{g^n(x) - \widetilde{x}}{D\,g^n(x)}, \qquad \widehat{G}_n(x) := \frac{1}{G_n(x)}, \qquad x \neq \widetilde{x}$$

A standard calculation shows

$$\widehat{G}_{n+1}(x) - \widehat{G}_n(x) = \frac{1}{h(g^n(x))} D\,h(g^n(x)) D\,g^n(x),$$

thus

$$\left|\widehat{G}_{n+1}(x) - \widehat{G}_n(x)\right| \leq \text{const } (r+\varepsilon)^n, \quad \text{for } |x - \widetilde{x}| < \delta.$$

In a similar way, we obtain

$$D^2 g^{n+1}(x) = D\,g(g^n(x))\, D^2 g^n(x) + (Dg^n(x))^2\, D^2 g(g^n(x)).$$

So

$$\left|D^2 g^{n+1}(x)\right| \leq \text{const } (r+\varepsilon)^n, \quad \text{for } |x - \widetilde{x}| < \delta.$$

Some additional computations supply

$$\left|\frac{d}{dx}\left(\widehat{G}_{n+1}(x) - \widehat{G}_n(x)\right)\right| \leq \text{const } (r+\varepsilon)^{n-1}, \quad \text{for } |x - \widetilde{x}| < \delta.$$

According to the above equations the sequence $\left(\widehat{G}_n(x)\right)_n$ converges uniformly on an interval $(\widetilde{x}, \widetilde{x}+\delta)$ to a smooth function $\widehat{f}$ defined on $(\widetilde{x}, \widetilde{x}+\delta)$. $\widehat{G}_1(x)$

is singular at $x = \widetilde{x}$ (in fact every $\widehat{G}_n(x)$ is singular as well) hence $\widehat{f}(x)$ is singular at $x = \widetilde{x}$.

Now, from

$$G_n(x) - G_{n+1}(x) = G_{n+1}(x)\, G_n(x)\, \frac{1}{h(g^n(x))} D\, h(g^n(x))\, D\, g^n(x),$$

we conclude that for a sufficiently small δ the sequence $(G_n(x))_n$ converges uniformly on an interval $[\widetilde{x}, \widetilde{x}+\delta)$ to a smooth function f defined on $[\widetilde{x}, \widetilde{x}+\delta)$ which vanishes at $x = \widetilde{x}$. As in Theorem (2) f can be extended to the whole domain Ω with $Dg(x)f(x) = f(g(x))$.

At this point we consider the smooth field on $\overline{\Omega}$ defined by (10). As usual, we suppose that g is a given diffeomorphism, and $\widetilde{x}$ is a fixed point of g with $D\,g(\widetilde{x}) < 1$. We claim that g is the time one map of the flow Φ_f. Recall that, $t \to g^{-1}(\Phi_f(t,x))$ is a solution of $x' = f(x)$. Observe that any solution of $x' = f(x)$ has rank Ω or is an equilibrium. Thus for $x_0 \in \Omega$ given, there exists $t_0 \in \mathbb{R}$ such that $g^{-1}(\Phi_f(t_0, x_0)) = x_0$, or $g(x_0) = \Phi_f(t_0, x_0)$. Note that $t_0 \neq 0$. Now, let $x \in \Omega$. Obviously, there exists $s \in R$ such that $\Phi_f(s, x_0) = x$. Then, in virtue of (9) and well known properties of the flows, we obtain

$$\begin{aligned} g(x) &= g(\Phi_f(s, x_0)) = \Phi_f(s, g(x_0)) = \Phi_f(s, \Phi_f(t_0, x_0)) \\ &= \Phi_f(t_0, \Phi_f(s, x_0)) = \Phi_f(t_0, x). \end{aligned}$$

The above equation and Lemma 1 imply that g is the time one map of the flow $\Phi_{t_0 f}$, so $D(t_0 f)(\widetilde{x}) = \ln D\, g(\widetilde{x})$. On the other side, from (8) we know that $D\, f(\widetilde{x}) = \ln D\, g(\widetilde{x})$, hence $t_0 = 1$ and g is the time one map of Φ_f. ☑

Example 1. Let a and b be positive.

$$g(x) := \frac{(a+b)\,x}{a\,x+b}$$

defines a diffeomorphism on $[0,1]$. See that $D\,g(0) = \frac{a+b}{a}$ and $D\,g(1) = \frac{b}{a+b}$. An easy computation shows that

$$g^n(x) = \frac{(a+b)^n x}{((a+b)^n - b^n)\,x + b^n}, \quad D\,g^n(x) = \frac{(a+b)^n\, b^n}{(((a+b)^n - b^n)\,x + b^n)^2}.$$

Then

$$\begin{aligned} &\lim_{n\to\infty} \frac{g^n(x) - 1}{D\,g^n(x)} \ln D\,g(1) \\ &= (x-1)\ln\left(\frac{b}{a+b}\right) \lim_{n\to\infty}\left(\left(1 - \left(\frac{b}{a+b}\right)^n\right)x + \left(\frac{b}{a+b}\right)^n\right) \\ &= \ln\left(\frac{b}{a+b}\right) x\,(1-x). \end{aligned}$$

Hence, the field $f(x) := \ln\left(\frac{b}{a+b}\right) x\,(1-x)$ defines the flow whose time one map is g.

An interesting case occurs when the diffeomorphism $g : [a,b] \longleftrightarrow [a,b]$ has as only fixed points a and b and both are hyperbolic. We obtain a symmetry property that relates the iterations $g^n(x)$ and $g^{-n}(x)$. Indeed, for any $x \in (a,b)$ we have:

$$\ln D\,g\,(a) \lim_{n\to\infty} \frac{g^n(x)-a}{D\,g^n(x)} = \ln D\,g\,(b) \lim_{n\to\infty} \frac{g^{-n}(x)-b}{D\,g^{-n}(x)} \quad \text{if} \quad g'(a) < 1,$$

$$\ln D\,g\,(a) \lim_{n\to\infty} \frac{g^{-n}(x)-a}{D\,g^{-n}(x)} = \ln D\,g\,(b) \lim_{n\to\infty} \frac{g^{n}(x)-b}{D\,g^{n}(x)} \quad \text{if} \quad g'(a) > 1.$$

We come back to the boundary value problems of Section 1 and state a sharper version of them. To do this, we denote with $C^2\left([0,1]\times[a,b]\right)$ the set of functions with partial derivatives up to order 2 and uniformly continuous on $(0,1)\times(a,b)$.

Theorem 4. *Let $g : [a,b] \longleftrightarrow [a,b]$ be a diffeomorphism. Φ is a flow and satisfies $\Phi(1,\cdot) = g(\cdot)$ iff Φ is a classical solution of the boundary value problem*

$$\begin{cases} \dfrac{\partial\Phi}{\partial x}\dfrac{\partial^2\Phi}{\partial t^2} - \dfrac{\partial\Phi}{\partial t}\dfrac{\partial^2\Phi}{\partial t\partial x} = 0, \\ \Phi(0,x) = x, \quad \Phi(1,x) = g(x), \quad \textit{for} \quad x \in (a,b), \\ \Phi(t,a) = a, \quad \Phi(t,b) = b, \quad \textit{for} \quad t \in (0,1) \end{cases} \tag{11}$$

Proof. Suppose $\Phi \in C^2\left([0,1]\times[a,b]\right)$ is a classical solution of the above boundary value problem. As $\frac{\partial}{\partial x}\Phi(0,x) = 1$ for all $x \in [a,b]$, we can choose an $\varepsilon > 0$ such that $\frac{\partial}{\partial x}\Phi(t,x) = 1$ for all $t \in (0,\varepsilon)$. For $t \in (0,\varepsilon)$ we have

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial\Phi}{\partial t}}{\frac{\partial\Phi}{\partial x}}\right) = \frac{\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial t\partial x}}{\left(\frac{\partial\Phi}{\partial x}\right)^2} = 0.$$

For fixed $t \in (0,\varepsilon)$, let us define

$$f(x) := \left(\frac{\partial\Phi}{\partial x}\right)^{-1} \frac{\partial\Phi}{\partial t}(t,x), \qquad x \in [a,b].$$

$f \in C^1[a,b]$ and $f(a) = f(b) = 0$. Let $\Psi \equiv \Phi_f$ be the only solution of the initial value problem

$$\frac{\partial\Psi}{\partial x}(t,x)\,f(x) - \frac{\partial\Psi}{\partial t}(t,x) = 0, \qquad \Psi(0,x) = x.$$

Since Ψ is solution of this initial value problem as well, $\Psi(t,x) = \Phi(t,x) = \Phi_f(t,x)$ for each $(t,x) \in [0,1]\times[a,b]$. This way we have shown that any solution Φ of the boundary value problem (11) is a flow Φ_f (for some field f).

On the other hand, if Φ is a flow stemming from a field f and $\Phi(1,\cdot) = g(\cdot)$, then we conclude from Theorem 1 that Φ is a classical solution of (11). ☑

As a consequence of our discussion we have the following result.

Theorem 5. *If $g : [a,b] \longleftrightarrow [a,b]$ is a C^3 diffeomorphism with hyperbolic fixed point at $x = a$ and $x = b$ and no additional fixed points, then there exists a unique solution of the boundary value problem (11).*

Before finishing this section, we point out that flows $\Phi_f(t,x)$ are somehow related with the wave equation and the Goursat problem. To see this, we remark that any one-dimensional invariant flow stemming from a smooth vector field can be expressed in the form

$$\Phi(t,x) = \alpha^{-1}(\alpha(x) + t),$$

where $\alpha : \Omega \longleftrightarrow \mathbb{R}$ is a diffeomorphism (see [2] and [4]). Now, define

$$\tau : \Omega \times \Omega \to R, \quad \tau(x,y) = t \quad \text{iff} \quad \Phi(t,x) = y.$$

If $x > y$ then $\tau(x,y)$ can be interpret as the required time to go from x to y following the solution $\Phi(t,x)$. Obviously the flow can be recovered from τ. For $\tau(x,y) = \alpha(y) - \alpha(x)$, τ is a smooth solution of the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial xy}\tau(x,y) = 0, & \text{for } (x,y) \in \Omega \times \Omega, \quad x < y < g(x), \\ \tau(x,x) = 0, & \text{for } x \in \Omega, \\ \tau(x,g(x)) = 1, & \text{for } x \in \Omega. \end{cases}$$

Nevertheless, solutions of the above boundary value problem do not necessarily determine a smooth vector field f such that $\Phi_f(1,\cdot) = g(\cdot)$. The reason is that there are infinitely many smooth solutions α of the Jabotisnsky equation (see [4]) $(\alpha(x) + 1) = \alpha(g(x))$.

3.2. Circles

In this section our attention is focussed on the unite circle S^1. We partially answer the question of when a given diffeomorphism $g : S^1 \leftrightarrow S^1$ is the time one map of a flow Φ_f associated with an autonomous system

$$\theta' = f(\theta), \qquad \theta \in S^1. \tag{12}$$

We only consider diffeomorphisms having no fixed points. Such diffeomorphisms are orientation preserving. Our approach is constructive and does not rest on the existence of solutions of the Abel equation. To solve a related question, Zdun [10] establishes conditions to guarantee that g can be embedded in a continuous flow; but such flows do not necessary stem from a smooth autonomous system.

We remark that if a field f on S^1 has no equilibriums, then each solutions of its associated autonomous system (12) is periodic. Moreover, any two solutions have the same period which is called the period of the flow.

It is convenient to define the covering map

$$\pi : \mathbb{R} \to S^1, \qquad \pi(x) = \exp(2\pi i x).$$

It can be shown that for every diffeomorphism g (having no fixed points), there exists a unique smooth lift $G : R \to R$, satisfying (see [5] for more details)

$$\pi(G(s)) = g(\pi(s)), \qquad 0 < G(0) < 1.$$

Let us suppose that g is the time one map of a flow Φ and that $\Phi(1,\cdot) = g(\cdot)$. We can assume that the period p of the flow is greater than 1.

Next, for a given diffeomorphism g we can define a strictly increasing sequence of positive integers (n_k) with $n_0 = 1$ and such that

$$G^{n_k}(0) \geq k, \quad G^j(0) < k, \quad \text{for } j = n_{k-1}, \dots, n_k - 1.$$

n_k can be interpreted as the minimum number of iterations to complete k turns around S^1.

Lemma 3. *Let g be a diffeomorphism on S^1 without fixed points and (n_k) be the sequence defined above. The sequence $\left(\frac{k}{n_k}\right)$ converges to the rotation number of g. Moreover, if g is the time one map of a flow Φ with period $p > 1$, i.e. $g(\cdot) = \Phi(1,\cdot)$, then the period and the rotation number are reciprocal.*

Proof. Let G the lift of g as defined previously. Since the rotation number ρ of a diffeomorphism f with lift F is defined to be the fractional part of $\lim_{n\to\infty} \frac{F^n(x)}{n}$ (it can be shown that this limit does not depend upon the choice of x nor upon the choice of the lift F, see [5]), we have $\rho = \lim_{n\to\infty} \frac{G^n(0)}{n}$.

Next, notice that the sequence (n_k) satisfies

$$G^{n_k}(0) \geq k, \quad \text{and} \quad G^n(0) < k \quad \text{for } n < n_k.$$

As a consequence we obtain

$$\frac{G^{n_k}(0)}{n_k} \geq \frac{k}{n_k} \quad \text{and} \quad \frac{G^{n_k-1}(0)}{n_k - 1} < \frac{k}{n_k - 1},$$

thus

$$\frac{n_k}{G^{n_k}(0)} < \frac{n_k}{k} < \frac{n_k - 1}{G^{n_k-1}(0)} + \frac{1}{k}.$$

Letting k go to ∞ we get the first claim.

Suppose now that g is the time one map of a flow with period p. In this case

$$n_k - 1 < kp < n_k.$$

Hence the sequence converges toward p.

In what follows we present some results related to flows on S^1. Particularly we shall show how to recover a flow from a certain function $\tau : [0,1) \to [0,p)$. Again $\tau(x)$ can be interpreted as the minimum time required by a moving point obeying (12) to go from $\pi(0)$ to $\pi(x)$. Indeed, given a positively oriented flow Φ with period p, we can define a function τ having such traits as follows

$$\tau(x) = t \quad \text{iff} \quad \Phi(t, \pi(0)) = \pi(x), \quad 0 \le t < p. \tag{13}$$

It is worth to note that:

P1. τ is a smooth function satisfying $\tau(0) = 0$ and $\tau'(x) > 0$ for all $x \in [0,1)$.

P2. The rule $\tau(x) = \tau(x-1) + p$ smoothly extends τ to the whole real axis $\mathbb{R}$.

P3. Let τ be defined on $\mathbb{R}$ according to **P2**. If $p > 1$ and $g(\theta) = \Phi(1, \theta)$, then $\tau(G(x)) = \tau(x) + 1$ for all $x \in [0,1)$.

The problem we solve is how to obtain the flow from a function τ satisfying **P1** and **P2**. The case of a negatively oriented flow can be handled in an analogous way.

Lemma 4. *If τ satisfies* **P1** *and* **P2**, *then*

$$\Phi(t, \pi(x)) = \pi\left(\tau^{-1}(\tau(x) + t)\right), \quad 0 \le t < p, \quad 0 \le x < 1$$

defines by periodic extension a (positively oriented) flow on S^1.

Proof. It is checked by a straightforward computation.

We are now in position to tackle the main problem in this section: to determine the flow from a given diffeomorphism.

Lemma 5. *If g is the time one map of a flow then any orbit of g is either finite or dense in S^1.*

Proof. If there exists a θ_0 such that its orbit is finite it can be seen that the orbit of any θ is also finite and has the same cardinality. Suppose $\{g^n(\theta_0)\}$ infinite for $\theta_0 \in S^1$. This means the period p is an irrational number. Let θ be another point in S^1 and t, $0 < t < p$ such that $\Phi(t, \theta_0) = \theta$. Applying the Jacobi Theorem we have that for a given $\delta > 0$ there exist k, $n \in \mathbb{N}$ which satisfy $|n - (t + kp)| < \delta$. Therefore we can find n such that $g^n(\theta_0)$ is as close to θ as we wish. We discuss the case of finite orbits.

Theorem 6. *If the rotation number of g is rational, g is the time one map of a flow iff there exists a positive integer m such that $g^m = I$, where I is the identity map on S^1.*

Proof. Let us assume that the rotation number of g is rational. In this case it is known that g has periodic points. So there are a θ_0 in S^1 and a positive integer m which satisfy $g^m(\theta_0) = \theta_0$. If g is the time one map of a flow we have

$\Phi(m,\theta_0) = g^m(\theta_0) = \theta_0$, therefore m is a multiple of the flow period. It follows that $g^m(\theta) = \theta$ for any $\theta \in S^1$.

Suppose now that $g^m = I$, m the smallest positive integer which satisfies such condition. Consider the orbit of $\alpha = \pi(0) : \{\alpha, g(\alpha), \ldots, g^{m-1}(\alpha)\}$. We have a reordering k_i of the set $\{0,1,\ldots,m-1\}$, determined by the condition $0 < a_0 < a_1 < \ldots < a_{m-1} < a_m = 1$ if $\pi(a_i) = g^{k_i}(\alpha)$. On the other hand, $h = g^{k_i}$ is a diffeomorphism on S^1 (with a lift H) which satisfies

$$h(\pi(a_i)) = \pi(a_{i+1}), \quad i = 0,1,\ldots,m-2, \quad h(\pi(a_{m-1})) = \pi(a_0).$$

Let $\tau : [a_0, a_1] \leftrightarrow [0,1]$ be a function satisfying

$$\tau'(x) > 0, \quad \tau'(0) = H'(0)\,\tau'(a_1).$$

Then the recursive rule

$$\tau(x) = \tau(H^{-1}(x)) + 1, \quad a_k < x \le a_{k+1}, \quad 1 \le k \le m-1$$

defines a function $\tau : [0,1] \leftrightarrow [0,m]$ which satisfies **P1**, **P2**, and **P3** (changing g by h in **P3**). Then we have a flow Φ, which stems from a field f, that satisfies $h(\theta) = \Phi(1,\theta)$. Now if $g(\alpha) = \pi(a_i)$, it follows that $g = h^i$ and $\Phi(i,\theta) = g(\theta)$. The condition $\tau'(0) = H'(0)\,\tau'(a_1)$ guarantees that the given extension of τ has a (continuous) derivative at the points $t = a_k$.

Remark 1. From the proof of the last theorem it is clear that the flow Φ is determined by τ. As there are infinitely many functions τ which satisfy the conditions required in the theorem, the flow is not uniquely determined by g.

Remark 2. If g is the time one map for a flow with no equilibrium points then the iterations g^n have lifts G_n $n \in \mathbb{N}$, $0 \le G_n(0) < 1$, whose graphics are equivalence classes in $\{(x,y) \in \mathbb{R}^2 : x \le y < x+1\}$. Moreover they are level curves for

$$(x,y) \to \tau(y) - \tau(x).$$

Example 2. (See [5], p. 109). The diffeomorphism on S^1 given by $g(\theta) = \theta + \pi + \frac{1}{2}\sin\theta$ is not the time one map of any flow (we wrote π to mean $\pi(\frac{1}{2}) = e^{i\pi}$). In fact, $g^2(\pi) = \pi$ and $g^2(\cdot)$ it is not the identity on S^1.

Theorem 7. *Let (x_n) be the sequence in $[0,1)$ defined by $\pi(x_n) = g^n(\pi(0))$. If the rotation number ρ of g is an irrational number, then g is the time one map of a positively oriented flow if and only if the orbit associated to $\pi(0)$ is dense in S^1 and the function*

$$x_n \to \tau_n, \quad \tau_n = n \mod p, \quad p = \frac{1}{\rho},$$

can be extended to $[0,1]$ as a smooth function τ which satisfies $\tau'(0) = \tau'(1)$ and $\tau'(x) > 0$ for any x in $[0,1]$.

Proof. If g is the time one map of a flow and its rotation number is irrational then the function τ defined in (13) gives the required extension.

Suppose now that the extension τ does exist. In this case it is easily seen that τ satisfies the **P2** condition. Let Φ be the flow determined by τ according to Lemma 4. It remains only to show that $\Phi(1, g^n(\pi(0))) = g(g^n(\pi(0)))$ for any n.

$$\Phi(1, g^n(\pi(0))) = \Phi(1, \pi(x_n)) = \pi\left(\tau^{-1}(\tau(x_n)+1)\right) = \pi\left(\tau^{-1}(\tau_n+1)\right).$$

Now,

$$\tau_n + 1 = \begin{cases} \tau_{n+1}, & \text{if } \tau_n + 1 < p, \\ \tau_{n+1} + p, & \text{if } \tau_n + 1 > p. \end{cases}$$

In the case $\tau_n + 1 = \tau_{n+1}$ we have

$$\Phi(1, g^n \pi(0)) = \pi\left(\tau^{-1}(\tau_{n+1})\right) = \pi(x_{n+1}) = g^{n+1}(\pi(0)).$$

In the other case we get the same conclusion by using **P2**.

Example 3. For $r > 1$ let $y = G_r(x)$, $G_r : \mathbb{R} \to \mathbb{R}$ the function implicitly defined by

$$ry + \frac{1}{2\pi}\sin 2\pi y = 1 + r\,x + \frac{1}{2\pi}\sin 2\pi x.$$

Now we define the diffeomorphism $g_r : S^1 \to S^1$ by $g_r(\pi(x)) = \pi(G_r(x))$. Note that G_r is a lift of g_r.

It can be seen that there exists an integer m such that $g_r^m = I$ (I the identity on S^1), if and only if r is rational. In fact $x_m = G_r^m(x_0)$ is the only number that satisfies:

$$\begin{aligned} r\,x_m + \frac{1}{2\pi}\sin 2\pi x_m &= 1 + r\,x_{m-1} + \frac{1}{2\pi}\sin 2\pi x_{m-1} \\ &\;\vdots \\ &= m + r\,x_0 + \frac{1}{2\pi}\sin 2\pi x_0. \end{aligned} \tag{14}$$

As $g_r^m(\pi(x)) = \pi(G_r^m(x))$ we have $g_r^m = I$. It means that $G_r^m(x) = x + k$, k an integer number and therefore $x_m = x_0 + k$. From this and using (14) we get $r\,k = m$. Conversely, if r is a rational number, $r = \frac{m}{k}$, we get $G_r^m(x_0) = x_0 + k$, for $x_0 \in \mathbb{R}$, whence $g_r^m = I$. So, in view of Theorem 6, if r is rational we can conclude that g_r is the time one map of a flow. Indeed it is easy to check that for any $r > 1$, $g_r(\theta) = \Phi_r(1, \theta)$, where Φ_r is the flow on S^1 associated to the field

$$f_r(\theta) = \frac{1}{r + \cos\theta}(-\sin\theta, \cos\theta).$$

On the other hand, in view of Theorem 7, we have that if r is an irrational number greater than 1 the orbits of g_r are dense sets in S^1. Figure 1 shows the graphs of the lifts $G_{r,m}$ for $r = \frac{\pi}{2}$ and $m = 1, \dots, 9$.

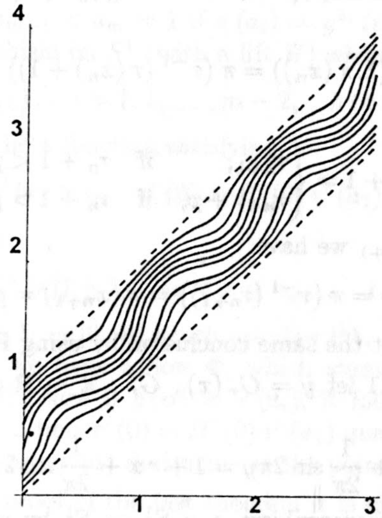

FIGURE 1. $G_{r,m}$ for $r = \frac{\pi}{2}$.

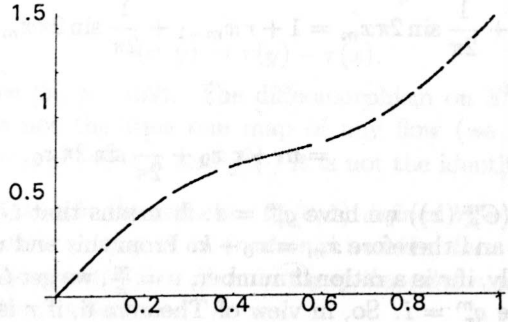

FIGURE 2. $x_n \to \tau_n$, $\tau_n = n \mod \frac{\pi}{2}$, $n = 0, \dots, 300$ associated to g_r for $r = \frac{\pi}{2}$.

Finally, in Figure 2 we present the graph of the function $x_n \to \tau_n, \quad \tau_n = n \mod \frac{\pi}{2}$, $n = 0, \dots, 300$ associated to g_r for $r = \frac{\pi}{2}$. Note that this function admits an extension as the one described in Theorem 7.

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