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Metric tensors for homogeneous, isotropic, 5-dimensional pseudo Riemannian models

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ABSTRACT. In this paper we study the metric tensor of a homogeneous, isotropic, 5-dimensional pseudo Riemannian space, solving the corresponding Einstein equations when the spatial component is flat, spherical or pseudo spherical.

Keywords and phrases. Einstein's equations; metric tensor; pseudo Riemannian space; curvature; stress-energy and Ricci tensors; flat, spherical and pseudo spherical manifolds; Christoffel's symbols.

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1. Introduction

Since the publication of [2] and [3] in 1921 and 1926, respectively, and the outcome of the Kaluza-Klein unified field theory, the study of Einstein's equations in 5-dimension has been an important issue among researchers. In particular, the question of whether the strength of the gravitational interaction has changed over the history of the universe has been widely discussed and remains unanswered. To find solutions of the field equations in 5-dimensional gravitation theory is important both to understand the meaning of the five dimensions and to make predictions which may be used to test the theory as a whole. Under restrictive conditions imposed on the flat case, Wesson [7] and Ma [4] have done work in this direction.

In this paper we explicitely calculate, under relatively mild assumptions, the metric tensor of a homogeneous, isotropic 5-dimensional pseudo Riemannian space, solving the corresponding Einstein equations when the spatial component is flat, spherical or pseudo spherical. The last two cases will be dealt with in Section 4, and the first in Section 5.

2. Preliminaries

A simplified general expression for the metric tensor for a homogeneous, isotorpic, 5-dimensional pseudo Riemannian space is given by

$$ds^2 = e^{\nu(t)}dt^2 - e^{\omega(t)}\frac{1}{[1+\frac{k}{4}(x^2+y^2+z^2)]^2}(dx^2+dy^2+dz^2) + e^{\mu(t)}d\psi^2 \quad (1)$$

where $k = 1, 0, -1$ is the curvature of the spatial part of the space. A transformation of coordinates allows to write it equivalently as

$$ds^2 = c^2 d\tau^2 - \frac{A^2(\tau)}{[1+\frac{k}{4}(x^2+y^2+z^2)]^2}(dx^2+dy^2+dz^2) + e^{\zeta(\tau)}d\psi^2 \qquad (2)$$

where $A^2(\tau)$ is the factor of expansion of the universe and (τ, x, y, z, ψ) is a synchronic system of coordinates. This is the form of the metric tensor we shall deal with.

The non-zero Christoffel's symbols for (2) are

$$\Gamma^{\tau}_{xx} = \Gamma^{\tau}_{yy} = \Gamma^{\tau}_{zz} = \frac{A\dot{A}}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]^2},$$

$$\Gamma^{x}_{\tau x} = \Gamma^{x}_{x\tau} = \Gamma^{y}_{\tau y} = \Gamma^{y}_{y\tau} = \Gamma^{z}_{\tau z} = \Gamma^{z}_{z\tau} = A\dot{A},$$

$$\Gamma^{x}_{xx} = \Gamma^{y}_{xy} = \Gamma^{y}_{yx} = \Gamma^{z}_{xz} = \Gamma^{z}_{zx} = -\frac{k}{2}\frac{x}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]},$$

$$\Gamma^{y}_{yy} = \Gamma^{x}_{yx} = \Gamma^{x}_{xy} = \Gamma^{z}_{yz} = \Gamma^{z}_{zy} = -\frac{k}{2}\frac{y}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]},$$

$$\Gamma^{z}_{zz} = \Gamma^{x}_{zx} = \Gamma^{x}_{xz} = \Gamma^{y}_{zy} = \Gamma^{y}_{yz} = -\frac{k}{2}\frac{z}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]},$$

$$\Gamma^{x}_{yy} = \Gamma^{x}_{zz} = \frac{k}{2}\frac{x}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]},$$

$$\Gamma^y_{xx} = \Gamma^y_{zz} = \frac{k}{2}\frac{y}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]},$$

$$\Gamma^z_{xx} = \Gamma^z_{yy} = \frac{k}{2}\frac{z}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]},$$

$$\Gamma^\psi_{\psi\tau} = \Gamma^\psi_{\tau\psi} = \frac{1}{2}\dot\zeta$$

and

$$\Gamma^\tau_{\psi\psi} = -\frac{1}{2}\frac{e^\zeta}{c^2}\dot\zeta.$$

Einstein's equation depends on the stress-energy tensor, that in our case we assume to be that of a fluid. It is known that the isotropy condition means that the spatial components of this tensor vanish, *i.e.*, that it is rotation invariant, and the homogeneity can be thought as meaning that the stress-energy tensor corresponds to a perfect fluid. As a local property we assume that pressure effects can be neglected (see [5]).

Under the above assumptions, the components of the Einstein equation are

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}, \tag{3}$$

where R stands for the curvature and T_{ab} and R_{ab} are respectively the components of the stress-energy and Ricci tensors.

Thus the non-zero components of the Ricci tensor,

$$R_{\mu\kappa} = \sum_{\lambda,\eta}(\partial\Gamma^\lambda_{\mu\lambda}/\partial x^\kappa - \partial\Gamma^\lambda_{\mu\kappa}/\partial x^\lambda + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta}),$$

are

$$R_{\tau\tau} = -\frac{3\ddot A}{A} - \frac{\ddot\zeta}{2} - \frac{\dot\zeta^2}{4},$$

$$R_{xx} = R_{yy} = R_{zz} = \frac{2kc^2 + 2\dot A^2 + A\ddot A + (1/2)\dot\zeta A\dot A}{c^2[1+\frac{k}{4}(x^2+y^2+z^2)]^2}$$

and

$$R_{\psi\psi} = \frac{e^\zeta}{2c^2}\left(-\ddot\zeta - 3\dot\zeta\frac{\dot A}{A} + \dot\zeta^2\right).$$

The curvature $R = \sum_{a,b} g^{ab}R_{ab}$ becomes

$$R = \frac{6\ddot A}{c^2A} + \frac{\ddot\zeta}{c^2} + \frac{\dot\zeta^2}{2c^2} + \frac{6k}{A^2} + \frac{6\dot A^2}{A^2c^2} + \frac{3\dot\zeta\dot A}{c^2A},$$

and the T_{00} component of the stress-energy tensor is given by

$$T_{00} = \rho(\tau)\,c^2$$

where $\rho(\tau)$ is the density of the universe (see [5]).

The independent components of Einstein's equation can be expressed by the system

$$6kc^2 + 6\dot{A}^2 + 3\dot{\zeta}\dot{A}A = 16\pi G\rho A^2, \tag{4}$$

$$4kc^2 + 4\dot{A}^2 + 8\ddot{A}A + 4\dot{\zeta}\dot{A}A + 2\ddot{\zeta}A^2 + \dot{\zeta}^2A^2 = 0, \tag{5}$$

$$\ddot{A}A + kc^2 + \dot{A}^2 = 0, \tag{6}$$

where G is the Newton gravitational constant.

3. Determination of the universe scale factor

To solve the above system, we first observe that equation (6) is integrable. In fact, writing

$$\dot{A}^2 + \ddot{A}A = d(A\dot{A})/d\tau,$$

we obtain at once that

$$A(\tau) = [-kc^2\tau^2 + 2\tau\beta - A_0]^{\frac{1}{2}}, \tag{7}$$

where A_0 and β are constant to be determined. Now, the initial condition $A(0) = 0$ fixes the constant $A_0 = 0$, and we introduce the new constant $\tilde{\tau} = \beta/c^2 > 0$ to ensure that the factor of expansion $A^2(\tau)$ and its derivative are positive for all $\tau > 0$ if $k \leq 0$ and for $0 < \tau < \tilde{\tau}/k$ if $k > 0$, implying in the latter case a finite expansion period $(0, \tilde{\tau}/k]$ followed by contraction on $(\tilde{\tau}/k, 2\,\tilde{\tau}/k)$. Thus,

$$A(\tau) = [c^2(-k\tau^2 + 2\tilde{\tau}\tau)]^{\frac{1}{2}}. \tag{8}$$

4. Determination of ζ for $k \neq 0$

Only the function ζ in (2) remains to be determined. To do so, we take kc^2 from (6) and replace it in (5) to get

$$2\ddot{\zeta} + \dot{\zeta}^2 + \frac{4\dot{\zeta}\dot{A}}{A} + \frac{4\ddot{A}}{A} = 0. \tag{9}$$

The change of variables $\eta = \dot{\zeta}$ leads to the Ricatti equation

$$\dot{\eta} = -\frac{\eta^2}{2} - \frac{2\eta\dot{A}}{A} - \frac{2\ddot{A}}{A}, \tag{10}$$

and setting $\eta = 2\dot{v}/v$ yields

$$A\ddot{v} + 2\dot{A}\dot{v} + \ddot{A}v = 0 \tag{11}$$

or, equivalently,

$$\frac{d^2(Av)}{d\tau^2} = 0. \tag{12}$$

Integrating we obtain

$$v = \frac{B_1\tau + B_2}{A}, \tag{13}$$

where B_1 and B_2 are constants to be determined. From the formulas for the changes of variables and from the result (13) above we have

$$\dot{\zeta} = \frac{2B_1}{B_1\tau + B_2} - \frac{2\dot{A}}{A} \tag{14}$$

and thus

$$\zeta = \ln\left[\frac{(B_1\tau + B_2)^2}{2\tilde{\tau}\tau - k\tau^2}\right] + D_1. \tag{15}$$

We choose $D_1 = i\pi$, in such a way that the signature of the metric tensor becomes (1, 4), which corresponds to a Lorentzian space, the curvature and the Ricci tensor still remaining invariant.

To determine B_1 and B_2, we require that ζ and its derivates vanish at $\tau = \tau_0$, where τ_0 is the "present-value" of the variable τ. This leads, observing that $H_0 = \dot{A}_0/A_0$ is the Hubble constant, to

$$\zeta = \ln\left[\frac{(H_0\tau + 1 - H_0\tau_0)^2(2\tilde{\tau}\tau_0 - k\tau_0^2)}{2\tilde{\tau}\tau - k\tau^2}\right] + i\pi. \tag{16}$$

Replacing in (4), we obtain for the density function ρ the expression

$$\rho(\tau) = \frac{3}{8\pi G}\frac{(1-\tau_0 H_0)^2}{\tau(5\tau\tau_0 H_0 + 2\tau_0 - 4\tau_0^2 H_0 - 4\tau\tau_0^2 H_0^2 + 2\tau_0^3 H_0^2 - \tau^2 H_0 - \tau + 2\tau^2\tau_0 H_0^2)}, \tag{17}$$

and the metric tensors are:

If $k = 1$ (spherical case),

$$\begin{aligned} ds^2 = c^2 d\tau^2 &- \frac{c^2(2\tau\tilde{\tau} - \tau^2)}{[1 + \frac{1}{4}(x^2 + y^2 + z^2)]^2}(dx^2 + dy^2 + dz^2) \\ &- \frac{(H_0\tau + 1 - H_0\tau_0)^2(2\tilde{\tau}\tau_0 - \tau_0^2)}{2\tilde{\tau}\tau - \tau^2}d\psi^2. \end{aligned} \tag{18}$$

If $k = -1$ (pseudo spherical case)

$$\begin{aligned} ds^2 = c^2 d\tau^2 &- \frac{c^2(2\tau\tilde{\tau} + \tau^2)}{[1 - \frac{1}{4}(x^2 + y^2 + z^2)]^2}(dx^2 + dy^2 + dz^2) \\ &- \frac{(H_0\tau + 1 - H_0\tau_0)^2(2\tilde{\tau}\tau_0 + \tau_0^2)}{2\tilde{\tau}\tau + \tau^2}d\psi^2. \end{aligned} \tag{19}$$

5. Determination of ζ for $k = 0$

In this case, equation (5) can be reduced to

$$\frac{2\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{\zeta}}{A} + \frac{\ddot{\zeta}}{2} + \frac{\dot{\zeta}^2}{4} = 0. \tag{20}$$

Changing variable $\dot{\zeta} = \eta$ we obtain

$$\frac{2\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\dot{A}\eta}{A} + \frac{\dot{\eta}}{2} + \frac{\eta^2}{4} = 0, \tag{21}$$

and letting $\eta = 2\dot{v}/v$, we get

$$2\ddot{A}v + \frac{\dot{A}^2 v}{A} + 2\dot{v}\dot{A} + \ddot{v}A = 0. \tag{22}$$

Replacing in (8) with $k = 0$ yields

$$-\frac{1}{4}\beta^2 v + \dot{v}\beta^2\tau + \ddot{v}\beta^2\tau^2 = 0, \tag{23}$$

and to solve for v we propose a solution of the form $C_1\tau^n + C_2\tau^m$. We find that

$$v = C_1\tau^{\frac{1}{2}} - C_2\tau^{-\frac{1}{2}} \tag{24}$$

where C_1 and C_2 are constants to be determined. Now, using that

$$\dot{\zeta} = \frac{2\dot{v}}{v} \tag{25}$$

leads to

$$\zeta = \ln\left(C_1\tau^{\frac{1}{2}} - C_2\tau^{-\frac{1}{2}}\right) + D_2, \tag{26}$$

and since the Christoffel symbols are invariant with respect to the election of g_{55} (space like or time like) we choose, by the same reasons as in the case $k \neq 0$, $D_2 = i\pi$.

Again, we demand that ζ and its derivatives vanish at $\tau = \tau_0$, , to get

$$\zeta = \ln\left[\frac{1}{2}\left(\frac{\tau}{\tau_0}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{\tau_0}{\tau}\right)^{\frac{1}{2}}\right]^2 + i\pi. \tag{27}$$

Replacing in (4), we obtain that

$$\rho(\tau) = \frac{3}{8\pi G}\frac{1}{2\tau(\tau+\tau_0)}, \tag{28}$$

and the metric tensor for this manifold is

$$ds^2 = c^2 d\tau^2 - 2c^2\tau\tilde{\tau}(dx^2 + dy^2 + dz^2) - \left[\frac{1}{2}\left(\frac{\tau}{\tau_0}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{\tau}{\tau_0}\right)^{-\frac{1}{2}}\right]^2 d\psi^2. \tag{29}$$

6. Conclusion

We have presented the complete analytical solutions for a 5-dimensional dust-filled universe. We have added the extra coordinate following the recipe given by Wesson [6]. These solutions can be used, for instance, to test the relation proposed in [4], [7], between the fifth dimension and the particles rest masses [1].

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