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Lifting theorems for some classes of two parameter martingales

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ABSTRACT. By means of nonstandard analysis we establish some lifting theorems for two parameter stochastic processes, for two parameter martingales and for weak, strong and i-martingales. We also prove that the standard part of an internal martingale is a standard larc martingale (a two parameter version of a càdlàg martingale). A basic nonstandard two parameter stochastic integral is introduced. An integral representation of Wong and Zakai proves to be a very useful tool for our purposes.

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1. Introduction

A good introduction to nonstandard analysis can be found in [1]. The main features that we need in our work are the following.

We assume the existence of a set ${}^*\mathbb{R} \supseteq \mathbb{R}$, called the set of the nonstandard real numbers, and of a mapping $* : V(\mathbb{R}) \to V({}^*\mathbb{R})$, where $V_1(S) = S$,

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$V_{n+1}(S) = V_n(S) \cup \mathfrak{P}(V_n(S))$, $\mathfrak{P}(A)$ denoting the set of subsets of A, and $V(S) = \cup_{n\in\mathbb{N}} V_n(S)$, with three basic properties. To state this properties we introduce the following notions:

An **elementary statement** is a statement Φ built up from " $=$ ", " $\in$ ", the predicate and functional symbols, the logical conectives "*and*", "*or*", "*not*" and "*implies*", and the bounded quantifiers $(\forall u \in v)$, $(\exists u \in v)$.

An **internal object** A is an element of $V(^*\mathbb{R})$ such that $A = {}^*S$, $S \in V(\mathbb{R})$. A set in $V(^*\mathbb{R})$ which is not internal is called external.

(1) **Extension Principle.** The set $^*\mathbb{R}$ is a proper extension of $\mathbb{R}$ and $* : V(\mathbb{R}) \to V(^*\mathbb{R})$ is an embedding such that $^*r = r$ for all $r \in \mathbb{R}$.
(2) **The Saturation Property**: Let $\{R_n : n \in \mathbb{N}\}$ be a sequence of internal objects, $\{S_m : m \in \mathbb{N}\}$ one sequence of internal sets. If for each $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that $R_n \in S_m$ for all $n \geq N_m$, then $\{R_n : n \in \mathbb{N}\}$ can be extended to an internal sequence $\{R_\eta : \eta \in {}^*\mathbb{N}\}$ such that $R_\eta \in \cap_m S_m$ for every $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$.
(2′) **General Saturation Principle**: Let κ be an infinite cardinal. A non-standard extension is called κ-saturated if for every family $\{X_i\}_{i\in I}$, $\text{card}(I) < \kappa$, with the finite intersection property, the intersection $\cap_{i\in I} X_i$ is nonempty; i.e., this intersection contains some internal object.
(3) **Transfer Principle**: Let $\Phi(X_1, \ldots, X_m, x_1, \ldots, x_n)$ be an elementary statement in $V(\mathbb{R})$. Then, for $A_1, \ldots, A_m \subseteq \mathbb{R}$ and $r_1, \ldots, r_n \in \mathbb{R}$, $\Phi(A_1, \ldots, A_m, r_1, \ldots, r_n)$ is true in $V(\mathbb{R})$ if and only if the statement $\Phi(^*A_1, \ldots, {}^*A_m, {}^*r_1, \ldots, {}^*r_n)$ is true in $V(^*\mathbb{R})$.

The system $(^*\mathbb{R}, {}^*+, {}^*\cdot, {}^*\leq)$ is a field that extends $\mathbb{R}$ as an ordered field. In general we will omit the $*$ for the operations and the order relation.

In $^*\mathbb{R}$ we can distinguish three kinds of numbers:

(a) $x \in {}^*\mathbb{R}$ is infinitesimal, if $|x| < r$ for each $r \in \mathbb{R}^+$.
(b) $x \in {}^*\mathbb{R}$ is a finite number, if there is a real number $r \in \mathbb{R}^+$ such that $|x| < r$.
(c) $x \in {}^*\mathbb{R}$ is an infinite number, if $|x| > r$ for all $r \in \mathbb{R}^+$.

To each finite number $x \in {}^*\mathbb{R}$ we can associate a unique real number $r := st(x) := {}^o x$ such that $x = r + \varepsilon$, where ε is infinitesimal. We say that x is infinitely closed to y, and denote it by $x \approx y$, if and only if $x - y$ is infinitesimal.

In general we use capital letters H, F, X, etc. for internal functions and processes, while h, f, x, etc. are used for standard ones.

For a given set A, $^*A$ stands for the elementary extension of A, and $ns(^*A)$ denotes the nearstandard points in $^*A$. If s is in $ns(^*A)$, the standard part of s is written as $st(s)$ or ${}^o s$. For a given function f, $^*f$ means the elementary extension of f.

We say that the set T is S-dense if $\{{}^o\underline{t} : \underline{t} \in T, {}^o\underline{t} < \infty\} = [0, \infty)$, and $ns(T) := \{\underline{t} \in T : {}^o\underline{t} < \infty\}$. With T we denote an internal S-dense subset of ${}^*[0, \infty)$. The elements of T, or more generally, of ${}^*[0, \infty)$, are denoted with $\underline{s}$, $\underline{t}$, $\underline{u}$, etc. The real numbers in $[0, \infty)$ are denoted by s, t, u, etc. We will work with different sets T, so we will always specify the definition of such T.

With $\mathbb{N}$ we denote the set of nonzero natural numbers $\{1, 2, 3, \dots\}$, and $\mathbb{N}_o = \mathbb{N} \cup \{0\}$. Elements of $\mathbb{N}_o$ are denoted with n, m, l, etc., while elements in ${}^*\mathbb{N} \setminus \mathbb{N}$ will be denoted with η, N, etc.

We say that a set T is hyperfinite if it is internal and its cardinality is an $N \in {}^*\mathbb{N}$.

If $(\Omega, \mathfrak{A}, \overline{P})$ is the internal measure space where Ω is an hyperfinite set, $\mathfrak{A}$ is the algebra of hyperfinite subsets of Ω and $\overline{P}$ is an internal measure, the corresponding Loeb space (see [1]) is $\underline{\Omega} = (\Omega, L(\mathfrak{A}), L(\overline{P}))$, and $L(\overline{P})$ will be the unique measure extending ${}^o\overline{P}$ to the σ-algebra $\sigma(\mathfrak{A})$ generated by $\mathfrak{A}$. $L(\mathfrak{A})$ will stand for the $L(\overline{P})$ completion of $\sigma(\mathfrak{A})$. In general we writte P for $L(\overline{P})$, and it will be a probability.

To say that $F : A \to B$ is an internal function means that the domain, range and graph of the function are internal concepts.

In this paper we give conditions for the existence of liftings of two parameter stochastic processes and two parameter martingales. We also give conditions ensuring that for some internal martingales the corresponding standard parts are standard martingales.

The terminology and notations are the usual in nonstandard analysis: see for example [1]. In particular, we assume to have saturation, as is usually done when discussing stochastic processes in the context of nonstandard analysis.

In order to simplify notation and proofs, we consider stochastic processes defined on $[0, 1]^2$ and with values in $\mathbb{R}$, instead of processes defined on $[0, \infty)^2$ with values on $\mathbb{R}^d$. In general, we only consider nearstandard points in ${}^*[0, \infty)^2$. If T is an S-dense set on $[0, \infty)$, then an internal stochastic process $X : T^2 \times \Omega \to {}^*\mathbb{R}^d$ will have a property P if and only if each of its components has property P. Therefore, proofs may be reduced to the one dimensional case.

The set $[0, 1]^2$ is equipped with the partial orders

$$(s_1, t_1) \leq (s_2, t_2) \iff s_1 \leq s_2 \text{ and } t_1 \leq t_2,$$

and

$$(s_1, t_1) \Delta (s_2, t_2) \iff s_1 \leq s_2 \text{ and } t_1 \geq t_2.$$

We use the notation $(s_1, t_1) < (s_2, t_2)$ to express that $(s_1, t_1) \leq (s_2, t_2)$ and $s_1 < s_2$ or $t_1 < t_2$, whereas $(s_1, t_1) \wedge (s_2, t_2)$ will mean $(s_1, t_1) \Delta (s_2, t_2)$ and $s_1 < s_2$ or $t_1 > t_2$. Also, $(s_1, t_1) \ll (s_2, t_2)$ will stand for $s_1 < s_2$ and $t_1 < t_2$.

Let $\rho : [0, 1]^2 \longrightarrow [0, 1]^2$, $\rho(s, t) = (\rho_1(s, t), \rho_2(s, t))$, be such that each $\rho_i \in \Lambda[0, 1]$, $i = 1, 2$, where $\Lambda[0, 1]$ is the set of time deformations of $[0, 1]$

(see [14]). We call ρ a deformation of $[0,1]^2$ and $\Lambda[0,1]^2$ will denote the set of all such deformations. We define on this set the measure $d(\rho)$ of amount of deformation by $d(\rho) = d_1(\rho) + d_2(\rho)$, where

$$d_i(\rho) = \sup_{r,s\in[0,1]} \left|\log \frac{\rho_i(r) - \rho_i(s)}{r - s}\right|.$$

1.1. Definition. A function $x : [0,1]^2 \longrightarrow \mathbb{R}$ is a larc in $[0,1]^2$, if for each $(s_o, t_o) \in [0,1]^2$ the four limits:

$$\lim_{\substack{s\to s_o^+\\ t\to t_o^+}} x(s,t) = x(s_o,t_o), \qquad \lim_{\substack{s\to s_o^+\\ t\to t_o^-}} x(s,t) = x(s_o,t_o^-),$$

$$\lim_{\substack{s\to s_o^-\\ t\to t_o^+}} x(s,t) = x(s_o^-,t_o^+), \qquad \lim_{\substack{s\to s_o^-\\ t\to t_o^-}} x(s,t) = x(s_o^-,t_o^-)$$

exist. They are called the quadrantal limits.

We denote by D^2 the set of all larcs in $[0,1]^2$. In this set we define a metric k_o by

$$k_o(x,y) =$$
$$\inf\Big\{\varepsilon \in \mathbb{R}^+ : (\exists \rho \in \Lambda[0,1]^2)\Big(\sup_{r\in[0,1]^2} |x(r) - y(\rho(r))| < \varepsilon \text{ and } d(\rho) < \varepsilon\Big)\Big\},$$

$x, y \in D^2$. Then, (D^2, k_o) is a separable and complete metric space. By $\mathcal{J}_2$ we denote the topology induced by this metric.

Note: Points in $[0,1]^2$ will be usually denoted by (s,t), (s_1,t_1), etc. Points in ${}^*[0,1]^2$, by $(\underline{s},\underline{t})$, $(\underline{s}_1,\underline{t}_1)$, etc.

To each point $(\underline{s},\underline{t}) \in {}^*[0,1]^2$ we assign the following sets:

$$Q^1_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} \geq \underline{s} \text{ and } \underline{v} \geq \underline{t}\},$$
$$Q^2_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} < \underline{s} \text{ and } \underline{v} \geq \underline{t}\},$$
$$Q^3_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} < \underline{s} \text{ and } \underline{v} < \underline{t}\},$$
$$Q^4_{(\underline{s},\underline{t})} = \{(\underline{u},\underline{v}) \in {}^*[0,1]^2 : \underline{u} \geq \underline{s} \text{ and } \underline{v} < \underline{t}\}.$$

1.2. Definition. Let $F \in {}^*D^2$ be such that $F(\underline{s},\underline{t}) \in ns({}^*\mathbb{R})$ for $(\underline{s},\underline{t}) \in {}^*[0,1]^2$. Then:

(a) F is of class SD^2, if for each $(s,t) \in [0,1]^2$ there are points $(\underline{s}_1,\underline{t}_1) \approx (\underline{s}_2,\underline{t}_2) \approx (\underline{s}_3,\underline{t}_3) \approx (\underline{s}_4,\underline{t}_4) \approx (s,t)$ such that:

(i) If $(\underline{u}_1, \underline{v}_1) \approx (s,t), (\underline{u}_1, \underline{v}_1) \in Q^1_{(\underline{s}_1, \underline{t}_1)}$, then $F(\underline{u}_1, \underline{v}_1) \approx F(\underline{s}_1, \underline{t}_1)$.

(ii) If $(\underline{u}_2, \underline{v}_2) \approx (s,t), (\underline{u}_2, \underline{v}_2) \in Q^2_{(\underline{s}_2, \underline{t}_2)}$, then $F(\underline{u}_2, \underline{v}_2) \approx F(\underline{s}_2^-, \underline{t}_2)$.

(iii) If $(\underline{u}_3, \underline{v}_3) \approx (s,t), (\underline{u}_3, \underline{v}_3) \in Q^3_{(\underline{s}_3, \underline{t}_3)}$, then $F(\underline{u}_3, \underline{v}_3) \approx F(\underline{s}_3^-, \underline{t}_3^-)$.

(iv) If $(\underline{u}_4, \underline{v}_4) \approx (s,t), (\underline{u}_4, \underline{v}_4) \in Q^4_{(\underline{s}_4, \underline{t}_4)}$, then $F(\underline{u}_4, \underline{v}_4) \approx F(\underline{s}_4, \underline{t}_4^-)$.

(b) F is of class SD^2J, or a larc lift, if (a) holds with $(\underline{s}_1, \underline{t}_1) = (\underline{s}_2, \underline{t}_2) = (\underline{s}_3, \underline{t}_3) = (\underline{s}_4, \underline{t}_4)$, and $F(\underline{s}, \underline{t}) \approx F(0,0)$ for all $(\underline{s}, \underline{t}) \approx (0,0)$ in ${}^*[0,1]^2$.

(c) F is S-continuous (SC) if $F(\underline{s}, \underline{t}) \approx F(\underline{u}, \underline{v})$ whenever $(\underline{s}, \underline{t}) \approx (\underline{u}, \underline{v})$ and $(\underline{s}, \underline{t}), (\underline{u}, \underline{v}) \in T^2$, where $T = \{k\delta t : \delta t = \frac{1}{N!}, N \in {}^*\mathbb{N} \setminus \mathbb{N}, k = 0, 1, \ldots, N!\}$.

A function $F : T^2 \longrightarrow {}^*\mathbb{R}$ is of class SD^2 (SD^2J, SC) in T^2 if it is the restriction to T^2 of an SD^2 (SD^2J, SC) function F on ${}^*[0,1]^2$.

1.3. Definition. The standard part of an SD^2 function F on T^2 is the function $st(F)$ defined by

$$st(F)(s,t) = \lim_{{}^o(\underline{s},\underline{t}) \downarrow (s,t)} {}^oF(\underline{s}, \underline{t}), \qquad (s,t) \in [0,1]^2.$$

1.4. Proposition. *Suppose $F : T^2 \longrightarrow {}^*\mathbb{R}$ is the restriction of a function in ${}^*D^2$ to T^2 and $F(\underline{s}, \underline{t}) \in ns({}^*\mathbb{R})$, the set of nearstandard points in ${}^*\mathbb{R}$, for all $(\underline{s}, \underline{t}) \in T^2$. Then F is SD^2 if and only if $st(F)$ exists and belongs to D^2.*

Proof. First assume that F is of class SD^2 and fix $\varepsilon > 0$ and $(s,t) \in [0,1]^2$. There exists $(\underline{s}_1, \underline{t}_1) \in T^2$, $(\underline{s}_1, \underline{t}_1) \approx (s,t)$, such that if $(\underline{u}, \underline{v}) \approx (s,t)$, $(\underline{u}, \underline{v}) \in T^2$ and $(\underline{u}, \underline{v}) \in Q^1_{(\underline{s}_1, \underline{t}_1)}$, then $F(\underline{u}, \underline{v}) \approx F(\underline{s}_1, \underline{t}_1)$. Let S be the set of points $n \in {}^*\mathbb{N}$ such that if $(\underline{u}, \underline{v}) \in T^2 \cap Q^1_{(\underline{s}_1, \underline{t}_1)}$ and $\|(\underline{u}, \underline{v}) - (\underline{s}_1, \underline{t}_1)\| < 1/n$ then $|F(\underline{u}, \underline{v}) - F(\underline{s}_1, \underline{t}_1)| < \varepsilon$. Then $S \supseteq {}^*\mathbb{N} \setminus \mathbb{N}$ and is internal. Thus there is $n_o \in \mathbb{N}$ such that if $(\underline{u}, \underline{v}) \in T^2 \cap Q^1_{(\underline{s}_1, \underline{t}_1)}$ and $\|(\underline{u}, \underline{v}) - (\underline{s}_1, \underline{t}_1)\| < 1/n_o$ then $|F(\underline{u}, \underline{v}) - F(\underline{s}_1, \underline{t}_1)| < \varepsilon$. Let $\delta = \frac{1}{\sqrt{2}n_o}$. Then

$$(\underline{u}, \underline{v}) \in [(\underline{s}_1, \underline{t}_1), (\underline{s}_1 + \delta, \underline{t}_1 + \delta)) \cap T^2 \Rightarrow |F(\underline{u}, \underline{v}) - F(\underline{s}_1, \underline{t}_1)| < \varepsilon.$$

Also

$${}^o(\underline{u}, \underline{v}) \in ((s,t), (s+\delta, t+\delta)) \Rightarrow (\underline{u}, \underline{v}) \in [(\underline{s}_1, \underline{t}_1), (\underline{s}_1 + \delta, \underline{t}_1 + \delta)).$$

Thus

$$|F(\underline{u}, \underline{v}) - F(\underline{s}_1, \underline{t}_1)| < \varepsilon \Rightarrow |{}^oF(\underline{u}, \underline{v}) - {}^oF(\underline{s}_1, \underline{t}_1)| < \varepsilon.$$

Therefore $st(F)(s,t)$ exists and is given by

$$st(F)(s,t) = \lim_{{}^o(\underline{u},\underline{v}) \downarrow (s,t)} {}^oF(\underline{u}, \underline{v}) = {}^oF(\underline{s}_1, \underline{t}_1).$$

In conclusion, if $(u,v) \in [0,1]^2$, there is $(\underline{u}_1, \underline{v}_1) \approx (u,v)$ such that

$$st(F)(u,v) = {}^oF(\underline{u}_1, \underline{v}_1).$$

We now claim that $st(F)$ is continuous from the right. We see that if $(u,v) \in ((s,t),(s+\delta,t+\delta))$ and $(\underline{u}_1,\underline{v}_1) \approx (u,v)$ with $st(F)(u,v) = {}^oF(\underline{u}_1,\underline{v}_1)$, then $(\underline{u}_1,\underline{v}_1) \in ((\underline{s},\underline{t}),(\underline{s}+\delta,\underline{t}+\delta))$ with $(\underline{s},\underline{t}) \approx (s,t)$ and $st(F)(s,t) = {}^oF(\underline{s},\underline{t})$. Thus

$$|F(\underline{u}_1,\underline{v}_1) - F(\underline{s},\underline{t})| < \varepsilon \Rightarrow |{}^oF(\underline{u}_1,\underline{v}_1) - {}^oF(\underline{s},\underline{t})| < \varepsilon,$$

and therefore

$$|st(F)(u,v) - st(F)(s,t)| < \varepsilon$$

whenever $(u,v) \in [(s,t),(s+\delta,t+\delta))$. In a similar way we prove the existence of the other quadrantal limits.

On the other hand, if $st(F)$ exists and belongs to D^2, fix $(s,t) \in [0,1]^2$. We have

$$\lim_{\substack{u\to s^+\\ v\to t^+}} st(F)(u,v) = st(F)(s,t) = \lim_{{}^o(\underline{u},\underline{v})\downarrow(s,t)} {}^oF(\underline{u},\underline{v}).$$

For $n > 0$, there is $\delta_n > 0$ ($\delta_n < 1/n$) such that

$${}^o(\underline{u},\underline{v}) \in ((s,t),(s+\delta_n,t+\delta_n)) \Rightarrow |{}^oF(\underline{u},\underline{v}) - st(F)(s,t)| < 1/n.$$

Let

$$D_n = \left\{(\underline{u},\underline{v}) \in T^2 : (s,t) \ll (\underline{u},\underline{v}) \ll (s+\delta_n, t+\delta_n)\right\}.$$

Then D_n is an internal set and if $F_n =$

$$\left\{(\underline{a},\underline{b}) \in D_n : (\underline{u},\underline{v}) \in D_n \text{ and } (\underline{u},\underline{v}) \geq (\underline{a},\underline{b}) \Rightarrow |F(\underline{u},\underline{v}) - st(F)(s,t)| < \frac{1}{n}\right\},$$

then $F_n \neq \varnothing$ and $\{F_n : n \in \mathbb{N}\}$ has the finite intersection property. Thus, by saturation, $\bigcap_{n\in\mathbb{N}} F_n \neq \varnothing$. Take $(\underline{s}_1,\underline{t}_1) \in \bigcap_{n\in\mathbb{N}} F_n$. Then $(\underline{s}_1,\underline{t}_1) \approx (s,t)$, and for all $(\underline{u},\underline{v}) \geq (\underline{s}_1,\underline{t}_1)$ and $(\underline{u},\underline{v}) \approx (s,t)$, we have $F(\underline{u},\underline{v}) \approx st(F)(s,t)$. In particular, $F(\underline{s}_1,\underline{t}_1) \approx st(F)(s,t)$, so that, when $(\underline{u},\underline{v}) \approx (s,t)$, then $F(\underline{u},\underline{v}) \approx F\underline{s}_1,\underline{t}_1)$. Now let $A = \lim_{\substack{u\to s^-\\ v\to t^+}} st(F)(u,v)$ whenever the limit exists. Given $n \in \mathbb{N}$ there exists δ_n ($\delta_n < 1/n$) such that

$$(\underline{u},\underline{v}) \in T^2 \text{ and } (s-\delta_n,t) \ll (\underline{u},\underline{v}) \ll (s,t+\delta_n) \Rightarrow |F(\underline{u},\underline{v}) - A| < 1/n.$$

Let $D_n = \left\{(\underline{u},\underline{v}) \in T^2 : (s-\delta_n,t) \ll (\underline{u},\underline{v}) \ll (s,t+\delta_n)\right\}$ and

$$F_n = \{(\underline{a},\underline{b}) \in D_n : (\underline{u},\underline{v}) \in D_n \text{ and } \underline{u} < \underline{a},\ \underline{v} \geq \underline{b} \Rightarrow |F(\underline{u},\underline{v}) - A| < 1/n\}.$$

We have that $F_n \neq \varnothing$ and that $\{F_n \mid n \in \mathbb{N}\}$ posseses the property of finite intersection. Thus, by saturation, $\bigcap_{n\in\mathbb{N}} F_n \neq \varnothing$. Let $(\underline{s}_2,\underline{t}_2) \in \bigcap_{n\in\mathbb{N}} F_n$. If $(\underline{u},\underline{v}) \approx (s,t)$ and $\underline{u} < \underline{s}_2$, $\underline{v} \geq \underline{t}_2$, then $F(\underline{u},\underline{v}) \approx A$. In particular, if $\underline{s}_2 - \delta t < \underline{s}_2$, $\underline{t}_2 = t_2$ ($\delta t = \frac{1}{N!}$), then $F(\underline{s}_2 - \delta t, t_2) \approx A$, and so, if $(\underline{u},\underline{v}) \approx (s,t)$ and $\underline{u} < \underline{s}_2$, $\underline{v} \geq t_2$, then $F(\underline{u},\underline{v}) \approx F(\underline{s}_2 - \delta t, t_2)$.

In a similar way we find $(\underline{s}_3,\underline{t}_3)$ and $(\underline{s}_4,\underline{t}_4)$ as in Definition 1.2 (a) (iii) and (iv). Consequently, F is SD^2. ☑

1.5. Proposition. *Function F is SD^2J on T^2, and for every $(s,t) \in [0,1]^2$ there is a $(\underline{s},\underline{t}) \approx (s,t)$ such that*

(i) $(\underline{u},\underline{v}) \approx (s,t)$ *and* $\underline{u} < \underline{s}$, $\underline{v} < \underline{t} \Rightarrow F(\underline{u},\underline{v}) \approx st(F)(s^-,t^-)$.
(ii) $(\underline{u},\underline{v}) \approx (s,t)$ *and* $\underline{u} < \underline{s}$, $\underline{v} \geq \underline{t} \Rightarrow F(\underline{u},\underline{v}) \approx st(F)(s^-,t)$.
(iii) $(\underline{u},\underline{v}) \approx (s,t)$ *and* $\underline{u} \geq \underline{s}$, $\underline{v} < \underline{t} \Rightarrow F(\underline{u},\underline{v}) \approx st(F)(s,t^-)$.
(iv) $(\underline{u},\underline{v}) \approx (s,t)$ *and* $\underline{u} \geq \underline{s}$, $\underline{v} \geq \underline{t} \Rightarrow F(\underline{u},\underline{v}) \approx st(F)(s,t)$.

In particular, if $st(F)$ is continuous at (s,t), then $F(\underline{u},\underline{v}) \approx F(\underline{s},\underline{t})$ for all $(\underline{u},\underline{v}) \approx (\underline{s},\underline{t}) \approx (s,t)$.

Proof. From Proposition 1.4, we have that $st(F)$ exists and belongs to D^2. Following the same steps as in the proof above we find, for example, that given $(s,t) \in [0,1]^2$,

$$\lim_{\substack{u \to s^- \\ v \to t^+}} st(F)(u,v) = st(F)(s^-,t)$$

exists, and from this, we have that there is $(\underline{s}_2,\underline{t}_2) \in T^2$, $(\underline{s}_2,\underline{t}_2) \approx (s,t)$, such that, whenever $(\underline{u},\underline{v}) \in T^2$, $(\underline{u},\underline{v}) \approx (s,t)$ and $\underline{u} < \underline{s}_2$ and $\underline{v} \geq \underline{t}_2$, then $F(\underline{u},\underline{v}) \approx F(\underline{s}_2,\underline{t}_2) \approx st(F)(s^-,t)$. Analogously we infer for the other three limits that F is SD^2J, so that the points $(\underline{s}_i,\underline{t}_i)$ are the same $(i = 1,2,3,4)$, and we have the result. ☑

1.6. Theorem. *The class of functions in ${}^*D^2$ which are nearstandard in the $\mathcal{J}_2$ topology is SD^2J, and $st|_{SD^2J}$ is the standard part map for the $\mathcal{J}_2$ topology.*

The proof is similar to that of the one parameter case. We have to carry on the same analysis for the two coordinates simultaneously.

2. Lifting theorems for two parameter martingales

2.1. Definition. An internal stochastic process X is of class SD^2 (SD^2J, SC) if for almost all w the mapping $X((\cdot,\cdot),w) : T^2 \longrightarrow {}^*\mathbb{R}$ is of class SD^2 (SD^2J, SC).

We now extend the notion of standard part of an internal SD^2 function to an internal process with sample paths in SD^2 via the following: a process $st(X)$ with sample paths in D^2 is defined by fixing $x_o \in \mathbb{R}$ and letting

$$st(X)(s,t)(w) = \begin{cases} st(X(\cdot,\cdot,w))(s,t), & \text{if } X(\cdot,\cdot,w) \in SD^2, \\ x_o, & \text{otherwise.} \end{cases}$$

An SD^2 (SD^2J) lifting of a stochastic process $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ is an internal stochastic process X of class SD^2 (SD^2J) such that $st(X)$ and x are indistinguishable.

When not likely to generate confussion, we will write $X(\underline{s},\underline{t})$, $x(s,t)$, etc., instead of $X(\underline{s},\underline{t},w)$, $x(s,t,w)$, etc.

2.2. Theorem. *A stochastic process $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ has sample paths in D^2 a.s. (i.e., for P-almost all w) (and $\{|x(s,t)|^p,\ (s,t) \in [0,1]^2\}$ is uniformly integrable for some real number $p \geq 1$) if and only if it has an SD^2J lifting X (such that $|X(\underline{s},\underline{t})|^p$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$).*

$\overline{E}(Y)$ will denote the internal expectation of the internal random variable Y, $E(y)$ will denote the expectation of the random variable y. For the meaning of S-integrability and uniform integrability as well, the reader can consult [1] and [3] respectively.

Proof. First we show the "only if" part. From Theorem 1.6, if X is of class SD^2J then $X \in ns_{\mathcal{J}^2}({}^*D^2)$ and $st(X) = st_{\mathcal{J}_2}(X) = x$, $x \in D^2$. Now, if $|X(\underline{s},\underline{t})|^p$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$, we claim that

$$\{|x(s,t)|^p,\ (s,t) \in [0,1]^2\}$$

is uniformly integrable for some $p \geq 1$. In fact, we observe that

(i) The set $A = \{\overline{E}(|X(\underline{s},\underline{t})|^p) : (\underline{s},\underline{t}) \in T^2\}$ is an internal set which takes only finite values. Let

$$B = \{n \in {}^*\mathbb{N} : \overline{E}|X(\underline{s},\underline{t})|^p < n\, \forall (\underline{s},\underline{t}) \in T^2\}.$$

The set B is internal and $B \supseteq {}^*\mathbb{N} \setminus \mathbb{N}$. Then there is $n \in \mathbb{N}$ such that $n \in B$, an so the set A is bounded. Therefore, the set $\{E(|x(s,t)|^p) : (s,t) \in [0,1]^2\}$ is uniformly bounded.

(ii) From the properties of S- integrability it follows that for $\varepsilon > 0$ in $\mathbb{R}$ the set

$$\left\{\delta \in {}^*\mathbb{R}^+ : \forall A \in \mathfrak{A}, \text{ internal, if } \overline{P}(A) < \delta \text{ then } \int_A (|X(\underline{s},\underline{t})|^p) < \varepsilon\right\}$$

is internal and contains all the positive infinitesimals. Then it contains a positive real δ, and thus we obtain for $\varepsilon > 0$ in $\mathbb{R}$ that there exists $\delta \in \mathbb{R}^+$ such that if $P(A) < \delta$, $A \in L(\mathfrak{A})$, then $E(|x(s,t)|^p) < \varepsilon$.

From these two observations we conclude that $\{|x(s,t)|^p,\ (s,t) \in [0,1]^2\}$ is uniformly integrable.

Now we consider the "if" part. We may assume that $x(\cdot,\cdot,w) \in D^2$ for all $w \in \Omega$. Since D^2 is a separable metric space, the lifting theorem in Anderson [2] implies that there is an internal stochastic process $X' : {}^*[0,1]^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ such that $st_{\mathcal{J}_2}(X'(\cdot,\cdot,w)) = x(\cdot,\cdot,w)$ a.s. (i.e., for P-almost all w). By Theorem 1.6, X' and $X = X'|_{T^2\times\Omega}$ are SD^2J, and thus X is the desired lifting.

To be precise, we should prove that the mapping $\Phi : \Omega \to D^2$, $w \mapsto X((\cdot,\cdot),w)$, is measurable. The proof of this is similar to that of the one parameter case (see for example, [14]). So, Φ is P- measurable, and therefore it has a lifting $\Psi : \Omega \longrightarrow {}^*D^2$, $w \mapsto X'(\cdot,\cdot,w) \in D^2$. Now X' defines a stochastic process $X' : [0,1]^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ such that $st_{\mathcal{J}_2}(X'(\cdot,\cdot,w)) = x(\cdot,\cdot,w)$ a.s., and from Theorem 1.6, X' is SD^2J. Let $X = X'|_{T^2\times\Omega}$. Then X is SD^2J, and is the desired limit.

Additionally, if $\{|x(s,t)|^p : (s,t) \in [0,1]^2\}$ is uniformly integrable for some $p \geq 1$, let Y be the SD^2J lifting of x obtained before, and define

$$x^N = \begin{cases} x, & \text{if } |x| \leq N, \\ N|x|^{-1}x, & \text{if } |x| \geq N. \end{cases}$$

The corresponding lifting is

$$Y^N = \begin{cases} Y, & \text{if } |Y| \leq N, \\ N|Y|^{-1}Y, & \text{if } |Y| \geq N. \end{cases}$$

We have that $x^N(\cdot,w) \in D^2$ and Y^N is a bounded SD^2J lifting of x^N. The proof that $Y^N(s,t)$ is S- integrable is also similar to that of the one parameter case. By saturation we can find $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that the above statement holds for Y^ν, for all $(\underline{s},\underline{t}) \in T^2$, all $\varepsilon > 0$ and M that depends on ε. Then $X = Y^\nu$ is an SD^2J lifting of x such that $|X(\underline{s},\underline{t})|^p$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$. ☑

Remark 1. A standard filtration in two parameters is a filtration that satisfies the following conditions:

F1. For (s,t) and (s',t')in $[0,1]^2$ such that $s \leq s'$, $t \leq t'$, $\mathfrak{F}_{(s,t)} \subseteq \mathfrak{F}_{(s',t')}$.
F2. $\mathfrak{F}_{(0,0)}$ is P- complete.
F3. For each (s,t), $\mathfrak{F}_{(s,t)} = \cap_{(s',t')\gg(s,t)}\mathfrak{F}_{(s',t')}$.

Additionaly we say that the filtration satisfies $F4$, the Cairoli-Walsh condition, if for (s,t) and (s',t') such that $s \leq s'$ and $t \geq t'$ it follows that $\mathfrak{F}_{(s,t)}$ and $\mathfrak{F}_{(s',t')}$ are conditionally independent. Conditional independence is equivalent to the following condition: if (s,t) and (s',t') are such that $s \leq s'$ and $t \geq t'$, and x is an $\mathfrak{F}_{(s',t)}$ - measurable random variable, then $E(x|\mathfrak{F}_{(s,t)}) = E(x|\mathfrak{F}_{(s,t')})$. Condition F4 is also equivalent to each one of the following:

(a) If $(s,t)\underline{\Delta}(s',t')$ and X is a random variable, then

$$E(E(X|\mathfrak{F}_{(s,t)})|\mathfrak{F}_{(s',t')}) = E(E(X|\mathfrak{F}_{(s',t')})|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')}).$$

(b) If $(s,t)\underline{\Delta}(s',t')$ and X is an $\mathfrak{F}_{(s',t')}$- measurable random variable, then $E(X|\mathfrak{F}_{(s,t)}) = E(X|\mathfrak{F}_{(s,t')})$.

2.3. Definition.

(i) Let $L \in {}^*\mathbb{N} \setminus \mathbb{N}$, $N = L!$, $\delta t = 1/N$. The hyperfinite line is

$$T = \{0, \delta t, 2\delta t, ..., (N-1)\delta t, 1\}.$$

(ii) Let $\Omega = \{-1,1\}^{T^2} = \{w : T^2 \longrightarrow \{-1,1\} \mid w \text{ is internal }\}$. The internal hyperfinite cardinal of Ω is $2^{(N+1)^2}$.

(iii) Given $(\underline{s},\underline{t}) \in T^2$, the equivalence relation $w \approx_{(\underline{s},\underline{t})} w'$ in Ω is

$$w \approx_{(\underline{s},\underline{t})} w' \Leftrightarrow w(\underline{s}',\underline{t}') = w'(\underline{s}',\underline{t}')$$

for all $(\underline{s}',\underline{t}') \leq (\underline{s},\underline{t})$, $(\underline{s}',\underline{t}') \in T^2$.

(iv) By means of the equivalence relation above we define for $(\underline{s},\underline{t}) \in T^2$,

$$\mathfrak{B}_{(\underline{s},\underline{t})} = \{A \subseteq \Omega \mid A \text{ is internal and closed under } \approx_{(\underline{s},\underline{t})}\}.$$

This is an internal $^*\sigma$-algebra.

(v) An internal two parameter filtration is an internal family $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$ of internal *sub-σ-algebras of $\mathfrak{B}$ that satisfy property $\overline{F1}$, i.e., property F1 in the nonstandard sense.

The filtration is $\overline{P}$-complete if $\mathfrak{B}_{(0,0)}$ is complete.

Let $(\Omega, \mathfrak{A}, \overline{P})$ be an internal probability space and let

$$(\Omega, \mathfrak{F}, P) = (\Omega, L(\Omega), L(\overline{P})).$$

As we have seen in (v) of Definition 2.3 above, an internal filtration on T^2 is a collection of *sub σ-algebras of $\mathfrak{A}$, $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$, such that, whenever $(\underline{s},\underline{t}) \leq (\underline{s}',\underline{t}')$, then $\mathfrak{B}_{(\underline{s},\underline{t})} \subseteq \mathfrak{B}_{(\underline{s}',\underline{t}')}$.

2.4. Definition. The standard part of $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ is the filtration $\{\mathfrak{F}_{(s,t)} : (s,t) \in [0,1]^2\}$ defined by

$$\mathfrak{F}_{(s,t)} = \left(\bigcap_{\substack{^\circ(\underline{s},\underline{t}) \gg (s,t) \\ (\underline{s},\underline{t}) \in T^2}} \sigma\left(\mathfrak{B}_{(\underline{s},\underline{t})}\right) \right) \vee \mathfrak{N},$$

where $\mathfrak{N}$ is the class of P-null sets of $\mathfrak{F}$ and $\vee$ stands for the smallest σ-algebra containing $\bigcap_{\substack{^\circ(\underline{s},\underline{t}) \gg (s,t) \\ (\underline{s},\underline{t}) \in T^2}} \sigma\left(\mathfrak{B}_{(\underline{s},\underline{t})}\right)$ and $\mathfrak{N}$.

The standard filtration $\{\mathfrak{F}_{(s,t)}\}_{(s,t) \in [0,1]^2}$ satisfies properties F1 to F4.

2.5. Theorem. *Let $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$ be an internal filtration. A process $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ is $\mathfrak{F}_{(s,t)}$-adapted and has almost all sample paths in D^2 (and $\{|x(s,t)|^p : (s,t) \in [0,1]^2\}$ is uniformly integrable for some $p \geq 1$) if and only if x has an SD^2J lifting X that is $\{\mathfrak{B}_{(\underline{s} \vee \Delta' t, \underline{t} \vee \Delta' t)} : (\underline{s},\underline{t}) \in T^2\}$- adapted for some positive infinitesimal $\Delta' t \in T$ (and for which $|X(\underline{s},\underline{t})|^p$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$).*

Proof. By Theorem 2.2, if x has an SD^2J lifting X, x has sample paths in D^2 a.s. If $X(\underline{s},\underline{t})$ is $\mathfrak{B}_{(\underline{s} \vee \Delta' t, \underline{t} \vee \Delta' t)}$- measurable, the process $x(^\circ\underline{s}, ^\circ\underline{t})$ is $\sigma\left(\mathfrak{B}_{(\underline{s} \vee \Delta' t, \underline{t} \vee \Delta' t)}\right)$- measurable, and so $x(^\circ\underline{s}, ^\circ\underline{t})$ is $\mathfrak{F}_{(^\circ\underline{s}, ^\circ\underline{t})}$- measurable. This proves the sufficient part. The proof of the "necessity part" is similar to that of the one parameter case. ☑

2.6. Theorem. *If $X : T^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ is an internal map of class SD^2, then there is a positive infinitesimal $\Delta' t \in T$ such that if $T' = \{k\Delta' t : k \in {}^*\mathbb{N},\ k\Delta' t \leq 1\} \cup \{1\}$ then $X|_{(T')^2 \times \Omega}$ is of class $SD^2 J$.*

Proof. By Theorem 2.2, X is SD^2 if and only if $st(X)$ exists and belongs to D^2. Therefore, there exits an $SD^2 J$ lifting Y of $st(X) = x$. Let $(s,t) \in [0,1]^2$. As in Proposition 2.17 we may choose an infinitesimal $\underline{\delta}_n \in T$ such that

$$\overline{P}\left(\left\{ w : \sup_{\substack{\underline{\delta}_n \leq \varepsilon \leq \varepsilon_n \\ \varepsilon \in T}} |X(\underline{s}+\varepsilon, \underline{t}+\varepsilon)(w) - Y(\underline{s},\underline{t})(w)| \geq 1/n) \right\} < 1/n \right).$$

We can extend the sequence $\{\underline{\delta}_n : n \in \mathbb{N}\}$ to ${}^*\mathbb{N}$, and then find $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $\underline{\delta} \in T$ such that $\underline{\delta} = \max_{n \leq \nu} \underline{\delta}_n \approx 0$ and

$$\overline{P}\left(\left\{ w : \sup_{\substack{\underline{\delta} \leq \varepsilon \leq \varepsilon_\nu \\ \varepsilon \in T}} |X(\underline{s}+\varepsilon, \underline{t}+\varepsilon)(w) - Y(\underline{s},\underline{t})(w)| > 0 \right\}\right) \approx 0.$$

Therefore,

$$P\left(\left\{ w : \sup_{\substack{\underline{\delta} \leq \varepsilon \leq \varepsilon_\nu \\ \varepsilon \in T}} |{}^o X(\underline{s}+\varepsilon, \underline{t}+\varepsilon)(w) - x({}^o\underline{s}, {}^o\underline{t})(w)| > 0 \right\}\right) = 0.$$

Now, for each $n \in \mathbb{N}$, let $(s,t) = (k/n, m/n)$, $1 \leq k, m \leq n$. Then, from the above argument, for each $n \in \mathbb{N}$ there exists $\underline{\rho}_n \approx 1/n$, $\underline{\rho}_n \in T$, and $\underline{\rho}_n > 1/n$, such that

$${}^o X(k\underline{\rho}_n, m\underline{\rho}_n) = x(k/n, m/n) \text{ a.s.}, \quad \text{for } 1 \leq k, m \leq n.$$

Hence, for all $n \in \mathbb{N}$ we have $0 < \underline{\rho}_n < 2/n$ and

$$\overline{P}\left(\left\{ w : \max_{1 \leq k,m \leq n} |X(k\underline{\rho}_n, m\underline{\rho}_n)(w) - Y(k\underline{\rho}_n, m\underline{\rho}_n)(w)| > 1/n \right\}\right) < 1/n.$$

Now the set of $n \in {}^*\mathbb{N}$ such that $0 < \underline{\rho}_n < 2/n$ and

$$\overline{P}\left(\left\{ w : \max_{1 \leq k,m \leq n} |X(k\underline{\rho}_n, m\underline{\rho}_n)(w) - Y(k\underline{\rho}_n, m\underline{\rho}_n)(w)| > 1/n \right\}\right) < 1/n$$

contains $\mathbb{N}$. Then, by overflow (if A is an internal set and $A \supseteq \mathbb{N}$, there exists $H \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $H \in A$), there is $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\underline{\rho}_\nu \approx 0$, $\underline{\rho}_\nu > 0$ and

$$\overline{P}\left(\left\{ w : \max_{1 \leq k,m \leq \nu} |X(k\underline{\rho}_\nu, m\underline{\rho}_\nu)(w) - Y(k\underline{\rho}_\nu, m\underline{\rho}_\nu)(w)| > 1/\nu \right\}\right) < 1/\nu.$$

Let $N_1 = \{w : \max_{1 \leq k,m \leq \nu} |X(k\underline{\rho}_\nu, m\underline{\rho}_\nu) - Y(k\underline{\rho}_\nu, m\underline{\rho}_\nu)| > 0\}$. From above, N_1 is a P- null; and since Y is $SD^2 J$, also $X|_{(T')^2 \times \Omega}$ is $SD^2 J$, where $T' = \{k\underline{\rho}_\nu : k \in {}^*\mathbb{N},\ k\underline{\rho}_\nu \leq 1\} \cup \{1\}$. ☑

2.7. Definition. Let $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$ be an internal filtration for which hypothesis $\overline{F1}$–$\overline{F4}$ hold (these are the corresponding internal conditions of $F1$–$F4$).

(i) An internal stochastic process $X : T^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ is a $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale if $\{(X(\underline{s},\underline{t}), \mathfrak{B}_{(\underline{s},\underline{t})}) : (\underline{s},\underline{t}) \in T^2\}$ is an internal martingale, i.e., is $\mathfrak{B}_{(\underline{s},\underline{t})}$-adapted and $\overline{E}\left(X(\underline{s}_2,\underline{t}_2)|\mathfrak{B}_{(\underline{s}_1,\underline{t}_1)}\right) = X(\underline{s}_1,\underline{t}_1)$ $\overline{P}$-a.s. whenever $(\underline{s}_1,\underline{t}_1) \le (\underline{s}_2,\underline{t}_2)$.

(ii) X is an S- martingale with respect to $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ if X is a $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale and $|X(\underline{s},\underline{t})|^p$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$ and some $p \ge 1$.

(iii) X is a *-martingale after Δt (in the terminology of [14]) for $\Delta t \approx 0$, $\Delta t \in T$, if $\{X(\underline{s},\underline{t}), \mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in (T')^2\}$ is an internal martingale, where $T' = \{k\Delta t : k \in {}^*\mathbb{N}, k\Delta t < 1\} \cup \{1\}$.

(iv) X is an S-Δt- martingale for some $\Delta t \in T$, $\Delta t \approx 0$, if X is SD^2J, S-integrable for all $(\underline{s},\underline{t}) \in (T')^2$ and a *- martingale after Δt.

Remark 2. From Theorem 2.6 we see that if X is an S-martingale and X is SD^2, there exists an infinitesimal $\Delta t \in T$ such that X is a Δt- martingale.

From now on, we will restrict ourselves to larc processes vanishing on the axis and L^p bounded for $p \ge 1$.

2.8. Definition. Let $\{\mathfrak{F}_{(s,t)} : (s,t) \in [0,1]^2\}$ be the standard part of $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in T^2\}$

(i) A stochastic process $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ is an $\{\mathfrak{F}_{(s,t)}\}$- larcmartingale if it is $\mathfrak{F}_{(s,t)}$- adapted, p-uniformly integrable for some $p \ge 1$, and $x((\cdot,\cdot),w) \in D^2$ a.s.; i.e. x is larc, and for $(s,t) \le (u,v)$,

$$E\left(x(u,v)|\mathfrak{F}_{(s,t)}\right) = x(s,t) \ P\text{-a.s.}$$

(ii) If x is an $\mathfrak{F}_{(s,t)}$- larcmartingale and $\{\mathfrak{B}_{(\underline{s},\underline{t})}\}$ is an internal filtration, a $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale lifting of x is an SD^2J lifting X of x for which there exists a positive infinitesimal $\Delta t \in T$ such that X is a Δt- martingale and $st(X) = x$ a.s.

2.9. Theorem. *If X is a Δt- martingale, then $st(X) = x$ is a larcmartingale.*

Proof. Since $X(\underline{s},\underline{t})$ is S- integrable for all $(\underline{s},\underline{t}) \in (T')^2$, then $x(s,t)$ is uniformly integrable (see Theorem 2.2.). Also, $X(1,1)$ is a lifting of $x(1,1)$. For fixed $(s,t) \in [0,1]^2$, there exists $(\underline{u}_1,\underline{v}_1) \approx (s,t)$, $(\underline{u}_1,\underline{v}_1) \in (T')^2$ (X is a martingale after Δt, $T' = \{k\Delta t : k \in {}^*\mathbb{N}, \ k\Delta t < 1\} \cup \{1\}$) such that for all $(\underline{s},\underline{t}) \ge (\underline{u}_1,\underline{v}_1)$, $(\underline{s},\underline{t}) \approx (s,t)$, $\overline{E}(X(1,1)|\mathfrak{B}_{(\underline{s},\underline{t})})$ is a lifting of $E\left(x(1,1)|\mathfrak{F}_{(s,t)}\right)$ (see [7], Proposition 3.2). Now X is SD^2J. Then by Proposition 1.5, there exists $(\underline{u}_2,\underline{v}_2) \in (T')^2$, $(\underline{u}_2,\underline{v}_2) \approx (s,t)$, such that, whenever $(\underline{s},\underline{t}) \ge (\underline{u}_2,\underline{v}_2)$, then $X(\underline{s},\underline{t}) \approx x(s,t)$. Let us take $\underline{u} = \max\{\underline{u}_1,\underline{u}_2\}$,

$\underline{v} = \max\{\underline{v}_1, \underline{v}_2\}$. If $(\underline{s}, \underline{t}) \approx (s,t)$, $(\underline{s}, \underline{t}) \in (T')^2$ and $(\underline{s}, \underline{t}) \geq (\underline{u}, \underline{v})$, then $\overline{E}\left(X(1,1)|\mathfrak{B}_{(\underline{s},\underline{t})}\right) = X(\underline{s}, \underline{t}) \approx x(s,t)$ a.s. Therefore

$$x(s,t) = {}^oX(\underline{s}, \underline{t}) = {}^o\overline{E}\left(X(1,1)|\mathfrak{B}_{(\underline{s},\underline{t})}\right) = E\left(x(1,1)|\mathfrak{F}_{(s,t)}\right) \text{ a.s.}$$

Finally, since X is SD^2J and X is a lifting of x, x has sample paths in D^2 a.s (Theorem 2.2.). Thus x is a larcmartingale. ☑

2.10. Definition. Let $F : T^2 \longrightarrow {}^*\mathbb{R}$ be internal. We say that $r \in \mathbb{R}$ is the $S^{(++)}$-limit of F at (s,t), if for all standard $\varepsilon > 0$ there is a standard $\delta > 0$ such that if $(\underline{s}, \underline{t}) \in T^2$ and $(s,t) \ll {}^o(\underline{s}, \underline{t}) \ll (s+\delta, t+\delta)$ then $|F(\underline{s}, \underline{t}) - r| < \varepsilon$. We also write

$$r = S - \lim_{\substack{\underline{s}\downarrow s \\ \underline{t}\downarrow t}} F(\underline{s}).$$

The $S^{(ij)}$-limits, $i = +, -$, $j = +, -$ are similarly defined.

We can extend this definition to the two parameter S-quadrantal limits.

2.11. Theorem. *If $\{X(\underline{s}, \underline{t})\}$ is an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$-martingale and ${}^o\overline{E}(|X(1,1)|)$ $< +\infty$, then X is SD^2.*

Proof. We shall prove that $st(X)(s,t)$ exists and $st(X)(s,t) \in D^2$, P-a.e.
(a) We show first that $st(X)$ exists. If for some $w \in \Omega$, $st(X)(s,t)$ does not exist for $(s,t) \in [0,1]^2$, there exists a decreasing sequence $\{(\underline{s}_n, \underline{t}_n)\}$, $(\underline{s}_n, \underline{t}_n) \in T^2$, with $X(\underline{s}_1, \underline{t}_1) = X(1,1)$, ${}^o(\underline{s}_n, \underline{t}_n) \gg (s,t)$ and

$$A = S - \liminf_{n\to\infty} X(\underline{s}_n, \underline{t}_n) < S - \limsup_{n\to\infty} X(\underline{s}_n, \underline{t}_n) = B.$$

Therefore, there are subsequences $\{(\underline{u}_n, \underline{v}_n)\}$ and $\{(\underline{u}'_n, \underline{v}'_n)\}$ such that $A = S - \lim_{n\to\infty} X(\underline{u}'_n, \underline{v}'_n)$ and $B = S - \lim_{n\to\infty} X(\underline{u}_n, \underline{v}_n)$.

Now, $\{X(\underline{s}, \underline{t})\}$ is an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale and $\{(\underline{s}_n, \underline{t}_n)\}$ is a totally ordered set. Therefore, there exist rational numbers a, b such that $\{X(\underline{s}_n, \underline{t}_n)\}$ crosses the interval $[a,b]$ an infinite number of times. If $U_{a,b}$ is the number of upcrossings of the interval $[a,b]$ by $\{X(\underline{s}_n, \underline{t}_n)\}$ we have, from the upcrossing lemma (see [3]), that

$$\overline{E}[U_{a,b}] \leq \frac{\overline{E}[|X(1,1)|] + |a|}{b-a},$$

and this number is finite by assumption. Thus, for almost all w, $X(\cdot,\cdot,w)$ restricted to $\{(\underline{s}_n, \underline{t}_n)\}$ is such that $\cup[\{U_{a,b} \approx \infty\} : a < b \text{ in } \mathbb{Q}]$ is a P-null set. Then $\{w : st(X(\cdot,\cdot,w))(s,t) \text{ does not exist }\}$ has measure zero, and therefore $st(X)(s,t)$ exists a.e.

(b) Let us now show that the function $st(X)(s,t)$ is continuous from the right and has the other quadrantal limits. The argument to prove the first assertion readily follows from (a) and from the definition of $st(X)(s,t)$. Let us then examine the existence of the other limits.

(i) Suppose that $\lim_{\substack{u\to s^- \\ v\to t^-}} st(X)(u,v)$ does not exist. Then there is an increasing sequence $\{(s_n,t_n)\}$ in $[0,1]^2$, $(s_n,t_n) \ll (s,t)$, such that

$$\liminf_{n\to\infty} st(X)(s_n,t_n) < \limsup_{n\to\infty} st(X)(s_n,t_n),$$

and for each (s_n,t_n) there exists $(\underline{s}_n,\underline{t}_n) \approx (s_n,t_n)$ in T^2 such that

$$st(X)(s_n,t_n) = {}^oX(\underline{s}_n,\underline{t}_n)$$

(Proposition 1.4.). Similarly as before, $\{X(\underline{s},\underline{t})\}$ is an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale and $\{(\underline{s}_n,\underline{t}_n)\}$ is a totally ordered set. So, from the proof in (a),

$$\left\{ w : \lim_{\substack{u\to s^- \\ v\to t^-}} st(X)(u,v) \text{ does not exist} \right\}$$

has measure zero.

(ii) If $\lim_{\substack{u\to s^+ \\ v\to t^-}} st(X)(u,v)$ does not exist, for $(s,t) \in [0,1]^2$ there is $(\underline{s},\underline{t}) \in T^2$ such that $(\underline{s},\underline{t}) \approx (s,t)$ and $(s,t) \in [(\underline{s},\underline{t}),(\underline{s}+\delta t,\underline{t}+\delta t))$, and for each $r \in \mathbb{R}$ there exists $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, $(s_n,t_n) \in ((s,t),(s+1/n,t+1/n))$ can be found such that $|st(X)(s_n,t_n) - r| > \varepsilon$. We can choose (s_n,t_n) such that $s_{n+1} < s_n$ and $t_{n+1} > t_n$. Now, for each (s_n,t_n) there exists $(\underline{s}_n,\underline{t}_n) \approx (s_n,t_n)$ in T^2 such that $st(X)(s_n,t_n) = {}^oX(\underline{s}_n,\underline{t}_n)$.

We have ${}^o(\underline{s}_n,\underline{t}_n) \in ((s,t),(s+1/n,t+1/n))$ and $|X(\underline{s}_n,\underline{t}_n) - r| > \varepsilon$; that is, $X(\underline{s}_n,\underline{t}_n) - r > \varepsilon$ or $X(\underline{s}_n,\underline{t}_n) - r < -\varepsilon$, and so

$$\overline{E}\left(|X(\underline{s}_n,\underline{t}_n) - r| | \mathfrak{B}_{(\underline{s}+\delta t,\underline{t}+\delta t)}\right) > \varepsilon.$$

Since $\{X(\underline{s},\underline{t})\}$ is an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale and the filtration satisfies $\overline{F4}$, then

$$\overline{E}(X(\underline{s}_n,\underline{t}_n)|\mathfrak{B}_{(\underline{s}+\delta t,\underline{t}+\delta t)}) = \overline{E}\left(X((\underline{s}_n,\underline{t}_n)|\mathfrak{B}_{(\underline{s}+\delta t,\underline{t}_n)}\right) = X(\underline{s}+\delta t,\underline{t}_n).$$

Thus $X(\underline{s}+\delta t,\underline{t}_n) - r > \varepsilon$ and similarly $X(\underline{s}+\delta t,\underline{t}_n) - r < -\varepsilon$. That is to say, $|X(\underline{s}+\delta t,\underline{t}_n) - r| > \varepsilon$. We conclude that

$$\left\{ w : \lim_{\substack{u\to s^+ \\ v\to t^-}} st(X)(u,v) \text{ does not exist} \right\}$$

is a subset of $\{w : S - \lim_{\underline{t}_n \uparrow t} X((\underline{s}+\delta t,\underline{t}_n) \text{ does not exist }\}$, and since $X(\underline{s}+\delta t,\underline{t}_n)$ is a one parameter $\mathfrak{B}_{(\underline{s}+\delta t,\underline{t})}$-martingale, this set has measure zero.

(iii) The proofs for the other quadrantal limit are similar.

Therefore have that $\{X(\underline{s},\underline{t})\}$ is SD^2. ☑

2.12. Theorem. *If $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ is a larcmartingale with respect to $\{\mathfrak{F}_{(s,t)}\}$, then there is a $\mathfrak{B}_{(\underline{s},\underline{t})}$-martingale lifting for some infinitesimal Δt in T.*

Proof. Let $X(1,1)$ be a lifting of $x(1,1)$. Define $X(\underline{s},\underline{t}) = \overline{E}\left(X(1,1)|\mathfrak{B}_{(\underline{s},\underline{t})}\right)$, $(\underline{s},\underline{t}) \in T^2$. $X(\underline{s},\underline{t})$ is S- integrable for all $(\underline{s},\underline{t}) \in T^2$ (see [2], Theorem 12), is an internal $\mathfrak{B}_{(\underline{s},\underline{t})}$- martingale, and by Theorem 2.11, $X(\underline{s},\underline{t})$ is SD^2. Thus, by Theorem 2.6 there exists a positive infinitesimal $\Delta t \in T$ such that if $T' = \{k\Delta t : k \in {}^*\mathbb{N},\ k\Delta t < 1\} \cup \{1\}$ then $X|_{(T')^2\times\Omega}$ is SD^2J, which implies that X is a Δt - martingale. Finally,

$$st(X)(s,t) = \lim_{{}^\circ(\underline{s},\underline{t})\downarrow(s,t)} {}^\circ X(\underline{s},\underline{t}) \text{ a.s.} = \lim_{{}^\circ(\underline{s},\underline{t})\downarrow(s,t)} {}^\circ\overline{E}\left(X(1,1)|\mathfrak{B}_{(\underline{s},\underline{t})}\right) \text{ a.s.}$$

$$= \lim_{{}^\circ(\underline{s},\underline{t})\downarrow(s,t)} E\left(x(1,1)|\sigma\left(\mathfrak{B}_{(\underline{s},\underline{t})}\right)\right) \text{ a.s.} = E\left(x(1,1)|\mathfrak{F}_{(s,t)}\right) = x(s,t) \text{ a.s.},$$

the last identity being a consequence of the reverse martingale theorem (see [3]). Therefore X is a martingale lifting of x with respect to the internal filtration $\{\mathfrak{B}_{(\underline{s},\underline{t})} : (\underline{s},\underline{t}) \in (T')^2\}$. ☑

3. An stochastic integral

In two parameter stochastic analysis we use different classes of filtrations. We associate to each of them corresponding nonstandard internal filtrations as follows:

a) $\mathfrak{B}^1_{(\underline{s},\underline{t})} = \mathfrak{B}_{(\underline{s},1)}$ and $\mathfrak{B}^2_{(\underline{s},\underline{t})} = \mathfrak{B}_{(1,\underline{t})}$.

b) $\mathfrak{B}^*_{(\underline{s},\underline{t})} = \mathfrak{B}_{(\underline{s},1)} \vee \mathfrak{B}_{(1,\underline{t})}$ is the smallest ${}^*\sigma$-algebra containing the ${}^*\sigma$-algebras $\mathfrak{B}^1_{(\underline{s},\underline{t})}$ and $\mathfrak{B}^2_{(\underline{s},\underline{t})}$. $\mathfrak{B}^*_{(\underline{s},\underline{t})}$ is atomic and his atoms are $[w]_{(\underline{s},\underline{t})^*} = [w]_{(\underline{s},1)} \cap [w]_{(1,\underline{t})}$.

We say that X is an internal weak martingale, if it is $\mathfrak{B}_{(\underline{s},\underline{t})}$- adapted and for any rectangle R we have $E(X(R)|\mathfrak{B}_{(\underline{s},\underline{t})}) = 0$.

We say that X is an internal strong martingale, if it is $\mathfrak{B}^*_{(\underline{s},\underline{t})}$ adapted and $E(X(R)|\mathfrak{B}^*_{(\underline{s},\underline{t})}) = 0$ for any rectangle R.

We say that X is an internal i-martingale, $i = 1,2$, if it is $\mathfrak{B}^i_{(\underline{s},\underline{t})}$- adapted and for any rectangle R, $E(X(R)|\mathfrak{B}^i_{(\underline{s},\underline{t})}) = 0$. Let $T = \{0, \Delta t, 2\Delta t, \ldots, (N-1)\Delta t, 1\}$ and $S = \{0, \Delta s, 2\Delta s, \ldots, (M-1)\Delta s, 1\}$ be hyperfinite discrete time lines, where $N, M \in {}^*\mathbb{N}\setminus\mathbb{N}$, $\Delta t = 1/N \approx 0$, and $\Delta s = 1/M \approx 0$. Let $T_0 = \{(0,k\Delta t), (l\Delta s, 0),\ l = 1,2,\ldots,M,\ k = 1,2,\ldots,N\}$, $\Omega = \{-1,1\}^{S\times T - T_0}$ and P be the counting measure.

3.1. Definition of the Brownian sheet. The internal hyperfinite random walk $\chi : T^2 \times \Omega \to {}^*\mathbb{R}$ is defined by $\chi(\underline{s},\underline{t},w) = 0$ if $(\underline{s},\underline{t}) \in T_0$ and $\chi(\underline{s}+\Delta s, \underline{t}+\Delta t, w) = \chi(\underline{s},\underline{t}+\Delta t, w) + \chi(\underline{s}+\Delta s, \underline{t}, w) - \chi(\underline{s},\underline{t},w) + \sqrt{\Delta s}\sqrt{\Delta t}w(\underline{s}+\Delta s, \underline{t}+\Delta t)$;

i.e.,

$$\chi(\underline{s},\underline{t},w)=\sum_{(\underline{s}',\underline{t}')\ll(\underline{s},\underline{t})}\sqrt{\Delta s}\sqrt{\Delta t}w(\underline{s}',\underline{t}').$$

The notation of sum over $(\underline{s}',\underline{t}')\ll(\underline{s},\underline{t})$ means that we sum over those $(\underline{s}',\underline{t}')$ such that $s'<s$ and $t'<t$.

The increment of χ is defined by

$$\Delta\chi(\underline{s},\underline{t},w)=\chi(\underline{s}+\Delta s,\underline{t}+\Delta t,w)-\chi(\underline{s}+\Delta s,\underline{t},w)\\-\chi(\underline{s},\underline{t}+\Delta t,w)+\chi(\underline{s},\underline{t},w).$$

In particular we can take $T=S$

3.2. Theorem. *If we define* $b(s,t,w)={}^{o}\chi(\underline{s},\underline{t},w)$ *for* $(s,t)\approx(\underline{s},\underline{t})$, *then* $b(s,t,w)$ *is a Brownian sheet. That is:*

(a) *For every* $(s,t)\in[0,1]^2$, $b(s,t,w)$ *has normal distribution with zero mean and variance st.*

(b) *b has independent increments; that is, if R and R′ are disjoint rectangles in* $[0,1]^2$, *then* $b(R,w)$ *and* $b(R',w)$ *are independent, where*

$$b(((s,t),(s',t')],w)=b(s',t',w)-b(s,t',w)-b(s',t,w)+b(s,t,w)$$

for a rectangle $R=((s,t),(s',t')]$.

(c) *b is continuous as a function of* (s,t) *for almost all w.*

The proof is similar to that of the one parameter case. See [9].

Remark 3. From Definition 3.1 it follows easily that χ is nonanticipanting. And from Theorem 3.2, we can see that $\chi(\underline{s},\underline{t})$ is an internal strong martingale.

3.3. Definition. Let X be an internal stochastic process. We say that X is an increasing process if

(1) X is adapted and SD^2.

(2) $X(\underline{s},0)=0=X(0,\underline{t})$.

(3) For each rectangle $R=((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$, $(\underline{s},\underline{t}),(\underline{s}',\underline{t}')\in T^2$, $X(R)\geq 0$.

3.4. Definition. We say that M is an internal SL^2-Δt-martingale if M is a Δt-martingale with respect to some $\Delta t\in T$, $\Delta t\approx 0$, and for all $(\underline{s},\underline{t})\in T^2$, ${}^{o}E(M(\underline{s},\underline{t})^2)<\infty$.

Throughout this paper, all martingales are supposed to vanish on the axis.

3.5. Definition. Given two hyperfinite stochastic processes $X,Y:T^2\times\Omega\longrightarrow {}^*\mathbb{R}$, the stochastic integral of X with respect to Y is the stochastic process $\int XdY$ defined by

$$\left(\int XdY\right)(\underline{s},\underline{t},w) = \sum_{(\underline{s}',\underline{t}')\ll(\underline{s},\underline{t})} X(\underline{s}',\underline{t}',w)\Delta Y(\underline{s}',\underline{t}',w),$$

$\Delta X(\underline{s},\underline{t},w) = X(\underline{s}+\Delta t,\underline{t}+\Delta t,w) - X(\underline{s}+\Delta t,\underline{t},w) - X(\underline{s},\underline{t}+\Delta t,w) + X(\underline{s},\underline{t},w)$ is the two dimensional increment of a stochastic process X.

Given an internal SL^2-Δt-martingale M we define a new stochastic process $\langle M\rangle$, called the quadratic variation, by

$$\langle M\rangle(\underline{s},\underline{t}) = \sum_{(\underline{s}',\underline{t}')\ll(\underline{s},\underline{t})} \overline{E}\left(\Delta M^2(\underline{s}',\underline{t}')|\mathfrak{B}_{(\underline{s}',\underline{t}')}\right).$$

From the definition it follows easily that $\langle M\rangle$ is an internal submartingale. Theorem 2.11 also holds for submartingales, since in the proof we only use property $\overline{F}4$ and the upcrossing lemma. Then, $\langle M\rangle$ is SD^2. By Theorem 2.6, there exists an S-dense set $T' \subseteq T$ such that $\langle M\rangle|_{(T')^2\times\Omega}$ is SD^2J.

Let us now see that $M^2 - \langle M\rangle$ is a weak martingale. Let $R = ((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]$ be a rectangle in T^2. Then

$$\overline{E}((M^2 - \langle M\rangle)(R)|\mathfrak{B}_{(\underline{s},\underline{t})}) = \overline{E}\left(M^2(R)|\mathfrak{B}_{(\underline{s},\underline{t})}\right) -$$
$$\overline{E}\Big(\Big[\sum_{(a,b)\ll(\underline{s}',\underline{t}')} - \sum_{(a,b)\ll(\underline{s}',\underline{t})} - \sum_{(a,b)\ll(\underline{s},\underline{t}')} + \sum_{(a,b)\ll(\underline{s},\underline{t})}\Big]\overline{E}\left(\Delta M^2(a,b)|\mathfrak{B}_{(a,b)}\right)\Big|\mathfrak{B}_{(\underline{s},\underline{t})}\Big)$$
$$= \overline{E}\Big(\Big[M^2(R) - \sum_{(\underline{s},\underline{t})\le(a,b)\ll(\underline{s}',\underline{t}')} \overline{E}\left(\Delta M^2(a,b)|\mathfrak{B}_{(a,b)}\right)\Big]\Big|\mathfrak{B}_{(\underline{s},\underline{t})}\Big)$$
$$= \overline{E}\left(M^2(R)|\mathfrak{B}_{(\underline{s},\underline{t})}\right) - \sum_{(\underline{s},\underline{t})\le(a,b)\ll(\underline{s}',\underline{t}')} \overline{E}\left(\Delta M^2(a,b)\Big|\mathfrak{B}_{(\underline{s},\underline{t})}\right)$$
$$= \overline{E}\Big(M^2(R) - \Big[\sum_{(\underline{s},\underline{t})\le(a,b)\ll(\underline{s}',\underline{t}')} \Delta M^2(a,b)\Big]\Big|\mathfrak{B}_{(\underline{s},\underline{t})}\Big) = 0,$$

as follows from

$$\sum_{(\underline{s},\underline{t})\le(a,b)\ll(\underline{s}',\underline{t}')} \Delta M^2(a,b) = M^2(R).$$

Process $\langle M\rangle$ is also the unique increasing process such that $M^2 - \langle M\rangle$ is an internal weak-Δt-martingale. Moreover as $M^2 - \langle M\rangle$ is an internal weak martingale, we know from the representation in [17] that

$$M^2_{(\underline{s},\underline{t})} - \langle M\rangle_{(\underline{s},\underline{t})} = M^{(1)}_{(\underline{s},\underline{t})} + M^{(2)}_{(\underline{s},\underline{t})}$$

where $M^{(1)}$ is an internal 1-martingale and $M^{(2)}$ is an internal 2-martingale. Then

$$\overline{E}(\langle M\rangle) = \overline{E}(M^2) - \overline{E}(M^{(1)}) - \overline{E}(M^{(2)}) = \overline{E}(M^2),$$

and since M vanishes on the axis, then

$$\overline{E}(M^{(1)}_{(\underline{s},\underline{t})}) = \overline{E}\left(\overline{E}(M^{(1)}_{(\underline{s},\underline{t})}|\mathfrak{B}_{(0,1)})\right) = \overline{E}\left(\overline{E}(M^{(1)}_{(\underline{s},\underline{t})}|\mathfrak{B}_{(0,\underline{t})})\right) = \overline{E}(M^{(1)}_{(0,\underline{t})}) = 0.$$

The same holds for $M^{(2)}$. As a consequence we have that if M is a $SL^2 - \Delta t$-martingale, $\langle M\rangle$ is square S-integrable.

By Theorem 2.2, $st(M)$ is also a square integrable larcmartingale, so that $st(\langle M\rangle)$ vanishes on the axis, is ++ - continuous, and for any rectangle $R \subseteq [0,1]^2$ we have $st(\langle M\rangle)(R) \geq 0$. Furthermore, $M^2 - \langle M\rangle$ is a Δt-weak martingale. Then $st(M^2 - \langle M\rangle) = st(M)^2 - st(\langle M\rangle)$ is a weak larcmartingale, and finally we have $st(\langle M\rangle) = \langle st(M)\rangle$ a.s.

Generalizing the theory in Chapter IV of [1], we now define internal measures.

Let M be an internal $SL^2 - \Delta t$- martingale. We define an internal measure ν_M on $T^2 \times \Omega$ by

$$\nu_M(\{(\underline{s},\underline{t},w)\}) = \overline{E}\left(\Delta M(\underline{s},\underline{t})^2|\mathfrak{B}_{(\underline{s},\underline{t})}\right)\cdot P(\{w\}).$$

It follows that $\nu_M(T^2\times\Omega) = \overline{E}(\langle M\rangle(1,1))$. If $M = \chi$, the Anderson Brownian motion (see [2]), then

$$\nu_\chi(((\underline{s},\underline{t}),(\underline{s}',\underline{t}')]\times A) = (s'-s)(t'-t)\cdot P(A)$$

for $A \in \mathfrak{B}_{(\underline{s},\underline{t})}$. So, we have that $\nu_\chi = \lambda^2 \times P$, where P is the internal measure on Ω and λ is the internal counting measure on T.

3.6. Definition. Let M be an internal $SL^2 - \Delta t$-martingale. We say that a stochastic process X is in $SL^2(M)$ if it is nonanticipanting, 2-S-integrable with respect to ν_M, and such that

$$\int_{T^2\times\Omega} X^2 d\nu_M < \infty.$$

Remark 4. Suppresing w for shortness of notation, note that

$$\int_{T^2\times\Omega} X^2 d\nu_M = \sum_{w\in\Omega}\left[\sum_{(\underline{s},\underline{t})\ll(1,1)} X^2(\underline{s},\underline{t})\overline{E}(\Delta M(\underline{s},\underline{t})^2|\mathfrak{B}_{(\underline{s},\underline{t})})(w)\right]\cdot P(w)$$

$$= \overline{E}\left(\sum_{(\underline{s},\underline{t})\ll(1,1)} X^2(\underline{s},\underline{t})\overline{E}(\Delta M(\underline{s},\underline{t})^2|\mathfrak{B}_{(\underline{s},\underline{t})})\right)$$

$$= \overline{E}\left(\sum_{(\underline{s},\underline{t})\ll(1,1)} X^2(\underline{s},\underline{t})\Delta\langle M\rangle(\underline{s},\underline{t})\right) = \overline{E}\left(\int_{T^2} X^2 d\langle M\rangle\right).$$

If M is an internal SL^2-Δt-martingale, M is SD^2J, and therefore ν_M is absolutely continuous with respect to P (that is to say, if $L(P)(C) = 0$ then $L(\nu_M)(C\times T^2) = 0$).

3.7. Proposition. *If M is an internal SL^2-Δt-martingale and $X \in SL^2(M)$, then $\int X dM$ is an internal SL^2-Δt-martingale. If M is an SL^2-Δt- local martingale and $X \in SL^2(M)$, then $\int X dM$ is an internal SL^2-Δt-local martingale.*

Proof. The second assertion is a consequence of the first. Assume that M is an internal SL^2-Δt-martingale and $X \in SL^2(M)$. Then M is an internal Δt-martingale if and only if M is an internal 1-Δt-martingale and an internal 2-Δt-martingale. Now, according to the comments after Definition 4.3.2. and Proposition 4.4.4 in [1], $\int X dM$ is an internal 1-Δt-martingale and an internal 2-Δt-martingale. Thus, $\int X dM$ is an internal Δt-martingale. Furthermore

$$\begin{aligned}
\overline{E}\left(\left(\int X dM\right)^2((0,0),(1,1)]\right) &= \overline{E}\left(\left\langle\int X dM\right\rangle\right)\\
&= \overline{E}\left(\sum_{(\underline{s},\underline{t})\ll(1,1)} \overline{E}\left(\Delta\left(\int X dM\right)^2(\underline{s},\underline{t})|\mathfrak{B}_{(\underline{s},\underline{t})}\right)\right)\\
&= \overline{E}\left(\sum_{(\underline{s},\underline{t})\ll(1,1)} \overline{E}\left(X^2(\underline{s},\underline{t})\Delta M(\underline{s},\underline{t})^2|\mathfrak{B}_{(\underline{s},\underline{t})}\right)\right)\\
&= \overline{E}\left(\sum_{(\underline{s},\underline{t})\ll(1,1)} X^2(\underline{s},\underline{t})\overline{E}\left(\Delta M(\underline{s},\underline{t})^2|\mathfrak{B}_{(\underline{s},\underline{t})}\right)\right)\\
&= \overline{E}\left(\int_{T^2} X^2 d\langle M\rangle\right) = \int_{T^2\times\Omega} X^2 d\nu_M < \infty.
\end{aligned}$$

Then, $\int X dM$ is an internal SL^2-Δt-martingale. ☑

It follows from Proposition 3.7 that $\int X dM$ is SD^2. Thus, $st(\int X dM)$ makes sense.

Now we define a measure which is similar to the Doléan measure (see [1]). Given a square integrable larcmartingale N, we define a measure ν_N on the σ-algebra of predictable sets (the class of sets $((s,t),(s',t')] \times B$, where $B \in \mathfrak{F}_{(s,t)}$) by

$$\begin{aligned}
\nu_N(((s,t),(s',t')] \times B) &= E\left(I_B E\left(N^2((s,t),(s',t')]|\mathfrak{F}_{(s,t)}\right)\right),\\
\nu_N((s,0) \times B) &= 0,\\
\nu_N((0,t) \times B) &= 0.
\end{aligned}$$

Let x be a simple stochastic process. There exist real numbers a_i and disjoint rectangles R_i, $1 \leq i \leq n$, $R_i = ((s_i,t_i),(s'_i,t'_i)]$, $i \neq 1$, $R_1 = \{(0,0)\}$, such that $\cup_{i=1}^n R_i = [0,1]^2$ and

$$x = \sum_{i=1}^n a_i I_{((s_i,t_i),(s'_i,t'_i)]\times A_i},$$

where $A_i \in \mathfrak{F}_{(s_i,t_i)}$. From the definition of the stochastic integral we then have that

$$\left(\int_{[0,1]^2} xdN\right)(w) = \sum_{i=1}^{n} a_i I_{A_i}(w) N(R_i).$$

3.8. Definition. Let x be a simple stochastic process. We say that $x \in L^2(\nu_N)$, if $\int x^2 d\nu_N < \infty$.

Remark 5. Let $x = \sum_{i=1}^n a_i I_{R_i \times A_i}$ be a simple stochastic process, where $A_i \in \mathfrak{F}_{(s_i,t_i)}$ and $R_i = ((s_i,t_i),(s'_i,t'_i)]$. Then

$$\begin{aligned}
E\left(\left(\int_{[0,1]^2} xdN\right)^2\right) &= E\left(\left(\sum_{i=1}^{n} a_i I_{A_i} N(R_i)\right)^2\right) \\
&= \sum_{i=1}^{n} a_i^2 E(I_{A_i} N(R_i)^2) + 2\sum_{i<j} a_i a_j E(I_{A_i} I_{A_j} N(R_i) N(R_j)) \\
&= \sum_{i=1}^{n} a_i^2 E(I_{A_i} N(R_i)^2) \\
&= \sum_{i=1}^{n} a_i^2 E\left(I_{A_i} E(N^2((s_i,t_i),(s'_i,t'_i)] \| \mathfrak{F}_{(s_i,t_i)})\right) = \int x^2 d\nu_N.
\end{aligned}$$

As in the one parameter case, we see that the mapping $x \longrightarrow \int xdN$ acting on simple functions is an isometry. Since simple functions are dense in $L^2(\nu_N)$, we can extend the mapping $x \longrightarrow \int xdN$ to an isometry from $L^2(\nu_N)$ into $L^2(P)$. We still denote this extension by $\int_{[0,1]^2} xdN$. If $x \in L^2(\nu_N)$, we can see $\int xdN$ as a process by defining

$$\left(\int xdN\right)(s,t,w) = \left(\int_{[0,1]^2} I_{[0,(s,t)]} xdN\right)(w)$$

for all $(s,t) \in [0,1]^2$.

Since $x \rightarrow \int xdN$ is given as an L^2 limit, the stochastic integral is defined up to equivalences. The above definition extends, as in the one parameter case, the Ito integral. We have replaced the measure $\lambda^2 \times P$ by ν_N, restricting at the same time the class of integrands from adapted to predictable processes. In the case of the Ito integral, if X is an internal, nonanticipanting and square S-integrable stochastic process with respect to $\lambda^2 \times P$, then, for all $(\underline{s},\underline{t}) \in T^2$, the stochastic integral $\int_{[(0,0),(\underline{s},\underline{t})]} Xd\chi$ is finite.

Now, given $x \in L^2(\nu_{st(M)})$, we want to show that there is $Y \in SL^2(\nu_M)$ such that

$$\int xd(st(M)) = st\left(\int YdM\right).$$

First we give some important preliminary results.

3.9. Proposition. *Given an internal $SL^2 - \Delta t$- martingale M, let $m = st(M)$ be its standard part. Then ν_m is the restriction of $L(\nu_M) \circ St^{-1}$ to the predictable sets, where $St = st \times st \times id : T \times T \times \Omega \longrightarrow [0,1] \times [0,1] \times \Omega$.*

Proof. Let $B \in \mathfrak{F}_z$, where $z \in [0,1]^2$. Just observe that for each $\underline{z} \in T^2$, $\underline{z} \approx z$, $\underline{z} \geq z$, there exists an internal set $A \in \mathfrak{B}_{\underline{z}}$ such that $L(P)(A \Delta B) = 0$. The rest of the proof follows an argument entirely similar to that in the proof of the one parameter case. For more details, see [14]. ☑

4. Lifting theorems

4.1. Definition. Let M be an internal $SL^2 - \Delta t$-martingale and $x : [0,1]^2 \times \Omega \longrightarrow \mathbb{R}$ be a predictable process in $L^2(\nu_m)$. A 2-lifting of x with respect to M is an adapted process $X : T^2 \times \Omega \longrightarrow {}^*\mathbb{R}$ in $SL^2(M)$ such that ${}^oX(\underline{s}, \underline{t}, w) = x({}^o\underline{s}, {}^o\underline{t}, w)$ for $L(\nu_M)$-a.a.

4.2. Theorem. *Let M be an internal $SL^2 - \Delta t$-martingale and $m = st(M)$. If $x \in L^2(\nu_m)$, then x has a 2-lifting X with respect to M which is in $SL^2(M)$.*

Proof. By 4.3.9 in [1], x has an adapted lifting X. We now show that we can choose $X \in SL^2(M)$. For each $n \in \mathbb{N}$, let x_n be the truncation of x, that is

$$x_n = \begin{cases} x, & |x| \leq n, \\ n, & x > n, \\ -n, & x < -n. \end{cases}$$

If X_n is the corresponding truncation of X, we see that X_n is an adapted lifting of x_n. From Remark 5 and Proposition 3.9, we have that

$${}^o\int X_n^2 d\nu_M = \int {}^oX_n^2 dL(\nu_M) = \int x_n^2 d\nu_m.$$

Then, since $\int x_n^2 d\nu_m \to \int x^2 d\nu_m$, we can find $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that ${}^o\int X_\eta^2 d\nu_M = \int x^2 d\nu_m$. Finally, by Proposition A3.9 in the Appendix of [14], we have that $X_\eta \in SL^2(\nu_M)$. Thus, X_η is a 2-lifting of x with the required property. ☑

4.3. Proposition. *Let M be an internal SL^2-Δt-martingale and $m = st(M)$. Let $x \in L^2(\nu_m)$. If X and Y are 2-liftings of x, then there is a set Ω' of Loeb measure one such that for all $w \in \Omega'$ and all $(\underline{s}, \underline{t}) \in (T')^2$ ($T' = \{k\Delta t : k\Delta t \in T\} \cup \{1\}$),*

$${}^o\left(\int X dM\right)(\underline{s}, \underline{t}, w) = {}^o\left(\int Y dM\right)(\underline{s}, \underline{t}, w).$$

Proof. By Doob's inequality and Remark 1

$$\overline{E}\left(\max_{(\underline{s},\underline{t})}\left(\left(\int XdM\right)(\underline{s},\underline{t})-\left(\int YdM\right)(\underline{s},\underline{t})\right)^2\right)$$
$$\leq 4\overline{E}\left(\left(\int(X-Y)dM\right)^2\right)=4\overline{E}\left(\int_{(T')^2}(X-Y)^2d\langle M\rangle\right)$$
$$=4\int_{(T')^2\times\Omega}(X-Y)^2d\nu_M\approx 0,$$

where we have taken into account that $X-Y\in SL^2(\nu_M)$ and is infinitesimal a.e. Thus, the assertion follows. ☑

From Proposition 4.3 we see that the standard part of the integral does not depend on the lifting.

4.4. Theorem. *Let M be an internal $SL^2-\Delta t$- martingale and assume $x\in L^2(\nu_m)$, where $m=st(M)$. Then x has a 2-lifting $X\in SL^2(M)$ and*

$$\int xdm=\int xd(st(M))=st\left(\int XdM\right).$$

Proof. Suppose first that $x=\sum_{i=1}^n a_iI_{(z_i,z_i']\times B_i}$ is a simple stochastic process, where $B_i\in\mathfrak{F}_{z_i}$ for each i and $z_i,z_i'\in[0,1]^2$. We can choose $\underline{z}_i,\underline{z}_i'\in T^2$, $\underline{z}_i\approx z_i$ and $\underline{z}_i'\approx z_i'$ such that ${}^oM(\underline{z}_i)=st(M)(z_i)$ a.s., ${}^oM(\underline{z}_i')=st(M)(z_i')$ a.s. and such that there are $A_i\in\mathfrak{B}_{\underline{z}_i}$ with $L(P)(A_i\Delta B_i)=0$, $i=1,2,\ldots,n$. Define an internal stochastic process by

$$X=\sum_{i=1}^n a_iI_{(\underline{z}_i,\underline{z}_i']\times A_i}$$

From Theorem 4.2, we see that X is a 2-lifting of x and that we have

$$st\left(\int XdM\right)={}^o\sum_{i=1}^n a_iI_{A_i}M(\underline{z}_i,\underline{z}_i']=\sum_{i=1}^n a_iI_{B_i}{}^oM(\underline{z}_i,\underline{z}_i']$$
$$=\sum_{i=1}^n a_iI_{B_i}st(M)(z_i,z_i']=\int xd(st(M)).$$

This prove the assertion when x is simple. For the general case, the proof is similar to that of the one parameter case (see [14], Theorem 2.4.13). ☑

Remark 6. From the definition of the integral we have for a simple internal stochastic process $x=\sum_{i=1}^n a_iI_{((\underline{s}_i,\underline{t}_i),(\underline{s}_i',\underline{t}_i')]\times A_i}$ that

$$\left(\int XdM\right)(\underline{s},\underline{t})=\sum_{i=1}^n a_iI_{A_i}M(((\underline{s}_i,\underline{t}_i),(\underline{s}_i',\underline{t}_i')]\cap R_{(\underline{s},\underline{t})}).$$

4.5. Definition. Given $m : \Omega \times [0,1]^2 \longrightarrow \mathbb{R}$, a strong larcmartingale with respect to the filtration $(\mathfrak{F}_z)_{z \in [0,1]^2}$, we say that M is an internal strong SL^2-Δt-martingale lifting of m if M is an internal strong $SL^2 - \Delta t$-martingale with respect to $\mathfrak{B}_{(\underline{s},\underline{t})}$ and $st(M)(\underline{s},\underline{t}) = m({}^o\underline{s}, {}^o\underline{t})$ a.s.

Suppose $T = S$. It follows directly from the definition of χ that whenever $(\underline{s}', \underline{t}') \leq (\underline{s}, \underline{t})$, we have

$$\overline{E}(\chi(\underline{s},\underline{t}) - \chi(\underline{s}',\underline{t}') | \mathfrak{B}_{(\underline{s}',\underline{t}')}) = \overline{E}(\chi(\underline{s},\underline{t}) - \chi(\underline{s}',\underline{t}')) = 0.$$

Thus, χ is a martingale.

On the other hand, if $(\underline{s}', \underline{t}') \leq (\underline{s}, \underline{t})$, $\chi((\underline{s}',\underline{t}'), (\underline{s},\underline{t})])$ is independent of ${}^*\mathfrak{B}_{(\underline{s}',\underline{t}')} = \mathfrak{B}_{(\underline{s}',1)} \vee \mathfrak{B}_{(1,\underline{t})}$, so that $\overline{E}(\chi((\underline{s}',\underline{t}'),(\underline{s},\underline{t})]) | {}^*\mathfrak{B}_{(\underline{s}',\underline{t}')}) = 0$. Thus χ is a strong martingale. Furthermore

$$\sum_{w \in A} \chi^2(\underline{s},\underline{t},w) \cdot \overline{P}(\{w\}) = \sum_{w \in A} (\Delta t)(\Delta t) \cdot \overline{P}(\{w\}) = (\Delta t)^2 \cdot \overline{P}(A) \approx 0$$

whenever $\overline{P}(A) \approx 0$, and thus χ is an internal SL^2-Δt strong martingale.

If M is an internal $SL^2 - \Delta t$-martingale (respectively, a strong martingale) then, by Proposition 3.7, $\int X dM$ also is an internal SL^2-Δt- martingale (respectively, a strong martingale).

We are now able to establish the first lifting theorem for strong martingales.

4.6. Theorem. *Given a strong square integrable larcmartingale m, there exist an internal strong $SL^2 - \Delta t$-martingale M such that $st(M)(\underline{s},\underline{t}) = m({}^o\underline{s}, {}^o\underline{t})$.*

Proof. From the Wong and Zakai representation Theorem in [12], we have that for a square integrable strong martingale m there exists a stochastic process x in $L^2(\nu_b)$, b being the Brownian motion, such that

$$m(s,t) = \int_{R_{(s,t)}} x \, db \text{ a.s.}$$

where $R_{(s,t)}$ is the rectangle $((0,0),(s,t)]$. Now, the standard part of the hyperfinite random walk χ is a standard Brownian motion b and χ is an internal SL^2-strong martingale. Therefore for such x there is an internal 2-lifting $X \in SL^2(\chi)$ such that

$$\int x \, db = {}^o \int X d\chi,$$

and then we have that the process $M = \int X d\chi$ is a strong SL^2-Δt-martingale and is an internal lifting of $m = \int x \, db$. ☑

4.7. Definition. Given an i- larcmartingale $m : \Omega \times [0,1]^2 \longrightarrow \mathbb{R}$ with respect to the filtration $(\mathfrak{F}_z)_{z\in[0,1]^2}$, $i = 1, 2$, we say that M is an internal i-Δt-martingale lifting of m if M is an internal i-$SL^2 - \Delta t$-martingale with respect to $\mathfrak{B}_{(\underline{s},\underline{t})}$ for $i = 1, 2$, and $st(M)(\underline{s},\underline{t}) = m({}^o\underline{s}, {}^o\underline{t})$ a.s.

4.8. Theorem. *Let $x : \Omega \times [0,1]^2 \longrightarrow \mathbb{R}$ be an 1-larcmartingale with respect to $\mathfrak{F}_{(s,t)}$. Then, it has a 1-Δt-martingale lifting X with respect $\mathfrak{B}_{(\underline{s},\underline{t})}$.*

Proof. The filtration $\{\mathfrak{F}_{(s,t)} : (s,t) \in [0,1]^2\}$ satisfies $F4$. So, we have

$$x(s,t) = E(x(1,t)|\mathfrak{F}_{(s,t)}) = E(x(1,t)|\mathfrak{F}_{(s,1)}).$$

Now, from Theorem 2.2.2 in [14], there exists an SD^2J lifting Y of x that is S-integrable for some $p \geq 1$. Define $X(\underline{s},\underline{t}) = \overline{E}(Y(1,\underline{t})|\mathfrak{B}_{(\underline{s},1)})$. Then X is an internal 1-martingale.

Let $\underline{s}_n \in T$ be a sequence such that $0 < {}^o\underline{s}_n - \underline{s} < 1/n$ and ${}^o\underline{s}_{n+1} < {}^o\underline{s}_n$. For $\underline{t} \in T$ we have

$$\begin{aligned}\lim_{{}^o\underline{s}_n \to s} {}^oX(\underline{s_n},\underline{t}) &= \lim_{{}^o\underline{s}_n \to s} E(x(1,{}^o\underline{t})|\sigma(\mathfrak{B}_{(\underline{s}_n,\underline{t})}))\\ &= E(x(1,{}^o\underline{t})|\mathfrak{F}_{(s,\underline{t})}) = E(x(1,{}^o\underline{t})|\mathfrak{F}_{(s,1)})\\ &= E(x(1,t)|\mathfrak{F}_{(s,1)}) = x(s,t) = \underline{s}t(X)(s,t),\end{aligned}$$

for all $\underline{t} \approx t$. The second equality being a consequence of the reverse martingale theorem (see [3]).

By definition, X is S- integrable. We must show that X is SD^2J for some $\Delta t \approx 0$, $\Delta t \in T$. Clearly X is an internal 1-martingale. Thus, for fixed $\underline{t} \in T$, $X(\underline{s},\underline{t})$ is a one parameter internal martingale with respect to $\underline{s}$, and so is SD with respect to $\underline{s}$ and therefore $\lim_{{}^o\underline{s}\to s} {}^oX(\underline{s},\underline{t})$ exists and is equal to $\underline{s}t(X)(s,\underline{t})$. Let $t = m/n$, $1 \leq m \leq n$. From the definition of T, we can find $\underline{\rho}_n \in T$, $\underline{\rho}_n \approx 1/n$, $1/n < \underline{\rho}_n < 2/n$, and $m/n \approx m\underline{\rho}_n$, $1 \leq m \leq n$. We can extend the sequence $\{\underline{\rho}_n\}$ to a sequence in ${}^*\mathbb{N}$ such that for some $\nu_o \in {}^*\mathbb{N} \setminus \mathbb{N}$ and all $\nu \leq \nu_o$, we have $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $\underline{\rho}_\nu \approx 0$, $\underline{\rho}_\nu \in T$. To each n we associate the set $\{0, \underline{\rho}_n, 2\underline{\rho}_n, \ldots, (n-1)\underline{\rho}_n\}$. For each m there exist $\varepsilon^n_m \in T$ and $n_o \in \mathbb{N}$ such that $1/(n+n_o) < {}^o\varepsilon^n_m < 1/n$ and

$$|{}^oX(\underline{s}+\varepsilon, m\underline{\rho}_n) - x({}^o\underline{s}, m/n)| < 1/n,$$

for all $\varepsilon \in T$, $0 < {}^o\varepsilon < {}^o\varepsilon^n_m$. Let $\varepsilon_n = \min\{\varepsilon^n_0, \varepsilon^n_1, \ldots, \varepsilon^n_n\}$. We have

$$P\left(\left\{w : \sup_{\substack{0<{}^o\varepsilon<{}^o\varepsilon_n\\ \varepsilon\in T}} |{}^oX(\underline{s}+\varepsilon, m\underline{\rho}_n) - x({}^o\underline{s}, m/n)| > 1/n\right\}\right) < 1/n,$$

for all $m = 1, 2, ..., n$. By the permanence principle (given any internal set of objects $\{R_\eta : \eta \in {}^*\mathbb{N}\}$, and an internal set S such that $R_n \in S$ for every $n \in \mathbb{N}$, there is an $\eta_0 \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $R_\eta \in S$ for every $\eta < \eta_0$), for each $n \in \mathbb{N}$ there is an infinitesimal $\underline{\delta}_n \in T$ such that

$$\overline{P}\left(\left\{w : \sup_{\substack{\delta_n \leq \varepsilon \leq \varepsilon_n \\ \varepsilon \in T}} |X(\underline{s}+\varepsilon, m\underline{\rho}_n) - Y(\underline{s}, m\underline{\rho}_n)| > 1/n\right\}\right) < 1/n.$$

We can extend the sequence $\underline{\delta}_n$ to ${}^*\mathbb{N}$, and so we find $\nu_1 \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\nu_1 \leq \nu_o$. Let $\underline{\delta} = \max_{n \leq \nu_1} \underline{\delta}_n \approx 0$. Then $\underline{\delta} \in T$ and

$$\overline{P}\left(\left\{w : \sup_{\substack{\underline{\delta} < \varepsilon < \varepsilon_{\nu_1} \\ \varepsilon \in T}} |X(\underline{s}+\varepsilon, m\underline{\rho}_{\nu_1}) - Y(\underline{s}, m\underline{\rho}_{\nu_1})| > 0\right\}\right) \approx 0,$$

so that

$$P\left(\left\{w : \sup_{\substack{\underline{\delta} < \varepsilon < \varepsilon_{\nu_1} \\ \varepsilon \in T}} |{}^oX(\underline{s}+\varepsilon, m\underline{\rho}_{\nu_1}) - Y(\underline{s}, m\underline{\rho}_{\nu_1})| > 0\right\}\right) = 0$$

for all $m = 1, 2, ..., \nu_1$. Let $T' = \{0, \underline{\rho}_{\nu_1}, 2\underline{\rho}_{\nu_1}, ..., m\underline{\rho}_{\nu_1}, ..., \} \cup \{1\}$ and $s = k/n$, $1 \leq k \leq n$. Then, there exists $\eta_n \approx 1/n$, $\eta_n \in T'$ and $\eta_n > 1/n$, such that ${}^oX(k\eta_n, m\underline{\rho}_{\nu_1}) \approx x(k/n, {}^o(m\underline{\rho}_{\nu_1}))$ a.s. for all $m = 1, 2, ..., \nu_1$.

For all n, $1/n < \eta_n < 2/n$, and

$$\overline{P}\left(\left\{w : \max_{1 \leq k \leq n} |X(k\eta_n, m\underline{\rho}_{\nu_1}) - Y(k\eta_n, m\underline{\rho}_{\nu_1})| > 1/n\right\}\right) < 1/n.$$

The set of n's such that $1/n < \eta_n < 2/n$ and

$$\overline{P}\left(\left\{w : \max_{1 \leq k \leq n} |X(k\eta_n, m\underline{\rho}_{\nu_1}) - Y(k\eta_n, m\underline{\rho}_{\nu_1})| > 1/n\right\}\right) < 1/n$$

contains $\mathbb{N}$. Thus by the permanence principle, we find $\beta \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\eta_\beta \approx 0$, $\eta_\beta > 0$, $\eta_\beta \in T'$ and

$$\overline{P}\left(\left\{w : \max_{1 \leq k \leq \beta} |X(k\eta_\beta, m\underline{\rho}_{\nu_1}) - Y(k\eta_\beta, m\underline{\rho}_{\nu_1})| > 1/\beta\right\}\right) < 1/\beta,$$

for all $m = 1, 2, ..., \nu_1$. Let $\eta_\beta = a\underline{\rho}_{\nu_1}$. Then $m\rho_{\nu_1} = \frac{m}{a}\eta_\beta$. Let $b = \frac{m}{a}$, so that

$$N_1 = \left\{w : \max_{1 \leq k \leq \beta} |X(k\eta_\beta, b\eta_\beta) - Y(k\eta_\beta, m\eta_\beta)| > 0\right\},$$

is a P- null set Now take $X|_{(T'')^2 \times \Omega}$, where $T'' = \{k\eta_\beta : k\eta_\beta \leq 1\} \cup \{1\}$. Then, since Y is SD^2J, also $X|_{(T'')^2 \times \Omega}$ is SD^2J. ☑

An analogous result holds for 2- martingales (2-local martingales).

4.9. Definition. Given a weak larcmartingale $m : \Omega \times [0,1]^2 \longrightarrow \mathbb{R}$ with respect to the filtration $(\mathfrak{F}_z)_{z \in [0,1]^2}$, we say that M is an internal weak Δt-martingale lifting of m, if M is an internal weak $SL^2 - \Delta t$-martingale with respect to $\mathfrak{B}_{(\underline{s},\underline{t})}$ and $st(M)(\underline{s},\underline{t}) = m({}^o\underline{s}, {}^o\underline{t})$ a.s.

4.10. Theorem. *Let $x : \Omega \times [0,1]^2 \longrightarrow \mathbb{R}$ be a weak larcmartingale with respect to $\mathfrak{F}_{(s,t)}$. Then x has a weak Δt-martingale lifting X with respect $\mathfrak{B}_{(\underline{s},\underline{t})}$.*

Proof. We know that a weak martingale x can be represented as $x = m_1 + m_2$, where m_1 is a 1-martingale and m_2 is a 2-martingale (see[17]). So, by Theorem 2.4.15 there exist an internal 1-Δt-martingale lifting M_1 of m_1 and an internal 2-Δt-martingale lifting M_2 of m_2 . Then, $M_1 + M_2$ is an internal weak Δt-martingale lifting of x. ☑

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