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On generalized limits and Toeplitz's algorithm

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ABSTRACT. We study generalized limits in connection with the Toeplitz linear algorithm of convergence.

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The problem of assigning a sum, finite or infinite, to a divergent series was already studied by Euler. Several approaches have been made towards the solution, but none has been fully satisfactory. The problem rests on the construction of a generalized limit Lim_n, that is, a functional f on the space of real sequences $x = (x_n)$ such that:

1. $f(x) \in \overline{\mathbb{R}}[-\infty, \infty]$ for any sequence x.
2. $f(x) = \lim_n x_n$ for any convergent sequence $x = (x_n)$.
3. $f(ax + by) = af(x) + bf(y)$ for each couple of real numbers a, b and for each couple of sequences x, y, except in the case of indetermination.
4. $f(x) \geq 0$ for each $x \geq 0$, i. e., $x_n \geq 0$ for all $n \in \mathbb{N}$.

A solution to the problem is given (due to the compactness of $\overline{\mathbb{R}}$) by the limit $\lim_{n,\mathcal{U}}$ over a non trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ consisting of unbounded subsets of $\mathbb{N}$. This limit can be approximated by limits over filters $\mathcal{F}$ with countable basis in $\mathbb{N}$ which are finer than the Fréchet filter, coarser than the given ultrafilter $\mathcal{U}$. Then the inferior limit $\underline{\lim}_{n,\mathcal{F}} x_n$ and the superior limit $\overline{\lim}_{n,\mathcal{F}} x_n$ give a lower and upper bound for $\lim_{n,\mathcal{U}} x_n$. Moreover, for each sequence (x_n), there exists one such filter $\mathcal{F} \subseteq \mathcal{U}$ such that $\lim_{n,\mathcal{F}} x_n = \lim_{n,\mathcal{U}} x_n$.

Conversely, if Lim_n is a multiplicative generalized limit over all bounded real sequences, that is,

$$\mathop{\mathrm{Lim}}_n (x_n y_n) = \left(\mathop{\mathrm{Lim}}_n x_n\right)\left(\mathop{\mathrm{Lim}}_n y_n\right),$$

then there is an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that $\mathrm{Lim}_n x_n = \lim_{n,\mathcal{U}} x_n$ for any bounded sequence (x_n).

For any generalized limit $f \in \ell_\infty^*$ $(= (\ell_\infty)^*)$ there exists a finitely additive measure $\mu = \mu_f \geq 0$, defined over the subsets of $\mathbb{N}$, such that

$$f(x) = \int_{\mathbb{N}} x_n d\mu(n) \quad (x = (x_n) \in \ell_\infty),$$

$\mu(\mathbb{N}) = 1$ and $\mu(\{n\}) = 0$ for all $n \in \mathbb{N}$. Conversely, if $\mu \geq 0$ is a finitely additive measure defined over the subsets of $\mathbb{N}$, such that $\mu(\mathbb{N}) = 1$ and $\mu(\{n\}) = 0$ for any $n \in \mathbb{N}$, then the integral

$$f(x) = \int_{\mathbb{N}} x_n d\mu(n)$$

defined for all $x = (x_n) \in \ell_\infty$, gives a generalized limit $f = \mathrm{Lim}_n$ over ℓ_∞, i.e., over all bounded sequences of real numbers. Hence, $f(\chi_A) = \mu(A)$ for any subset A of $\mathbb{N}$ and $\|f\| = 1$.

If $(\lambda_i)_{i \in I}$ is a net of elements $\lambda_i = (\lambda_{in})_{n \in \mathbb{N}} \in \ell_1$ satisfying $\|\lambda_i\|_1 \leq M$ $(i \in \mathbb{N})$ for a given constant $M > 0$, $\lim_i \lambda_{in} = 0$ for any $n \in \mathbb{N}$ and $\lim_i \sum_{n=1}^\infty \lambda_{in} = 1$, then we have

$$\lim_i \sum_{n=1}^\infty \lambda_{in} x_n = \lim_n x_n$$

for all convergent sequences (x_n).

Hence,

$$f(x) = \lim_i \sum_{n=1}^\infty \lambda_{in} x_n$$

defines a linear extension of the usual limit $\mathrm{Lim}\, x_n$ $(\in \mathbb{R})$ over the sequences (x_n) for which the limit $\lim_i \sum_{n=1}^\infty \lambda_{in} x_n$ exists. This linear algorithm of convergence for $I = \mathbb{N}$ is due to Toeplitz (see [K]).

From the latter it turns out that if $\mathcal{U}$ is an ultrafilter of subsets of an infinite directed set I, and finer than the corresponding Fréchet filter, i. e., the filter of the sets $\{j : j > i\}$, we have that

$$f(x) = \lim_{i,\mathcal{U}} \sum_{n=1}^{\infty} \lambda_{in} x_n$$

is a functional defined not only for bounded sequences, but also for those sequences (x_n) for which the series $\sum_{n=1}^{\infty} \lambda_{in} x_n$ $(i \in I)$ are convergent.

This functional satisfies the above conditions 1-3, but it is not necessarily a generalized limit since it does not satisfy 4. Indeed, if $\lambda_{i,2k-1} \geq 0, \lambda_{i,2k} \leq 0$ for all $k \in \mathbb{N}, \sum_{k=1}^{\infty} \lambda_{i,2k-1} = 2$ and $\sum_{k=1}^{\infty} \lambda_{i,2k} = -1$ $(i \in \mathbb{N})$, for the sequences $x_n = 1 + (-1)^n \geq 0$ we have

$$\lim_{i} \sum_{n=1}^{\infty} \lambda_{in} x_n = -2 < 0.$$

On the contrary, if $\lambda_{in} \geq 0$ for any $i \in I$, $n \in \mathbb{N}$, then under the previous conditions we have that

$$f(x) = \lim_{i,\mathcal{U}} \sum_{n=1}^{\infty} \lambda_{in} x_n$$

is a generalized limit over all sequences (x_n) for which $\sum_{n=1}^{\infty} \lambda_{in} x_n$ is convergent for all $i \in I$. This holds in any case if each λ_i has finite support, i. e., if for each $i \in I$ there is an $n_i \in \mathbb{N}$ such that $\lambda_{in} = 0$ for $n \geq n_i$.

Now, we are going to prove the converse property.

Theorem 1. *Let*

$$S = \{a = (\alpha_k)_{k \in \mathbb{N}} \in \ell_1 : \|\alpha\|_1 = 1, \ \alpha_k \geq 0 \text{ for all } k \in \mathbb{N}\}$$

and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of elements $\lambda_n = (\lambda_{nk})_{k \in \mathbb{N}} \in S$ weakly dense in S. The, for any generalized limit f over ℓ_∞, there exists an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that

$$f(x) = \lim_{n,\mathcal{U}} \lambda_n(x) = \lim_{n,\mathcal{U}} \sum_{k=1}^{\infty} \lambda_{nk} x_k \quad (x = (x_k)_{k \in \mathbb{N}} \in \ell_\infty)$$

and

$$\mu(A) = \lim_{n,\mathcal{U}} \sum_{k \in A} \lambda_{nk} \quad (A \subseteq \mathbb{N})$$

for the measure $\mu = \mu_f$ associated with f.

Proof. Since ℓ_1 is separable and $\|f\|_{\ell_1^{**}} = 1$, by Goldstine's theorem, according to which the unit ball of a Banach space is weak*-dense in the unit ball of the bidual, there exists a new sequence $(\lambda_n)_{n\in\mathbb{N}}$ in ℓ_1 with $\|\lambda_n\|_1 = 1$ and an ultrafilter in $\mathcal{U}$ in $\mathbb{N}$ such that

$$\lim_{n,\mathcal{U}} \lambda_n(x) = f(x)$$

for all $x \in \ell_\infty$.

Let

$$\mu^+(A) = \lim_{n,\mathcal{U}} \sum_{k\in A} \lambda_{nk}^+ \quad (\lambda^+ = \sup\{\lambda, 0\})$$

and

$$\mu^-(A) = \lim_{n,\mathcal{U}} \sum_{k\in A} \lambda_{nk}^- \quad (\lambda^- = \sup\{-\lambda, 0\})$$

for each subset A of $\mathbb{N}$. Then

$$\mu^+(\mathbb{N}) + \mu^-(\mathbb{N}) = \lim_{n,\mathcal{U}} \sum_{k=1}^{\infty} |\lambda_{nk}| = 1$$

and

$$\mu^+(\mathbb{N}) - \mu^-(\mathbb{N}) = \mu(\mathbb{N}) = f(x_0) = 1$$

where $x_0 = (1, 1, \dots)$. It follows that $\mu^-(A) = \mu^-(\mathbb{N}) = 0$ for any subset A of $\mathbb{N}$. Hence

$$\mu(A) = \mu^+(A) = \lim_{n,\mathcal{U}} \sum_{k\in A} \lambda_{nk}^+ = \lim_{n,\mathcal{U}} \sum_{k\in A} |\lambda_{nk}| \quad (A \subseteq \mathbb{N})$$

and

$$f(x) = \lim_{n,\mathcal{U}} \sum_{k\in\mathbb{N}} |\lambda_{nk}|\, x_k \quad (x = (x_k) \in \ell_\infty),$$

from which it turns out that, for any generalized limit f over ℓ_∞, there exists a sequence $(\lambda'_n)_{n\in\mathbb{N}} \subset S$ and an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that

$$f(x) = \lim_{n,\mathcal{U}} \lambda'_n(x)$$

for all $x \in \ell_\infty$. Then the result follows immediately.

It is well known that for every $x \in \ell_\infty$ there exists a real continuous function $x' \in C(\beta\mathbb{N})$ (space of continuous functions on the Cech-Stone compactification $\beta\mathbb{N}$ of $\mathbb{N}$) defined by

$$x'(u) = \lim_{n,\mathcal{U}} x_n \quad (x = (x_n) \in \ell_\infty)$$

for each ultrafilter $\mathcal{U}$ in $\mathbb{N}$ and $u = \lim_{n,\mathcal{U}} n \in \beta\mathbb{N}$. We denote by x'' the restriction $x' \restriction_{\Omega'}$, where $\Omega = \beta\mathbb{N} \setminus \mathbb{N}$. Hence,

$$x''(u) = \lim_{n,\mathcal{U}} x_n \quad (x = (x_n) \in \ell_\infty)$$

for every non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ and $u = \lim_{n,\mathcal{U}} n \in \Omega$.

The mapping $x \mapsto x'$ from ℓ_∞ into $C(\beta\mathbb{N})$ is an isometric isomorphism, as well as the mapping $x \mapsto x''$ from ℓ_∞/c_0 into $C(\Omega)$. Therefore

$$\|x''\| = \overline{\lim_n} |x_n| = \|x\|_{\ell_\infty/c_0}$$

where $x = (x_n) \in \ell_\infty$.

Definition 2. A Toeplitz sequence is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ whose terms $\lambda_n = (\lambda_{nk})_{k\in\mathbb{N}}$ lie in

$$S = \{\alpha = (\alpha_n) \in \ell_1 : \|\alpha\|_1 = 1, \alpha_n \geq 0 \text{ for all } n \in \mathbb{N}\}$$

and such that $\lim_n \lambda_{nk} = 0$ for any $k \in \mathbb{N}$. A *Toeplitz matrix* is a matrix (λ_{nk}) with real entries such that

1. $\lambda_{nk} \geq 0$ for all $n, k \in \mathbb{N}$.
2. $\sum_{k=1}^{\infty} \lambda_{nk} = 1$ for all $n \in \mathbb{N}$.
3. $\lim_n \lambda_{nk} = 0$ for all $k \in \mathbb{N}$.

Then the rows of a Toeplitz matrix form a Toeplitz sequence.

We denote by F the set of all generalized limits over ℓ_∞ and by F_0 the set of the functionals $f \in \ell_\infty^*$ for which there exists a Toeplitz sequence $(\lambda_n)_{n\in\mathbb{N}}$ and a non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that

$$f(x) = \lim_{n,\mathcal{U}} \lambda_n(x)$$

for all $x \in \ell_\infty$, i. e., $f = \lim_{n,\mathcal{U}} \lambda_n$ in ℓ_∞^* for the weak*-topology. Then $F_0 \subseteq F$ and we say that the elements of F_0 are *Toeplitz generalized limits*.

From the isometry between the spaces ℓ_∞/c_0 and $C(\Omega)$ it follows that $F \subseteq (\ell_\infty/c_0)^*$, furnished with the weak*-topology, is isomorphic to the set of probability measures on Ω endowed with the weak*-topology. Therefore, F is a weak*-compact set and the extremal points of F are the limits $\lim_{n,\mathcal{U}} x_n$ over the non-trivial ultrafilters $\mathcal{U}$ in $\mathbb{N}$, corresponding to the δ of Dirac of $C(\Omega)^*$.

Theorem 3. *For any countable subset A of F_0, the w*-closure $\overline{A}^*$, rests in F_0. Hence, F_0 is countably weak*-compact.*

Proof. Let $A = \{f_i : i \in \mathbb{N}\} \subseteq F_0$. Since $f_i \in F_0$, there exists a Toeplitz sequence $\left(\lambda_n^i\right)_{n\in\mathbb{N}}$ such that $f_i \in \overline{\Lambda}_i^*$ where $\Lambda_i = \left\{\lambda_n^i : n \in \mathbb{N}\right\}$. It is clear that we can take $\lambda_{nk}^i < \frac{1}{i}$ for $k = 1, 2, \ldots, i$ $\left(\lambda_n^i = \left(\lambda_{nk}^i\right)_{k\in\mathbb{N}}\right)$. Let $\Lambda = \bigcup_{i=1}^{\infty} \Lambda_i = \left\{\lambda_n^i : i, n \in \mathbb{N}\right\}$, then Λ is a Toeplitz sequence and $A \subseteq \overline{\Lambda}^*$. Therefore, $\overline{A}^* \subseteq \overline{\Lambda}^* \cap F_0$.

Corollary 4. *F_0 is a norm-closed set in ℓ_∞^* (or in $(\ell_\infty/c_0)^*$).*

Theorem 5. $\overline{F}_0^* = F$.

Proof. We are going to prove first that $\overline{F}_0^*$ contains the convex hull of F_0, from which it follows that $\overline{F}_0^*$ is a convex set.

Let $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ be a Toeplitz sequence, then, for each $f \in \overline{\Lambda}^* \cap F_0$ there is an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that

$$f(x) = \lim_{n,\mathcal{U}} \lambda_n(x)$$

for all $x \in \ell_\infty$. Conversely, for any non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ we have that the functional

$$x \mapsto \lim_{n,\mathcal{U}} \lambda_n(x)$$

belongs to $\overline{\Lambda}^* \cap F_0$. From this follows the existence of a continuous map $u \mapsto f(u,.)$ from $\Omega = \beta\mathbb{N} \setminus \mathbb{N}$ into ℓ_∞^* such that

$$f(u,x) = \lim_{n,\mathcal{U}} \lambda_n(x)$$

where $u = \lim_{n,\mathcal{U}} n \in \Omega$ and $x \in \ell_\infty$, in such a way that $u \mapsto f(u,.)$ is a continuous mapping from Ω into $\overline{\Lambda}^* \cap F_0$ when the weak*-topology is considered. For the Toeplitz sequence $\Lambda^i = \{\lambda_n^i : n \in \mathbb{N}\}$, let $u \mapsto f^i(u,.)$ be the corresponding mapping.

Let $u_1, u_2, \ldots, u_m$ be distinct elements of Ω and $U_1, U_2, \ldots, U_m$ be distinct open-closed neighborhoods of $u_1, u_2, \ldots, u_m$, respectively, in Ω. Then there exist homeomorphisms $\varphi_1, \varphi_2, \ldots, \varphi_m$ from $\beta\mathbb{N}$ onto $\beta\mathbb{N}$, corresponding to permutations of $\mathbb{N}$, such that $\varphi_i(U_i) = U_i$. Let $u_i' = \varphi(u_i)$ for $i = 1, 2, \ldots, m$. Then, for $\alpha_i \geq 0$ $(i = 1, 2, \ldots, m)$ and $\sum_{i=1}^m \alpha_i = 1$ we have that the mapping

$$x \mapsto \sum_{i=1}^m \alpha_i f^i(u_i', x) = \lim_{n,\mathcal{U}_1} \sum_{i=1}^m \alpha_i \lambda^i_{\varphi_i(n)}(x) \quad (x \in \ell_\infty)$$

if $u_1 = \lim_{n,\mathcal{U}_1} n \in \Omega$, i. e.,

$$\sum_{i=1}^m \alpha_i f^i(u_i', .) \in \overline{F}_0^*.$$

Since this holds for any system of open-closed neighborhoods distinct from Ω, namely $U_1, U_2, \ldots, U_m$ and $u_i' \in U_i$, and the maps f^i are continuous, we have that

$$\sum_{i=1}^m \alpha_i f^i(u_i, .) \in \overline{F}_0^*,$$

from which it follows immediately that $\overline{F}_0^*$ contains the convex hull of F_0.

To conclude the proof we just have to apply the Krein-Milman theorem taking into consideration that F_0 contains the extremal points of F, since they are the limits $\lim_{n,\mathcal{U}} x_n$ over the non-trivial ultrafilters $\mathcal{U}$ in $\mathbb{N}$, and F is a convex weak*-compact set.

Proposition 6. *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive real numbers with $\sum_{i=1}^m \alpha_i = 1$ $(\alpha_i > 0)$ and let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ be non-trivial ultrafilters in $\mathbb{N}$ for which there exists a Toeplitz sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ such that*

$$\left(x \to \sum_{i=1}^m \alpha_i \lim_{n,\mathcal{U}_i} x_n\right) \in \overline{\Lambda}^* \quad (x = (x_n) \in \ell_\infty),$$

the for every sequence of real numbers (α_i') satisfying $\alpha_i' \geq 0$ and $\sum_{i=1}^m \alpha_i' = 1$ there exists a Toeplitz sequence $\Lambda' = (\lambda_n')_{n\in\mathbb{N}}$ such that

$$\left(x \to \sum_{i=1}^m \alpha_i' \lim_{n,\mathcal{U}_i} x_n\right) \in \overline{\Lambda'}^* \quad (x = (x_n) \in \ell_\infty).$$

Proof. We may assume that the ultrafilters $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m$ are different and, therefore, there exist disjoint subsets $M_1, M_2, \ldots, M_m$ of $\mathbb{N}$ such that $M_i \in \mathcal{U}_i$ for $i = 1, 2, \ldots, m$. Since

$$\left(x \to \sum_{i=1}^m \alpha_i \lim_{n,\mathcal{U}_i} x_n\right) \in \overline{\Lambda}^*$$

there exists an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that

$$\sum_{i=1}^m \alpha_i \lim_{n,\mathcal{U}_i} x_n = \lim_{n,\mathcal{U}} \lambda_n(x)$$

for every $x = (x_n) \in \ell_\infty$. Then

$$\alpha_i \lim_{n,\mathcal{U}_i} x_n = \lim_{n,\mathcal{U}} \lambda_n (x\chi_{M_i})$$

for $i = 1, 2, \ldots, m$, and hence

$$\sum_{i=1}^m \alpha_i' \lim_{n,\mathcal{U}_i} x_n = \lim_{n,\mathcal{U}} \sum_{i=1}^m \frac{\alpha_i'}{\alpha_i} \lambda_n (x\chi_{M_i}) = \lim_{n,\mathcal{U}} \sum_{i=1}^m \alpha_i' \frac{\lambda_n (x\chi_{M_i})}{\lambda_n (\chi_{M_i})}$$

for all $x = (x_n) \in \ell_\infty$, which proves our claim since $\left(\lambda_n (\cdot\chi_{M_i}) / \lambda_n (\chi_{M_i})\right)_{n\in\mathbb{N}}$ $(i = 1, 2, \ldots, m)$ are essentially Toeplitz sequences. Indeed, if

$$M_i' = \left\{n \in \mathbb{N} : \lambda_n(\chi_{M_i}) > \frac{\alpha_i}{2}\right\} \quad (\in \mathcal{U})$$

and we put

$$\lambda_n^i(x) = \frac{\lambda_n(x\chi_{M_i})}{\lambda_n(\chi_{M_i})} \quad \text{for } n \in M_i', \qquad \lambda_n^i(x) = \lambda_n(x) \quad \text{for } n \notin M_i',$$

then $(\lambda_n^i)_{i,n\in\mathbb{N}}$ is a Toeplitz sequence which, equivalently, can replace

$$\left(\frac{\lambda_n(\cdot\chi_{M_i})}{\lambda_n(\chi_{M_i})}\right)_{i,n\in\mathbb{N}}.$$

Given two non-trivial ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{N}$, we will write $\mathcal{U}_1 \simeq \mathcal{U}_2$, meaning that there is a Toeplitz sequence $\Lambda_\alpha = (\lambda_n)_{n\in\mathbb{N}}$ such that

$$\alpha \lim_{n,\mathcal{U}_1} x_n + (1-\alpha)\lim_{n,\mathcal{U}_2} x_n = \lim_{n,\mathcal{U}} \lambda_n(x)$$

for every $x = (x_n) \in \ell_\infty$ and some non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ and some $0 < \alpha < 1$, that is,

$$\left(x \to \alpha \lim_{n,\mathcal{U}_1} x_n + (1-\alpha)\lim_{n,\mathcal{U}_2} x_n\right) \in \overline{\Lambda}_\alpha^*.$$

Then from Proposition 6 it follows that the same holds for every $0 < \alpha < 1$.

Given two non-trivial ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{N}$, we will write $\mathcal{U}_1 \le \mathcal{U}_2$ meaning that there exists a Toeplitz sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,\mathcal{U}_1} x_n = \lim_{n,\mathcal{U}_2} \lambda_n(x)$$

for every $x = (x_n) \in \ell_\infty$. Since the product of two Toeplitz matrices is a Toeplitz matrix, $\mathcal{U}_1 \le \mathcal{U}_2 \le \mathcal{U}_3$ implies $\mathcal{U}_1 \le \mathcal{U}_3$.

Let φ be a permutation of $\mathbb{N}$ and $\mathcal{U}$ be a non-trivial ultrafilter in $\mathbb{N}$. Then $\varphi(\mathcal{U})$ denote the ultrafilter consisting of all the sets $\varphi(M)$ with $M \in \mathcal{U}$. It is easily seen that $\mathcal{U} \le \varphi(\mathcal{U}) \le \mathcal{U}$ for every permutation φ of $\mathbb{N}$ and every ultrafilter $\mathcal{U}$.

Proposition 7. *$\mathcal{U}_1 \simeq \mathcal{U}_2$ if and only if there exists an ultrafilter $\mathcal{U}$ such that $\mathcal{U}_1 \le \mathcal{U}$ and $\mathcal{U}_2 \le \mathcal{U}$.*

Proof. Assume that $\mathcal{U}_1 \simeq \mathcal{U}_2$, rejecting the trivial case $\mathcal{U}_1 = \mathcal{U}_2$. From the proof of Proposition 6 it follows that there exists an ultrafilter $\mathcal{U}$ such that $\mathcal{U}_1 \le \mathcal{U}$ and $\mathcal{U}_2 \le \mathcal{U}$.

Now suppose $\mathcal{U}_1 \le \mathcal{U}$ and $\mathcal{U}_2 \le \mathcal{U}$. Then there exist two Toeplitz sequences $(\lambda_n')_{n\in\mathbb{N}}$ and $(\lambda_n'')_{n\in\mathbb{N}}$ such that

$$\lim_{n,\mathcal{U}_1} x_n = \lim_{n,\mathcal{U}} \lambda_n'(x)$$

and

$$\lim_{n,\mathcal{U}_2} x_n = \lim_{n,\mathcal{U}} \lambda_n''(x)$$

for every $x = (x_n) \in \ell_\infty$. Then, since

$$\alpha \lim_{n,\mathcal{U}_1} x_n + (1-\alpha)\lim_{n,\mathcal{U}_2} x_n = \lim_{n,\mathcal{U}}[\alpha\lambda_n' + (1-\alpha)\lambda_n''](x)$$

for every $x = (x_n) \in \ell_\infty$ and $0 < \alpha < 1$, and $(\alpha\lambda'_n + (1-\alpha)\lambda''_n)_{n\in\mathbb{N}}$ is a Toeplitz sequence, it follows that $\mathcal{U}_1 \simeq \mathcal{U}_2$.

Theorem 8. *F_0 is a convex set if and only if $\mathcal{U}_1 \simeq \mathcal{U}_2$ for every pair of non-trivial ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{N}$.*

Proof. Of course, if F_0 is a convex set, $\mathcal{U}_1 \simeq \mathcal{U}_2$ for every pair of ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{N}$.

Suppose $\mathcal{U}_1 \simeq \mathcal{U}_2$ for every pair of ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbb{N}$. Let $f_1, f_2 \in F_0$, then there exist two Toeplitz sequences $(\lambda'_n)_{n\in\mathbb{N}}$ and $(\lambda''_n)_{n\in\mathbb{N}}$ such that

$$f_1(x) = \lim_{n,\mathcal{U}_1} \lambda'_n(x)$$

and

$$f_2(x) = \lim_{n,\mathcal{U}_2} \lambda''_n(x)$$

for every $x \in \ell_\infty$ and some ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$. Then, since $\mathcal{U}_1 \simeq \mathcal{U}_2$, there exist two Toeplitz sequences $(\mu'_n)_{n\in\mathbb{N}}$ and $(\mu''_n)_{n\in\mathbb{N}}$, and an ultrafilter $\mathcal{U}$ such that

$$\lim_{n,\mathcal{U}_1} x_n = \lim_{n,\mathcal{U}} \mu'_n(x), \quad \lim_{n,\mathcal{U}_2} x_n = \lim_{n,\mathcal{U}} \mu''_n(x)$$

for every $x = (x_n) \in \ell_\infty$. Now, since the product of the Toeplitz matrices corresponding to the pairs $((\mu'_n)_{n\in\mathbb{N}}, (\lambda'_n)_{n\in\mathbb{N}})$ and $((\mu''_n)_{n\in\mathbb{N}}, (\lambda''_n)_{n\in\mathbb{N}})$ is a Toeplitz matrix, we may assume $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$. Therefore,

$$\alpha f_1(x) + (1-\alpha) f_2(x) = \lim_{n,\mathcal{U}} \left[\alpha\lambda'_n + (1-\alpha)\lambda''_n\right](x)$$

for every $x \in \ell_\infty$ and $0 < \alpha < 1$. This proves that $\alpha f_1 + (1-\alpha) f_2 \in F_0$ for every $0 < \alpha < 1$, and hence, that F_0 is a convex (under the above hypothesis).

Problem. *Is F_0 a convex set?*

If F_0 is not a convex set, then $F_0 \neq F$ and F_0 is not weakly* compact. Therefore, for every Toeplitz sequence Λ there exists a generalized Toeplitz limit $f \notin \overline{\Lambda}^*$ ($f \in F_0$).

Proposition 9. *For every non-trivial ultrafilter $\mathcal{U}_0$ in $\mathbb{N}$, the cardinal number of the set*

$$P(\mathcal{U}_0) = \{\mathcal{U} : \mathcal{U} \leq \mathcal{U}_0\}$$

is c, and the cardinal number of the set

$$Q(\mathcal{U}_0) = \{\mathcal{U} : \mathcal{U} \geq \mathcal{U}_0\}$$

is 2^c. If A is a set of ultrafilters in $\mathbb{N}$ with cardinal number less than 2^c, then there exists an ultrafilter $\mathcal{U}_1$ such that $A \cap Q(\mathcal{U}_1) = \varnothing$.

Proof. Indeed, the cardinal number of $P(\mathcal{U}_0)$ is less than or equal to the cardinal number of the set of Toeplitz sequences, and therefore card $P(\mathcal{U}_0) \leq c$. On the other hand, since $P(\mathcal{U}_0)$ contains all the ultrafilters $\varphi(\mathcal{U}_0)$ corresponding to the permutations φ of $\mathbb{N}$, we have card $P(\mathcal{U}_0) \geq c$. In the same way, card $Q(\mathcal{U}_0) \geq c$.

Now we are going to see that card $Q(\mathcal{U}_0) = 2^c$. Let $\lambda_n(x) = x_k$ for

$$\binom{k}{2} < n \leq \binom{k+1}{2} \quad \left(\binom{k}{2} = \frac{k(k-1)}{2}\right).$$

Then $(\lambda_n)_{n\in\mathbb{N}}$ is a Toeplitz sequence. For each ultrafilter $\mathcal{U}_0$ let us put

$$M_i = \left\{n = \binom{k}{2} + i : k \in M\right\}$$

for every $M \in \mathcal{U}_0$. Then the sets M_i define ultrafilters $\mathcal{U}_i$. Since

$$M_i \cap M_j \cap \left\{n \in \mathbb{N} : n > \binom{k}{2}\right\} = \varnothing$$

for $1 \leq i < j \leq k$, it follows that the ultrafilters $\mathcal{U}_i$ are different. As in Theorem 5, let us consider the continuous map $u \mapsto f(u,.)$ from $\Omega = \beta\mathbb{N} \setminus \mathbb{N}$ into ℓ_∞^* such that

$$f(u,x) = \lim_{n,\mathcal{U}} \lambda_n(x)$$

for every $x \in \ell_\infty$ and $u = \lim_{n,\mathcal{U}} n \in \Omega = \beta\mathbb{N} \setminus \mathbb{N}$, so that we can identify u with $\mathcal{U}$. Then

$$\left\{\mathcal{U} = u \in \Omega : f(u,x) = \lim_{n,\mathcal{U}_0} x_n \text{ for all } x = (x_n) \in \ell_\infty\right\}$$

is an infinite closed set (because it contains the ultrafilters $\mathcal{U}_i$). From this it follows (according to [E-S, pag. 132]) that the cardinal number of this set is 2^c, and hence that $Q(\mathcal{U}_0) = 2^c$ (since card $\Omega = 2^c$ according to [E-S, pag. 132]). Finally, if card $A < 2^c$, then

$$\operatorname{card} \bigcup\{P(\mathcal{U}) : \mathcal{U} \in A\} \leq c \cdot \operatorname{card} A < 2^c \quad (= \operatorname{card}\Omega)$$

and therefore there exists a non-trivial ultrafilter $\mathcal{U}_1 \notin \bigcup\{P(\mathcal{U}) : \mathcal{U} \in A\}$, which implies $A \cap Q(\mathcal{U}_1) = \varnothing$.

Proposition 10. *Let us assume that F_0 is a convex set. Then, if $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is a sequence of non-trivial ultrafilters in $\mathbb{N}$, there exists an ultrafilter $\mathcal{U}$ such that $\mathcal{U}_n \leq \mathcal{U}$ for every $n \in \mathbb{N}$.*

Proof. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of disjoint infinite subsets of $\mathbb{N}$. Then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of permutations of $\mathbb{N}$ such that $M_n \in \varphi_n(\mathcal{U}_n)$ for every $n \in \mathbb{N}$. If F_0 is a convex set, Corollary 4 implies that

$$\left(x \to \sum_{n=1}^{\infty} 2^{-n} \lim_{k,\varphi_n(\mathcal{U}_n)} x_k\right) \in F_0 \quad (x = (x_n)_{n\in\mathbb{N}}).$$

Therefore, there exists a Toeplitz sequence $(\lambda_k)_{k\in\mathbb{N}}$ and an ultrafilter $\mathcal{U}$ in $\mathbb{N}$ such that

$$\sum_{n=1}^{\infty} 2^{-n} \lim_{k,\varphi_n(\mathcal{U}_n)} x_k = \lim_{k,\mathcal{U}} \lambda_k(x)$$

for every $x = (x_k) \in \ell_\infty$, and therefore

$$2^{-n} \lim_{k,\varphi_n(\mathcal{U}_n)} x_k = \lim_{k,\mathcal{U}} \lambda_k(x\chi_{M_n})$$

and

$$\lim_{k,\varphi_n(\mathcal{U}_n)} = \lim_{k,\mathcal{U}} 2^n \lambda_k(x\chi_{M_n}) = \lim_{k,\mathcal{U}} \frac{\lambda_k(x\chi_{M_n})}{\lambda_k(\chi_{M_n})}$$

for every $x = (x_n) \in \ell_\infty$. Since $\big(\lambda_k(.\chi_{M_n})/\lambda_k(\chi_{M_n})\big)_{n\in\mathbb{N}}$ is essentially a Toeplitz sequence for all $n \in \mathbb{N}$ (as in Proposition 6), it follows that $\mathcal{U}_n \leq \varphi_n(\mathcal{U}_n) \leq \mathcal{U}$ and $\mathcal{U}_n \leq \mathcal{U}$ for every $n \in \mathbb{N}$.

Theorem 11. *For any Toeplitz sequence* $(\lambda_k)_{n\in\mathbb{N}}$, *there exists some* $x \in \ell_\infty$ *for which* $\lim_n \lambda_n(x)$ *does not exist. That is to say, there exist two non-trivial ultrafilters* $\mathcal{U}_1, \mathcal{U}_2$ *in* $\mathbb{N}$ *and an element* $x \in \ell_\infty$ *such that*

$$\lim_{n,\mathcal{U}_1} \lambda_n(x) \neq \lim_{n,\mathcal{U}_2} \lambda_n(x).$$

Proof. Indeed, if the ordinary limit

$$\lim_n \lambda_n(\chi_A) = \mu(A)$$

existed for every set $A \subseteq \mathbb{N}$, then, according to the Hahn–Vitali–Saks–Nikodym theorem [D-U, I. 4.8], μ would turn out to be countably additive (as a limit of the countably additive measures

$$\mu_n : \mu_n(A) = \lambda_n(\chi_A) \quad (A \subseteq \mathbb{N})),$$

which would lead us to a contradiction: $1 - \mu(\mathbb{N}) = \sum_{n=1}^{\infty} \mu(\{n\}) = 0$. The result also follows from Schur property of ℓ_1.

Definition 12. Given a non-trivial ultrafilter $\mathcal{U}_0$ in $\mathbb{N}$ and a Toeplitz sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ we define the primitive set relative to $\mathcal{U}_0$ as the set $P = P(\mathcal{U}_0, \Lambda)$ consisting of all the ultrafilters $\mathcal{U}$ such that

$$\lim_{n,\mathcal{U}_0} x_n = \lim_{n,\mathcal{U}} \lambda_n(x)$$

for all $x = (x_n) \in \ell_\infty$.

Given a second Toeplitz sequence $\Lambda' = (\lambda'_n)_{n\in\mathbb{N}}$ we define the *reflexive set* relative to $\mathcal{U}_0$ as the set $R = R(\mathcal{U}_0, \Lambda, \Lambda')$ consisting of the ultrafilters $\mathcal{U}'$ such that

$$\lim_{n,\mathcal{U}'} x_n = \lim_{n,\mathcal{U}} \lambda'_n(x)$$

for every $x = (x_n) \in \ell_\infty$ and some $\mathcal{U} \in P(\mathcal{U}_0, \Lambda)$.

Proposition 13. *Identifying, as in the proof of Proposition 9, every non-trivial ultrafilter $\mathcal{U}$ in $\mathbb{N}$ with $u = \lim_{n,\mathcal{U}} n \in \Omega = \beta\mathbb{N} \setminus \mathbb{N}$, we have that every primitive set $P = P(\mathcal{U}_0, \Lambda)$ is closed and has empty interior, and every refelexive set $R = R(\mathcal{U}_0, \Lambda, \Lambda')$ is closed.*

Proof. Let $u \mapsto f(x, .)$ be a continuous map from $\beta\mathbb{N}$ into ℓ_∞^* such that $f(n,x) = \lambda(x)$. Then

$$\left\{u \in \Omega : f(u,x) = \lim_{n,\mathcal{U}_0} x_n\right\}$$

is a closed set and, hence, so is P.

On the other hand, if P had an interior point, there would exist an open-closed U in Ω such that

$$\lim_{n,\mathcal{U}_0} x_n = \lim_{n,\mathcal{U}} \lambda_n(x) \quad (x = (x_n) \in \ell_\infty)$$

for every $\mathcal{U} \in U$. Let M be a subset of $\mathbb{N}$ such that $\overline{M} \setminus M = U$, then that equality would hold for every non-trivial ultrafilter $\mathcal{U} \ni M$ and the limit of the sequence $(\lambda_n(x))_{n\in M}$ should exist for every $x \in \ell_\infty$, which contradicts Theorem 11.

Since $P = P(\mathcal{U}_0, \Lambda)$ is a compact set, the set of all the generalized limits f on ℓ_∞ such that

$$f(x) = \lim_{n,\mathcal{U}} \lambda'_n(x)$$

for some $\mathcal{U} \in P$ is a weakly* closed set and, hence, the set $R = R(\mathcal{U}_0, \Lambda, \Lambda')$ is closed as well.

Now, using the Continuum Hypothesis (CH) we are going to prove the following theorem.

Theorem 14. *Given a non-trivial ultrafilter $\mathcal{U}_0$ in $\mathbb{N}$, $\mathcal{U}_0 \simeq \mathcal{U}_1$ holds for every non-trivial ultrafilter $\mathcal{U}_1$ in $\mathbb{N}$ if and only if some reflexive set $R = R(\mathcal{U}_0, \Lambda, \Lambda')$ has an interior point.*

Proof. Suppose that $R = R(\mathcal{U}_0, \Lambda, \Lambda')$ has an interior point. Then there exists an open-closed non-empty set $U \subseteq R$. Let M be an infinite subset of $\mathbb{N}$ such that $\overline{M} \setminus M = U$ and let φ be a bijection from $\mathbb{N}$ onto M. If $\Lambda'' = (\lambda''_n)_{n\in\mathbb{N}}$ is a Toeplitz sequence such that $\lambda''_{\varphi(n),n} = 1$ and $\lambda''_{\varphi(n),k} = 0$ for every $k \neq n$, we have that for each $\mathcal{U}_1 \in \Omega$ there exists $\mathcal{U}' \in R$ such that

$$\lim_{n,\mathcal{U}_1} x_n = \lim_{n,\mathcal{U}_1} \lambda''_{\varphi(n)}(x) = \lim_{n,\mathcal{U}'} \lambda''_n(x)$$

for every $x = (x_n) \in \ell_\infty$. Therefore,

$$\lim_{n,\mathcal{U}_1} x_n = \lim_{n,\mathcal{U}} \sum_{i=1}^{\infty} \lambda'_{ni}\lambda''_{ik}x_k = \lim_{n,\mathcal{U}} \lambda''_n(x)$$

for every $x = (x_n) \in \ell_\infty$ if we choose $\mathcal{U} \in P$ so that

$$\lim_{n,\mathcal{U}'} x_n = \lim_{n,\mathcal{U}} \lambda'_n(x)$$

Then, by Proposition 7, we have $\mathcal{U}_0 \simeq \mathcal{U}_1$ for every $\mathcal{U}_1 \in \Omega$.

Now, assume that every reflexive set $R = R(\mathcal{U}_0, \Lambda, \Lambda')$ lacks interior points. It will suffice to prove that $\bigcup R(\mathcal{U}_0, \Lambda, \Lambda')$ does not have interior points either. With this aim, it is enough to prove that the union $\bigcup_{\alpha\in A} R_\alpha$ of non-dense closed sets does not contain a non-empty open-closed set $\mathcal{U}_0$ if card $A \leq c$. In order to do this, we proceed as in Baire's category theorem, using (CH) and Cantor's separability property: for every decreasing sequence of non-empty open-closed sets there exists a non-empty open-closed set which is contained in all sets of the sequence. Indeed, if A is the set of all ordinal numbers less than $c = \omega_1$, supposing that a decreasing family $(U_\alpha)_{\alpha<\alpha_0}$ of non-empty open-closed sets satisfying $R_\alpha \cap U_\alpha = \varnothing$ for every $\alpha < \alpha_0$ has been constructed and U is a non-empty open-closed set contained in $\bigcap_{\alpha<\alpha_0} U_\alpha$, since $R_{\alpha_0} \not\supseteq U$ we deduce that there exists a non-empty open-closed set $U_{\alpha_0} \subseteq U$ such that $R_{\alpha_0} \cap U_{\alpha_0} = \varnothing$. To conclude the proof it is enough to note that every $x \in \bigcap_{\alpha<c} U_\alpha$ does not belong to $\bigcup_{\alpha<c} R_\alpha$ and belongs to U_0 if we take $U_1 \subseteq U_0$.

Corollary 15 (CH). *Given a non-trivial ultrafilter $\mathcal{U}_0$ in $\mathbb{N}$, we have $\mathcal{U}_0 \simeq \mathcal{U}_1$ for every non-trivial ultrafilter $\mathcal{U}_1$ in $\mathbb{N}$ if and only if some reflexive set $R = R(\mathcal{U}_0, \Lambda, \Lambda') = \Omega$ $(= \beta\mathbb{N} \setminus \mathbb{N})$.*

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