A remark on exponential dichotomies

RAÚL NAULIN*
Universidad de Oriente, Cumaná, Venezuela

ABSTRACT. A proof of the existence of an exponential dichotomy for the linear system \( x'(t) = A(t)x(t) \) is given, based on the admissibility of the pair \((B(\infty), B_A(\infty))\), where \(B(\infty)\) is the space of continuous functions on the semi-axis \(J = [0, \infty)\), values in \(\mathbb{C}^n\) and having a limit as \(t \to \infty\), and \(B_A(\infty)\) is the space of bounded functions \(f\) on \(J\) such that \(A^{-1}f \in B(\infty)\).

Keywords and phrases. Exponential dichotomies, admissibility.

1991 Mathematics Subject Classification. Primary 34A30.

1. Introduction

In this paper we consider the system of differential equations

\[
x'(t) = A(t)x(t) + f(t), \quad t \in J := [0, \infty),
\]

where \(A(t)\) is a continuous matrix function with complex entries. The function \(f(t)\) belongs to a functional space we will define in the course of the paper.

Definition 1. Let \(\mathcal{C}\) and \(\mathcal{D}\) be function spaces. We say that the pair \((\mathcal{C}, \mathcal{D})\) is admissible for equation (1) if for each \(f\) in the space \(\mathcal{D}\) there exists a solution of (1) belonging to \(\mathcal{C}\).

*Supported by Proyecto CI-5-025-00730/95
Admissible pairs are important in the theory of differential equations (see [1], [2]), as they define the dichotomic behavior of the linear system

\[ x'(t) = A(t)x(t). \]  

(2)

Definition 2. We say that equation (2) has an exponential dichotomy on \( J \), if there exist a fundamental matrix \( \Phi \) of (2), a projection matrix \( P \) (i.e., \( PP = P \)) and positive constants \( K, \alpha \) such that

\[ |\Phi(t)P\Phi^{-1}(s)| \leq Ke^{\alpha(s-t)}, \quad t \geq s \geq 0, \]

\[ |\Phi(t)(I-P)\Phi^{-1}(s)| \leq Ke^{\alpha(t-s)}, \quad s \geq t \geq 0. \]

(3)

In this paper, we are concerned with the following classical result [1]:

**Theorem A.** Equation (2) has an exponential dichotomy on \( J \) if for any bounded and continuous function \( f(t) \) on \( J \), equation (1) has at least one bounded solution.

The aim of this paper is the characterization of exponential dichotomy by means of the admissibility of a pair of spaces of functions with limit at infinity.

### 2. Preliminaries

We will make use of the following spaces

\[ B \ := \ \{ f : J \to \mathbb{C}^n : f \text{ is bounded and continuous} \}, \]

\[ B(\infty) \ := \ \{ f \in B : \lim_{t \to \infty} f(t) \text{ exists} \}. \]

We call \( B(\infty) \) the space of functions with limit at infinity. These spaces, endowed with the norm \( |f|_\infty = \sup\{|f(t)| : t \in J\} \), become Banach spaces. Furthermore, if \( F : J \to \mathbb{C}^{n \times n} \) and \( F(t) \) is invertible for each \( t \in J \), we define

\[ B_F(\infty) := \{ f \in B : F^{-1}f \in B(\infty) \}. \]

To this space we give the norm \( |f|_F = |F^{-1}f|_\infty \). Provided that \( F \) is bounded on \( J \), also \( B_F(\infty) \) is a Banach space. If equation (2) has an exponential dichotomy, then for any \( f \in B \), equation (1) has the following bounded solution:

\[ x_f(t) = \int_0^t \Phi(t)P\Phi(s)f(s)\, ds - \int_t^{\infty} \Phi(t)(I-P)\Phi^{-1}(s)f(s)\, ds. \]

Let us introduce the following Green function:

\[ G(t, s) = \begin{cases} \Phi(t)P\Phi(s), & t \geq s, \\ -\Phi(t)(I-P)\Phi^{-1}(s), & s > t. \end{cases} \]

(4)
By means of this function we can write the solution $x_I$ in the form:

$$x_I(t) = \int G(t, s)f(s)ds.$$  \hfill (5)

If $A(t)$ is a bounded function, we will use the following identity:

$$\Phi(t)P\Phi^{-1}(0) - I = \int G(t, s)A(s)ds.$$  \hfill (6)

3. The main result

**Theorem 1.** If the function $A(t)$ is bounded on $J$ and the matrix $A(t)$ is invertible for each $t \in J$, then the following assertions are equivalent:

(A) The pair $(B, B)$ is admissible.

(B) The pair $(B(\infty), B_A(\infty))$ is admissible.

(C) Equation (2) has an exponential dichotomy on $J$.

**Proof.**

(A) $\Leftrightarrow$ (C). This follows from Theorem A. We observe that this equivalence holds without the requirements of invertibility of the matrices $A(t)$ or the boundedness of the function $A(t)$.

(A) $\Rightarrow$ (B). Let $f \in B_A(\infty)$. Since $f \in B_A$, formula (5) makes sense. Therefore $x_I$ defines a solution of (1) belonging to $B(\infty)$. We have to prove that $\lim_{t \to \infty} x_I$ exists. Using (6) we may write

$$x_I(t) = -A^{-1}(t)f(t) + \Phi(t)P\Phi^{-1}(0)A^{-1}(t)f(t) + I_1(t) + I_2(t),$$  \hfill (7)

where

$$I_1(t) := \int_0^t G(t, s)A(s)\left[A^{-1}(s)f(s) - A^{-1}(t)f(t)\right]ds,$$

$$I_2(t) := \int_t^\infty G(t, s)A(s)\left[A^{-1}(s)f(s) - A^{-1}(t)f(t)\right]ds.$$

Taking into account (3), we can estimate $I_i, \ i = 1, 2$. We have

$$|I_1(t)| \leq \int_0^t |\Phi(t)P\Phi^{-1}(s)||A(s)||A^{-1}(s)f(s) - A^{-1}(t)f(t)|ds$$

$$\leq \int_0^{t/2} \ldots + \int_{t/2}^t \ldots$$

$$\leq 2K|A|_\infty \alpha^{-1}e^{-\alpha t/2}|A^{-1}f|_\infty$$

$$+ \alpha^{-1}K \sup_{s \in [t/2, t]} |A^{-1}(s)f(s) - A^{-1}(t)f(t)|$$  \hfill (8)
and
\[
|I_2(t)| \leq |A|_\infty \int_t^\infty e^{\alpha(t-s)}|A^{-1}(s)f(s) - A^{-1}(t)f(t)|\,ds
\]
\[
\leq 2\alpha^{-1}|A|_\infty K \sup_{s \in [t, \infty)} |A^{-1}(s)f(s) - A^{-1}(t)f(t)|.
\]  
(9)

Since \( f \in B_A(\infty) \), it is clear from (8) and (9) that \( \lim_{t \to \infty} I_1(t) = 0 \). From (7) we obtain that \( \lim_{t \to \infty} x_f(t) = -(A^{-1}f)(\infty) \). Therefore, the function pair \((B(\infty), B_A(\infty))\) is admissible.

(B) \( \Rightarrow \) (A). Let \( S \) be the subspace of \( C^n \) of values of initial conditions of solutions of equation (2) belonging to \( B(\infty) \), and let \( U \) be a supplementary subspace of \( S \). We have the direct sum \( C^n = S \oplus U \). Then it is easy to prove that equation (1) has, for any \( f \in B_A(\infty) \), a unique solution, which we denote by \( T(f) \), that belongs to \( B(\infty) \) and is such that the initial condition satisfies \( T(f)(0) \in U \). It is also easy to verify that this correspondence is linear. Thus, we define this way a linear map \( T : B_A(\infty) \to B(\infty) \) such that \( T(f) \) satisfies (1) and \( T(f)(0) \in U \). This map has a closed graph (the proof of this assertion is exactly the same as that of Proposition 3.4 in [1]). Therefore, it is bounded, i.e., there exists a constant \( M \), such that
\[
|T(f)| \leq M|f|_A.
\]  
(10)

Let \( f \in B \) and for each \( n = 1, 2, \ldots \), let \( \theta_n(t) \) be a continuous function such that \( |\theta_n|_\infty = 1 \), \( \theta_n(t) = 1 \) if \( t \in [0, n] \) and \( \theta_n(t) = 0 \) if \( t \geq n + 1 \). Let \( \{f_n\} \) be the sequence in \( B_A \) defined by
\[
f_n(t) = \theta_n(t)f(t).
\]  
(11)

For each function \( f_n \), we consider the solution \( x_n = T(f_n) \) of the equation
\[
x'(t) = A(t)x(t) + f_n(t).
\]  
(12)

According to (10), for any index \( n \) we have
\[
|x_n|_\infty \leq M|f_n|_A \leq M|f|_A.
\]  
(13)

From (12) and (13) we obtain that the sequences \( \{x_n\} \) and \( \{x'_n\} \) are bounded on any compact subinterval of \( J \). From the Ascoli-Arzelà theorem, there exists then a subsequence \( \{x_{n_1}^k\} \) of \( \{x_n\} \) uniformly convergent on \([0, 1]\) to a continuous function \( u_1 \) on the interval \([0, 1]\). By the same argument, there exists a subsequence \( \{x_{n_2}^k\} \) of \( \{x_{n_1}^k\} \) converging uniformly on the interval \([0, 2]\) to a continuous function \( u_2 \) such that \( u_1 = u_2 \) on \([0, 1]\). Carrying out this process iteratively, we obtain, for any natural number \( N \), a subsequence \( \{x_{n_N}^k\} \) of \( \{x_{n_{N-1}}^k\} \), converging uniformly to a continuous function \( u_N \) on the interval \([0, N]\), and such that \( u_N = u_{N-1} \) on \([0, N-1]\). Defining \( u(t) = u_N(t) \) if \( t \in [0, N] \), we obtain that the diagonal sequence \( \{x_{n_N}^k\} \) converges uniformly to \( u \) on each compact subinterval.
of \( J \). From (13), we obtain that \( u \in \mathcal{B} \). From (11) and (12) it follows that \( u \) satisfies \( u' = A(t)u + f \). This means that \( u \) is a solution of (1) in the space \( \mathcal{B} \).

References


DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE ORIENTE
CUMANÁ 6101 A-285, VENEZUELA