Revista Colombiana de Matemáticas
Volumen 32 (1998), páginas 133–142

Two new conjectures concerning positive Jacobi polynomials sums

DIMITAR K. DIMITROV* & CLINTON A. MERLO†
Universidade Estadual Paulista, Brasil

ABSTRACT. A refinement of a conjecture of Gasper concerning the values of (α, β), $-1/2 < \beta < 0$, $-1 < \alpha + \beta < 0$, for which the inequalities

$$\sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1) \geq 0, \quad -1 \leq x \leq 1, \quad n = 1, 2, \ldots$$

hold, is stated. An algorithm for checking the new conjecture using the package *Mathematica* is provided. Numerical results in support of the conjecture are given and a possible approach to its proof is sketched.

Keywords and phrases. Jacobi polynomials, positive sums, Bessel functions, discriminant of a polynomial.
1991 Mathematics Subject Classification. Primary 33C45.

1. Introduction

The Jacobi polynomials are defined in terms of the hypergeometric function ${}_2F_1$ by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2),$$

*Research supported by Brazilian Science Fundation CNPq under Grant 300645/95-3.
†Research supported by a fellowship of the Brazilian Science Fundation CAPES.

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhamer symbol and

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. $$

Various special cases of the inequalities

$$ S_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)/P^{(\beta,\alpha)}(1) \geq 0, \; -1 \leq x \leq 1, \; n = 1,2,\ldots \tag{1} $$

have been proved. Fejér [11, 12] was the first to establish inequalities of this form for $\alpha = 1/2$, $\beta = -1/2$ and for $\alpha = \beta = 0$. Fejér conjectured that (1) also hold for $\alpha = \beta = 1/2$ and this was proved independently by Jackson [16] and Gronwall [15]. Feldheim [13] proved (1) for $\alpha = \beta \geq 0$. Some special cases of these inequalities were considered by Askey [1, 2] and Askey and Gasper [4] proved (1) for $\beta \geq 0$, $\alpha + \beta \geq -2$. The importance of the latter result is justified by the fact that de Branges [7] used (1) for $\beta = 0$, $\alpha = 2,4,6,\ldots$, in the final step of his proof of the celebrated Bieberbach conjecture. Gasper [14] proved inequalities (1) for $\beta \geq -1/2$, $\alpha + \beta \geq 0$.

Note that Bateman's integral formula (Bateman [6])

$$ \frac{P_n^{(\alpha-\mu,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha-\mu)}(1)} = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \int_{-1}^{x} \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\beta,\alpha)}(1)} \frac{(1+t)^\beta}{(1+x)^{\beta+\mu}} (x-t)^{\mu-1}\, dt, \tag{2} $$

which holds for $\mu > 0$, and $\beta > -1$, implies the following result.

Lemma 1. *If the inequalities (1) holds for (α,β), they hold for $(\alpha-\mu,\beta+\mu)$, $\mu > 0$ as well. Hence, if (1) fail for some (α,β) they fail for $(\alpha+\mu,\beta-\mu)$, $\mu > 0$.*

On the other hand $S_1^{(\alpha,\beta)}(x) = (\alpha+\beta+2)(1+x)/(2(\beta+1))$. Having in mind these observations, the above mentioned results of Askey and Gasper [4] and of Gasper [14] yield: Inequalities (1) hold for $\alpha \leq 0$, $\beta \geq \max\{0, -\alpha-2\}$ and $\alpha \geq 0$, $\beta \geq \max\{-1/2, -\alpha\}$, and fail for $\beta < \max\{-1/2, -\alpha-2\}$.

In 1993 Askey [3] drew attention to (1) for the rest of the (α,β)–plane, namely, for (α,β) in the parallelogram $D_1 = \{-1/2 \leq \beta < 0, \; -2 \leq \alpha+\beta < 0\}$. It was proved in [10] that (1) fail for $x = 1$ and for sufficiently large n, if $|\alpha - 3/2| - 1/2 \leq \beta < 0$. The latter and Bateman's integral (2) disprove inequalities (1) for the left hand half of D_1 and n large enough. Thus the only region in the (α,β)–plane for which inequalities (1) is still to be proved or disproved is the parallelogram

$$ D = \{(\alpha,\beta) : -1/2 < \beta < 0, \; -1 \leq \alpha+\beta < 0\}. $$

On the other hand, (1) hold for the upper boundary $\{\beta = 0,\ -1 \le \alpha < 0\}$ and fail for the lower boundary $\{\beta = -1/2,\ -1/2 \le \alpha < 1/2\}$ of D. Hence, by Bateman's integral, for any $\theta \in (-1, 0)$ there exists an $(\alpha', \beta') \in D$ with $\alpha' + \beta' = \theta$ such that (1) hols for $\{\alpha + \beta = \theta, \beta \ge \beta'\}$ and fail for $\{\alpha + \beta = \theta,\ \beta < \beta'\}$. The curve formed by the points (α', β') with this property will be denoted by γ. Also, denote by $J_\alpha(x)$ the Bessel function of the first kind with parameter α and let $j_{\alpha,2}$ be the second positive zero of $J_\alpha(x)$. The following conjecture is due to Gasper [14, p. 444].

Conjecture 1. *The subregion Δ of D for which the inequalities (1) holds is given by*

$$\Delta = \left\{ (\alpha, \beta) \in D : \beta \ge \beta(\alpha),\ \text{where } \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t)\, dt = 0 \right\}. \qquad (3)$$

It may be pointed out that Gaspers's conjecture is equivalent to the statement that

$$\gamma = \left\{ (\alpha, \beta(\alpha)) \in D : \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t)\, dt = 0 \right\}.$$

The conjecture is based on the well-known formula (see (1.8) in [3])

$$\lim_{n\to\infty} \left(\frac{\theta}{n}\right)^{\alpha-\beta+1} \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\cos(\theta/n))}{P_k^{(\beta,\alpha)}(1)} = 2^\alpha \Gamma(\beta+1) \int_0^\theta t^{-\beta} J_\alpha(t)\, dt, \quad \beta < \alpha + 1,$$

and on the following theorem.

Theorem 1. *Let $-1 < \alpha < 1/2$ and $\beta > -1/2$. Then the inequality*

$$\int_0^\theta t^{-\beta} J_\alpha(t)\, dt \ge 0$$

holds for any nonnegative θ if and only if

$$\int_0^{j_{\alpha,2}} t^{-\beta} J_\alpha(t)\, dt \ge 0.$$

The proof of this theorem for $\alpha \in (-1, -1/2)$ is due to Askey and Steinig [5] and the case $\alpha \in (-1/2, 1/2)$ was proved by Makai [17].

Very recently Brown, Koumandos and Wang [8, 9] verified Gasper's conjecture for the case when (α, β) lies on the lines $\alpha = \beta$ or $\alpha = -1/2$.

The objective of the present paper is to state a slight refinement of Conjecture 1 and to give numerical evidence of its truth.

2. The new conjecture

For any positive integer n, set

$$\Delta_n = \left\{ (\alpha, \beta) \in D : S_n^{(\alpha,\beta)}(x) \geq 0 \text{ for } x \in [-1, 1] \right\}.$$

Then Gasper's conjecture can be formulated in the equivalent form

$$\bigcup_{n=1}^{\infty} \Delta_n = \Delta,$$

where Δ is defined by (3).

We state

Conjecture 2. *For any positive integer n, $\Delta_{n+1} \subset \Delta_n$.*

Denote by γ_n the boundary of Δ_n which passes through D:

$\gamma_n = \{(\alpha, \beta) \in D : S_n^{(\alpha,\beta)}(x) \geq 0$ for all $x \in [-1, 1]$ and every (α, β) with $\alpha + \beta = \alpha_n + \beta_n$, $\beta \geq \beta_n$, and for some $x \in [-1, 1]$, $S_n^{(\alpha,\beta)}(x) < 0$ for (α, β) with $\alpha + \beta = \alpha_n + \beta_n$, $\beta < \beta_n\}$.

The curve γ_n is well defined because of Lemma 1.

An equivalent formulation of Conjecture 2 is that γ_{n+1} lies above γ_n for any positive integer n. The latter conjecture implies that of Gasper, because of (4) and Theorem 1.

In the next section we give explicit expresions for Δ_2 and Δ_3 or, equivalently, for γ_2 and γ_3. In Section 3 an algorithm to trace the curves γ_n is developed. Tables for the curves γ_n for $n = 4$ and 5 are given and the graphs of γ_n for $n = 2, 3, 4, 5$ are drawn. In Section 4 we discuss an idea of how Conjecture 2 might be proved.

3. The cases $n = 2$ and $n = 3$

In what follows we suppose that $(\alpha, \beta) \in D$. First we consider the case $n = 2$. Straightforward calculations show that

$$4(\beta + 1)(\beta + 2)\, S_2^{(\alpha,\beta)}(x) = a_2 x^2 + 2a_1 x + a_0,$$

where

$$\begin{aligned}
a_2 &= (\alpha + \beta + 3)(\alpha + \beta + 4), \\
a_1 &= 2(\alpha + 2)(\alpha + \beta + 3) + (\alpha + \beta + 2)(\beta + 2) - (\alpha + \beta + 3)(\alpha + \beta + 4) \\
&= (\alpha + 1)(\alpha + \beta + 4), \\
a_0 &= 2(\alpha + \beta + 2)(\beta + 2) + 4(\alpha + 1)(\alpha + 2) + (\alpha + \beta + 3)(\alpha + \beta + 4) \\
&\quad - 4(\alpha + 2)(\alpha + \beta + 3) = \alpha^2 + 3\beta^2 + 3\alpha + 7\beta + 4.
\end{aligned}$$

Obviously $S_2^{(\alpha,\beta)}(x)$ is convex and its minimum value is attained at $x_{\min} = -a_1/a_2 = -(\alpha+1)/(\alpha+\beta+3)$. Observe that $-1 < x_{\min} < 0$. Hence, $S_2^{(\alpha,\beta)}(x) \geq 0$ for $x \in [-1,1]$ if and only if it is non-negative for any real x. Since its leading coefficient is positive, then $S_2^{(\alpha,\beta)}(x)$ is non-negative if and only if its discriminant

$$(\alpha+1)^2(\alpha+\beta+4)^2 - (\alpha+\beta+3)(\alpha+\beta+4)\left(\alpha^2+3\beta^2+3\alpha+7\beta+4\right)$$

is non-positive. Thus,

$$\Delta_2 = \left\{(\alpha,\beta) \in D : \beta \geq \frac{-3\alpha - 10 + \sqrt{9\alpha^2+36\alpha+52}}{6}\right\}.$$

The case $n = 3$ may be treated similarly because $S_n^{(\alpha,\beta)}(-1) = 0$ for any odd n. Set $u = (x+1)/2$. Staightforward calculations show in fact that

$$\overline{S}_3^{(\alpha,\beta)}(u) = \frac{S_3^{(\alpha,\beta)}(x)}{u} = b_2u^2 - 2b_1u + b_0$$

where

$$\begin{aligned} b_2 &= (\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)/(\beta+1)(\beta+2)(\beta+3), \\ b_1 &= (\alpha+\beta+4)(\alpha+\beta+6)/(\beta+1)(\beta+2), \\ b_0 &= 2(\alpha+\beta+4)/(\beta+1), \end{aligned}$$

and we have to characterize the values of (α,β) in D for which $\overline{S}_3^{(\alpha,\beta)}(u) \geq 0$ for each $u \in [0,1]$. Since $\overline{S}_3^{(\alpha,\beta)}(u)$ attains its minimum at $u_{\min} = b_1/b_2 = (\beta+3)/(\alpha+\beta+5)$ and $u_{\min} \in [0,1]$, then $\overline{S}_3^{(\alpha,\beta)}(u) \geq 0$ for $u \in [0,1]$ and those (α,β) for which the discriminant

$$\left(\frac{(\alpha+\beta+4)(\alpha+\beta+6)}{(\beta+1)(\beta+2)}\right)^2 - 2\frac{(\alpha+\beta+4)^2(\alpha+\beta+5)(\alpha+\beta+6)}{(\beta+1)^2(\beta+2)(\beta+3)}$$

of $\overline{S}_3^{(\alpha,\beta)}(u)$ is non-negative. Therefore

$$\Delta_3 = \left\{(\alpha,\beta) \in D : \beta \geq \frac{-\alpha - 5 + \sqrt{\alpha^2+6\alpha+17}}{2}\right\}.$$

4. An algorithm to find Δ_n

The algorithm for tracing the curves γ_n is based on the following simple fact.

Lemma 2. *If* $(\alpha_n,\beta_n)\in\gamma_n$, *then there exists* $\xi\in(-1,1)$ *for which*

$$S_n^{(\alpha_n,\beta_n)}(\xi)=\frac{d}{dx}S_n^{(\alpha_n,\beta_n)}(\xi)=0.$$

Proof. Assume that for some (α_n,β_n) the polynomial $S_n^{(\alpha_n,\beta_n)}(x)$ is positive at the points of local extrema in $(-1,1)$. Then a continuity argument implies that there exists a neighborhood U of (α_n,β_n) such that for every (α,β) in U and for every $x\in(-1,1)$ the polynomial $S_n^{(\alpha,\beta)}(x)$ is positive. The latter contradicts the definition of γ_n. ☑

A well known necessary condition for a polynomial

$$p(x)=\sum_{\nu=0}^{n}a_\nu x^{n-\nu}$$

to have a double root is stated in the following lemma. We recall that the discriminant $D(p)$ of p is

$$D(p)=a_0^{2n-2}\prod_{1\le i<j\le n}(x_i-x_j)^2,$$

where $x_1,\dots,x_n$ are the roots (zeros) of p.

Lemma 3. *The discriminant* $D(p)$ *of the polynomial* p *can be represented as a* $(2n-1)\times(2n-1)$ *determinant in the form*

$$\frac{a_0D(p)}{(-1)^{n-1}}=\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n & & & \\ na_0 & (n-1)a_1 & \cdots & a_{n-1} & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & a_0 & a_1 & \cdots & & a_{n-1} & a_n \\ & & na_0 & (n-1)a_1 & \cdots & & a_{n-1} & \\ & & & na_0 & (n-1)a_1 & & \cdots & a_{n-1} \end{vmatrix}$$

Moreover, $D(p)=0$ *if and only if* $p(x)$ *has at least one root of multiplicity at least two.*

We refer to [18, Section 1.3.3] and the references therein for the proof of this lemma and for additional information about discriminants.

Lemmas 2 and 3 immidiately yield the following result.

Theorem 2. *Let* $S_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}a_k(\alpha_n,\beta_n)x^{n-k}$. *If* $(\alpha_n,\beta_n)\in\gamma_n$, *then*

$$D(\alpha_n,\beta_n):=D\left(S_n^{(\alpha_n,\beta_n)}\right)=0.$$

The basic steps of the algorithm to construct an approximation to the curve γ_n are:

1. Choose $k \in \mathbb{N}$.
2. Divide the interval $[-2, 1/2]$ into k subintervals by the mesh points $\alpha_n^{(i)} = -2 + 2.5i/k$, $i = 0, k$.
3. For any fixed $\alpha_n^{(i)}$ find all the solutions $\beta_{n,1}^{(i)}, \dots, \beta_{n,p}^{(i)} \in (-1/2, 0)$ of the equation $D\left(\alpha_n^{(i)}, \beta\right) = 0$.
4. Find that s, $1 \leq s \leq p$, for which
$$S_n^{\left(\alpha_n^{(i)}, \beta_{n,s}^{(i)}\right)}(x) \geq 0 \text{ for } x \in [-1, 1]$$
and
$$S_n^{\left(\alpha_n^{(i)}, \beta_{n,s}^{(i)}\right)}(\xi) = \frac{d}{dx} S_n^{\left(\alpha_n^{(i)}, \beta_{n,s}^{(i)}\right)}(\xi) = 0 \text{ for some } \xi \in (-1, 1)\,.$$
5. Choose $\beta_n^{(i)} = \beta_{n,s}^{(i)}$.
6. Approximate the data $\left(\alpha_n^{(i)}, \beta_n^{(i)}\right)$ by a smooth curve.

Table 1 in the next page contains the results of the algorithm for $n = 4$ and $n = 5$, for $k = 50$. The values of $\beta_4^{(i)}$ and $\beta_5^{(i)}$ which correspond to $\alpha_n^{(i)} = \alpha^{(i)} = -2 + 0.05i$, $i = 0, \dots, 50$, are:

The graphs of the approximations to the curves γ_n for $n = 2, 3, 4$ and 5 are drawn in Figure 1 at the end of the paper.

5. An idea for proving Conjecture 2

The graphs of the curves $\gamma_2, \gamma_3, \gamma_4$ and γ_5 show that Conjecture 2 holds for $n = 2, 3$ and 4. It is clear that Conjecture 2 would be proved if one proves that $S_n^{(\alpha,\beta)}$ is nonnegative on $[-1, -1]$ for any (α, β) for which $S_{n+1}^{(\alpha,\beta)}$ is nonnegative there. Another possible idea to prove Conjecture 2 is to show that for any $(\alpha_n, \beta_n) \in \gamma_n$ the inequality $S_{n+1}^{(\alpha_n,\beta_n)}(x) \geq 0$ fails for some $x \in [-1, 1]$. It turns out that for $n = 2, 3$ and 4 such x exists. Based on the graphs of $S_n^{(\alpha_n,\beta_n)}(x)$ and $S_{n+1}^{(\alpha_n,\beta_n)}(x)$ for various $(\alpha_n, \beta_n) \in \gamma_n$ we may state an additional conjecture which implies the truth of Conjecture 2, and thus, of Conjecture 1.

Conjecture 3. Let $(\alpha_n, \beta_n) \in \gamma_n$. Then there exists a unique $\xi_n \in (-1, 1)$ such that
$$S_n^{(\alpha_n,\beta_n)}(\xi_n) = \frac{d}{dx} S_n^{(\alpha_n,\beta_n)}(\xi_n) = 0.$$

i	$\alpha^{(i)}$	$\beta_4^{(i)}$	$\beta_5^{(i)}$	i	$\alpha^{(i)}$	$\beta_4^{(i)}$	$\beta_5^{(i)}$
0	−2.00	0	0				
1	−1.95	−0.0124665	−0.0100482	26	−0.70	−0.29347	−0.271235
2	−1.90	−0.0248627	−0.020186	27	−0.65	−0.303304	−0.281463
3	−1.85	−0.0371837	−0.0304035	28	−0.60	−0.313026	−0.291642
4	−1.80	−0.0494251	−0.0406914	29	−0.55	−0.322637	−0.30177
5	−1.75	−0.0615829	−0.051041	30	−0.50	−0.332137	−0.311845
6	−1.70	−0.0736534	−0.0614439	31	−0.45	−0.341526	−0.321856
7	−1.65	−0.0856334	−0.0718924	32	−0.40	−0.350807	−0.331828
8	−1.60	−0.0975197	−0.0823791	33	−0.35	−0.359997	−0.341732
9	−1.55	−0.10931	−0.0928969	34	−0.30	−0.36904	−0.351576
10	−1.50	−0.121001	−0.103439	35	−0.25	−0.377995	−0.361359
11	−1.45	−0.132592	−0.1114	36	−0.20	−0.386843	−0.371079
12	−1.40	−0.144079	−0.124573	37	−0.15	−0.395585	−0.380734
13	−1.35	−0.155462	−0.135135	38	−0.10	−0.404222	−0.390324
14	−1.30	−0.166739	−0.145734	39	−0.05	−0.412754	−0.399847
15	−1.25	−0.177909	−0.156312	40	0.00	−0.421183	−0.409303
16	−1.20	−0.18897	−0.166881	41	0.05	−0.429509	−0.418691
17	−1.15	−0.199922	−0.177438	42	0.10	−0.437734	−0.428009
18	−1.10	−0.210763	−0.110763	43	0.15	−0.445858	−0.437258
19	−1.05	−0.221493	−0.198469	44	0.20	−0.453883	−0.446436
20	−1.00	−0.232112	−0.208989	45	0.25	−0.46181	−0.455544
21	−0.95	−0.242619	−0.219454	46	0.30	−0.469638	−0.464579
22	−0.90	−0.253014	−0.229886	47	0.35	−0.477371	−0.473543
23	−0.85	−0.263296	−0.240284	48	0.40	−0.485008	−0.482435
24	−0.80	−0.273467	−0.250643	49	0.45	−0.49225	−0.491254
25	−0.75	−0.283524	−0.260961	50	0.50	−0.5	−0.5

TABLE 1. *The curves* γ_4 *and* γ_5

Moreover, there exist η_n' and η_n'' with $-1 < \xi_n < \eta_n' < \eta_n'' < 1$ such that

$$S_{n+1}^{(\alpha_n,\beta_n)}(x) < 0 \quad \text{for } x \in (\eta_n', \eta_n'').$$

Finally, we recall that Askey [3] conjectured that $\beta(\alpha)$ defined by (3) is a convex function, which is equivalent to assert that the curve γ is convex. It seems that every γ_n is a convex curve. If so, obviously γ would also be convex.

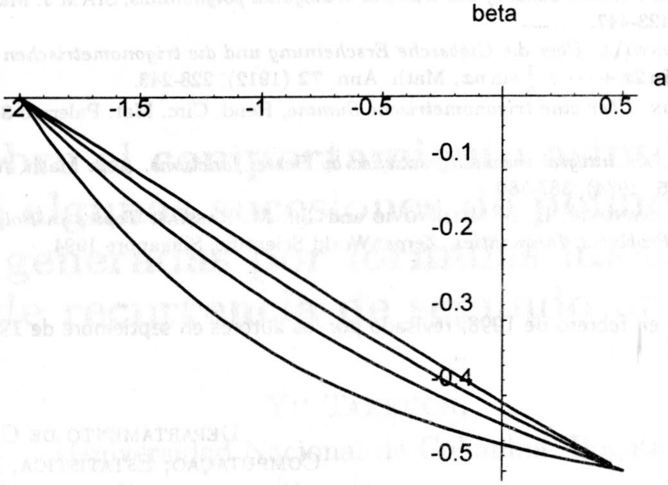

FIGURE 1. *The curves* γ_2, γ_3, γ_4 *and* γ_5.

References

[1] R. ASKEY, *Jacobi polinomial sums*, Tôhoku Math. J. **24** (1972), 109-119.

[2] R. ASKEY, *Orthogonal polynomials and special functions*, Regional Conf. Lect Appl. Math. **48,** SIAM, Philadelphia, 1975.

[3] R. ASKEY, *Problems which interest and/or annoy me*, J. Comp. Appl. Math. **48** (1993), 3–15.

[4] R. ASKEY AND G. GASPER, *Positive Jacobi polynomial sums*, II, Amer. J. Math. **98** (1976), 709-737.

[5] R. ASKEY AND STEINIG, *Some positive trigonometric sums*, Trans. Amer. Math. Soc. **187** (1974), 295-307.

[6] H. BATEMAN, *The solution of linear differential equations by means of definite integrals*, Trans. Camb. Phil. Soc. **21** (1909), 171–196.

[7] L. DE BRANGES, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[8] G. BROWN, S. KOUMANDOS and K. Y. WANG, *Positivity of more Jacobi polynomial sums,* Math. Proc. Camb. Phil. Soc. **119** (1996), 681-694.

[9] G. BROWN, S. KOUMANDOS and K. Y. WANG, *Positivity of basic sums of ultraspherical polynomials,* (submitted).

[10] D. K. DIMITROV and G. M. PHILLIPS, *A note on convergence of Newton interpolating polynomials*, J. Comp. Appl. Math. **51** (1994), 127-130; Erratum **51** (1994), 411.

[11] L. FEJER, *Sur les functions bornée et integrables*, C. R. Acad. Sci. Paris **131** (1900), 984-987.

[12] L. FEJER, *Sur le développement d'une function arbitraire suivant les functions de Laplace*, C. R. Acad. Sci. Paris **146** (1908), 224-227.

[13] E. FELDHEIM, *On the positivity of certain sums of ultraspherical polynomials*, J. Analyse Math. **11** (1963), 275-284.

[14] G. GASPER, *Positive sums of the classical orthogonal polynomials*, SIAM J. Math. Anal. **8** (1977), 423-447.

[15] T. H. GRONWALL, *Über die Gibbssche Erscheinung und die trigonometrischen Summen* $\sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{n}\sin nx$, Math. Ann. **72** (1912), 228-243.

[16] D. JACKSON, *Über eine trigonometrische Summe*, Rend. Circ. Mat. Palermo **32** (1911), 257-262.

[17] E. MAKAI, *An integral inequality satisfied by Bessel functions*, Acta Math. Acad. Sci. Hungar. **25** (1974), 387-380.

[18] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ and TH. M. RASSIAS, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore 1994.

(Recibido en febrero de 1998; revisado por los autores en septiembre de 1998)

DEPARTAMENTO DE CIÊNCIAS;
COMPUTAÇAO; ESTATÍSTICA, IBILCE
UNIVERSIDADE ESTADUAL PAULISTA
15054-000 SÃO JOSÉ DO RIO PRETO, SP, BRASIL
dimitrov@nimitz.dcce.ibilce.unesp.br
merlo@nimitz.dcce.ibilce.unesp.br