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Harmonizable locally spatially isotropic random fields

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ABSTRACT. In this paper, the local behavior of harmonizable spatially isotropic random fields is considered. Spectral representations are obtained for generalized and ordinary harmonizable spatially isotropic fields with spatially isotropic increments of order M. The representation of a field with spatially isotropic increments of order M is also presented.

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1. Introduction

In recent years the study of harmonizable processes has played a central role in the development of the theory of non-stationary processes. Crucial to this development is the pioneering work of Chang and Rao [2] on bimeasures and Morse-Transue integration. Their paper set the stage for the recent advances in the theory. A recent account of the development of harmonizable processes and some of their applications may be found in Swift [14]. That paper also contains a detailed bibliography of the existing work on harmonizable processes.

The corresponding theory for harmonizable fields and their applications is being developed by R. J. Swift in a series of papers [9]–[11], [13]–[15]. In this paper, spectral representations for local classes of fields is developed.

After the necessary background on harmonizable fields is presented, a brief account of the development of local fields is given. The notion of a generalized random field is central to this material and the required theory is detailed. The paper concludes with spectral representations of several classes of local fields.

2. Background

In the following, the probability space, (Ω, Σ, P), is always present.

The desired class of random functions is obtained by considering a random field $X : \mathbb{R}^n \to L_0^2(P)$ that is stationary. Recall that such a field can be expressed as:

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^n} e^{i\boldsymbol{\lambda}\cdot\boldsymbol{t}} dZ(\boldsymbol{\lambda}), \tag{1}$$

where $Z(\cdot)$ is a σ-additive stochastic measure on the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}^n$, with orthogonal values in the complex Hilbert space, $L_0^2(P)$, of centered random variables. The covariance, $r(\cdot,\cdot)$, of the field is

$$r(\boldsymbol{s},\boldsymbol{t}) = \int_{\mathbb{R}^n} e^{i(\boldsymbol{s}-\boldsymbol{t})\cdot\boldsymbol{\lambda}} dF(\boldsymbol{\lambda}),$$

where $E(Z(A)\overline{Z(B)}) = F(A \cap B)$, F a positive finite Borel measure on $\mathbb{R}^n$. Here $E(\cdot)$ denotes the expectation.

A generalization of the concept of stationarity is given by fields $X : \mathbb{R}^n \to L_0^2(P)$ with covariance $r(\cdot,\cdot)$ expressible as

$$r(\boldsymbol{s},\boldsymbol{t}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\boldsymbol{\lambda}\cdot\boldsymbol{s} - i\boldsymbol{\lambda}'\cdot\boldsymbol{t}} dF(\boldsymbol{\lambda},\boldsymbol{\lambda}'),$$

where $F(\cdot,\cdot)$ is a complex bimeasure, called the *spectral bimeasure* of the field, of bounded variation in the Vitali's sense or more inclusively in Fréchet's sense; in which case the integrals are strict Morse-Transue (cf. Rao, [6] and Chang and Rao [2]). The covariance as well as the field are termed *strongly* or *weakly harmonizable* respectively. Every weakly or strongly harmonizable field $X : \mathbb{R}^n \to L^2(P)$ has an integral representation given by (1), where $Z : \mathcal{B} \to L^2(P)$ is a stochastic measure (not necessarily with orthogonal values) and is called the *spectral measure* of the field. Both of these concepts reduce to the stationary case if F concentrates on the diagonal $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ of $\mathbb{R}^n \times \mathbb{R}^n$.

A general class of non-stationary processes which extends the ideas of the harmonizable class was first considered by Cramér in 1952.

Definition 2.1. A second-order random field $X : \mathbb{R}^n \to L^2(P)$ is of Cramér class (or class (C)) if its covariance function $r(\cdot, \cdot)$ is representable as

$$r(\boldsymbol{t}_1, \boldsymbol{t}_2) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(\boldsymbol{t}_1, \boldsymbol{\lambda}) \overline{g(\boldsymbol{t}_2, \boldsymbol{\lambda}')} \, dF(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \tag{2}$$

relative to a family $\{g(\boldsymbol{t}, \cdot), \boldsymbol{t} \in \mathbb{R}^n\}$ of Borel functions and a positive definite function $F(\cdot, \cdot)$ of locally bounded variation on $\mathbb{R}^n \times \mathbb{R}^n$, with each g satisfying the (Lebesgue) integrability condition:

$$0 \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(\boldsymbol{t}_1, \boldsymbol{\lambda}) \overline{g(\boldsymbol{t}_2, \boldsymbol{\lambda}')} \, dF(\boldsymbol{\lambda}, \boldsymbol{\lambda}') < \infty, \quad \boldsymbol{t} \in \mathbb{R}^n.$$

If $F(\cdot, \cdot)$ has a locally finite Fréchet variation, then the integrals in equation (2) are in the sense of (strict) Morse-Transue and the corresponding concept is termed *weak class (C)*.

3. Local classes of fields

In the modern statistical theory of turbulence, random fields with certain local properties are often considered. A useful addition to this theory is given by considering a random field $X(\boldsymbol{t})$ which is not necessarily of class (C), but whose increment field

$$I_{\boldsymbol{\tau}} X(\boldsymbol{t}) = X(\boldsymbol{t} + \boldsymbol{\tau}) - X(\boldsymbol{t})$$

is of class (C). Rao [7], obtained the spectral representations for these locally class (C) random fields. Rao showed that the representations are obtained by considering generalized (in the sense of Gel'fand and Vilenkin, [4]) random fields, since they provide the required differentiability structure. The notion of a generalized field will now be given for completeness.

Consider the space $\mathcal{K}$ of infinitely differentiable functions $h(\boldsymbol{t})$ having compact supports, which with compact convergence becomes a locally convex linear topological space. A generalized random field $\widetilde{X}$ is a linear functional $\widetilde{X} : \mathcal{K} \to \mathbb{C}$ such that if $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{K}, \phi_n \to 0$ in the topology of $\mathcal{K}$, then $\widetilde{X}(\phi_n) \to 0$ in probability, as $n \to \infty$.

The mean of a generalized field is the linear functional

$$m(h) = E(\widetilde{X}(h)), \quad h \in \mathcal{K}$$

and similarly its covariance is the bilinear (conjugate linear in the complex case) functional

$$r(h_1, h_2) = E(\widetilde{X}(h_1) \overline{\widetilde{X}(h_2)}), \quad h_i \in \mathcal{K}, \quad i = 1, 2.$$

Ordinary fields generate the corresponding generalized fields by the relation

$$\widetilde{X}(h) = \int_{\mathbb{R}^n} X(\boldsymbol{t})h(\boldsymbol{t})d\boldsymbol{t} \quad \text{for } h \in \mathcal{K},$$

The converse is not true unless an additional condition is assumed. That is, if a generalized field $\widetilde{X}(\cdot)$ has point values (also called "of function space type") then the reverse implication holds.

Using this, and results from the theory of generalized functions, one defines the derivative $\widetilde{X}^{(m_1,\dots,m_n)}(h)$ of a generalized field $\widetilde{X}(h)$ as

$$\widetilde{X}^{(m_1,\dots,m_n)}(h) = (-1)^M \widetilde{X}(h^{(m_1,\dots,m_n)}), \quad M = m_1 + \ldots + m_n.$$

Using these ideas, one can define the class of generalized class (C) fields as follows.

Definition 3.1. A generalized random field $\widetilde{X} : \mathcal{K} \to \mathbb{C}$ with zero mean and covariance functional $r(\cdot,\cdot)$ is of weak class (C) if it can be expressed as

$$r(h_1,h_2) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{h_1}(\boldsymbol{\lambda})\overline{\widehat{h_2}(\boldsymbol{\lambda}')}dF(\boldsymbol{\lambda},\boldsymbol{\lambda}') \tag{3}$$

where $F(\cdot,\cdot)$ is a function of locally bounded Fréchet variation satisfying where

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|dF(\boldsymbol{\lambda},\boldsymbol{\lambda}')|}{((1+||\boldsymbol{\lambda}||^2)\ (1+||\boldsymbol{\lambda}'||^2))^{\frac{p}{2}}} < \infty \tag{4}$$

where $p > 0$, $||\cdot||$ is the Euclidean length. Further the integrals relative to F are in the strict Morse-Transue sense and $\widehat{h_i}$ are the g-transforms of h_i, $i = 1,2$

$$\widehat{h_i}(\boldsymbol{\lambda}) = \int_{\mathbb{R}^n} h_i(\boldsymbol{t})g(\boldsymbol{t},\boldsymbol{\lambda})d\boldsymbol{t}. \tag{5}$$

This Definition was given by Rao [7] for class (C) fields and extended to weak class (C) by Swift [14]. Spectral bimeasures $F(\cdot,\cdot)$ which satisfy equation (4) are known as *tempered.*

It may be shown that such an $\widetilde{X}(\cdot)$ admits a representation

$$\widetilde{X}(h) = \int_{\mathbb{R}^n} \widehat{h}(\boldsymbol{\lambda})dZ(\boldsymbol{\lambda})$$

where $Z : \mathcal{B} \to L^2(P)$ is a vector measure such that

$$E(Z(A)\overline{Z(B)}) = \int_A\int_B dF(\boldsymbol{\lambda},\boldsymbol{\lambda}').$$

If in the representation (3) $g(\boldsymbol{t},\boldsymbol{\lambda}) = e^{i\boldsymbol{\lambda}\cdot\boldsymbol{t}}$, then the generalized random field $\widetilde{X}(\cdot)$ will be a weakly harmonizable random field. That is, the covariance has representation

$$r(h_1, h_2) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{h_1}(\boldsymbol{\lambda}) \overline{\widehat{h_2}(\boldsymbol{\lambda}')} dF(\boldsymbol{\lambda}, \boldsymbol{\lambda}'), \tag{6}$$

where $F(\cdot, \cdot)$ is a positive definite function which satisfies equation (4). Further, one notes that the integrals relative to F are in the strict Morse-Transue sense.

The theory of generalized fields will be used throughout the remaining sections of this paper.

A useful extension of the classes thus far considered was considered by Swift [14] and is given by the class of fields $X(\cdot)$ for which the increments of order M are of class (C). More specifically, if $\widetilde{X}(h)$, is an arbitrary generalized random field, then it is a random field with class (C) increments of order M if its generalized partial derivatives $\widetilde{X}^{(m_1, m_2, \ldots, m_n)}(h)$, where $m_1 + m_2 + \ldots + m_n = M$ are of class (C).

Swift showed that for a field to have class (C) increments, g must satisfy further conditions. Specifically for the case of class (C) increments of order M, it is required that g satisfies $g(\boldsymbol{t}, \boldsymbol{\lambda} - \boldsymbol{\lambda}') = g(\boldsymbol{t}, \boldsymbol{\lambda}) \overline{\beta(\boldsymbol{t}, \boldsymbol{\lambda}')}$ for all $\boldsymbol{t}, \boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathbb{R}^n$ where

$$\frac{\partial^M g(\boldsymbol{t}, 0)}{\partial \lambda_1^{m_1} \partial \lambda_2^{m_2} \ldots \partial \lambda_n^{m_n}} \neq 0 \quad \text{for } t_k \neq 0 \tag{7}$$

with

$$\frac{\partial^M \beta(\boldsymbol{t}, \boldsymbol{\lambda})}{\partial^M t_k} = \alpha_k(\boldsymbol{t}, \boldsymbol{\lambda}) \lambda_k, \tag{8}$$

and $\alpha_k(0, \boldsymbol{\lambda}) = \alpha_k \neq 0$. One notes that these restrictions are satisfied if $g(\boldsymbol{t}, \boldsymbol{\lambda}) = e^{i \boldsymbol{\lambda} \cdot \boldsymbol{t}}$.

Using this, Swift obtained the following representation of generalized random fields with class (C) increments of order M.

Theorem 3.1. *A generalized random field $\widetilde{X}(h)$ with Mth order class (C) increments has spectral representation:*

$$\widetilde{X}(h) = \int_{\mathbb{R}^n - \{0\}} \widehat{h}(\boldsymbol{\lambda}) dZ_Y(\boldsymbol{\lambda}) + (\alpha, \widehat{\nabla}_M \widehat{h}(0))$$

where $Z_Y(\cdot)$ is the spectral measure associated with its class(C) Mth order partial derivative field $Y(\cdot) = \widetilde{X}^{(m_1, m_2, \ldots, m_n)}(h)(\cdot)$ and $\widehat{h}$ is the g-transform (5) of h with g satisfying $g(\boldsymbol{t}, \boldsymbol{\lambda} - \boldsymbol{\lambda}') = g(\boldsymbol{t}, \boldsymbol{\lambda}) \overline{\beta(\boldsymbol{t}, \boldsymbol{\lambda}')}$ for all $\boldsymbol{t}, \boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathbb{R}^n$ and (7) and (8). More specifically,

$$Z_Y : \mathcal{B}(\mathbb{R}^n - \{0\}) \to L_0^2(P)$$

is a measure such that

$$F_Y(A,B) = E(Z_Y(A)\overline{Z_Y(B)}),$$

which is of finite Vitali variation, where $\mathcal{B}(\mathbb{R}^n - \{0\})$ *is the Borel* σ*-algebra of* $\mathbb{R}^n - \{0\}$. *Further,* $(\cdot,\cdot)$ *is the inner product and the* M*th order gradient is defined as:*

$$\widehat{\nabla}_M = (-1)^M \alpha \cdot (\partial^{m_1}/\partial\lambda_1^{m_1}, \partial^{m_2}/\partial\lambda_2^{m_2}, \ldots, \partial^{m_n}/\partial\lambda_n^{m_n}).$$

The covariance functional of $\widetilde{X}(\cdot)$ *is given by*

$$r(h_1,h_2) = \int_{\mathbb{R}^n-\{0\}}\int_{\mathbb{R}^n-\{0\}} \widehat{h}_1(\boldsymbol{\lambda})\overline{\widehat{h}_2(\boldsymbol{\lambda}')}dF(\boldsymbol{\lambda},\boldsymbol{\lambda}') + (A\,\widehat{\nabla}_M\widehat{h}_1(0),\ \widehat{\nabla}_M\widehat{h}_2(0)) \tag{9}$$

with A *a positive definite matrix.*

As noted before, the conditions upon g are satisfied when $g(\boldsymbol{t},\boldsymbol{\lambda}) = e^{i\boldsymbol{\lambda}\cdot\boldsymbol{t}}$, in which case the previous representation specializes to:

$$\widetilde{X}(h) = \int_{\mathbb{R}^n-\{0\}} \widehat{h}(\boldsymbol{\lambda})dZ_Y(\boldsymbol{\lambda}) + (-1)^M\left(\alpha, \widehat{\nabla}_M\widehat{h}(0)\right)$$

where $Z_Y(\cdot)$ is the spectral measure associated with its strongly harmonizable Mth order partial derivative field $Y(\cdot) = \widetilde{X}^{(m_1,m_2,\ldots,m_n)}(h)(\cdot)$ and $\widehat{h}$ is the Fourier transform of h and Z_Y, α a second-order random vector and $\widehat{\nabla}_M$ as defined above.

An important subclass of random fields satisfy an additional condition called *isotropy*. Isotropic random fields $X(\cdot)$, have covariance $r(\cdot,\cdot)$ which are invariant under rotation and reflection. Isotropic fields play an important role in the statistical theory of turbulence, where direction in space is unimportant [17]. Swift [9] obtained the representation of a weakly harmonizable isotropic covariance as

$$r(\boldsymbol{s},\boldsymbol{t}) = 2^\nu\Gamma\left(\frac{n}{2}\right)\int_0^\infty\int_0^\infty \frac{J_\nu(||\lambda\boldsymbol{s}-\lambda'\boldsymbol{t}||)}{||\lambda\boldsymbol{s}-\lambda'\boldsymbol{t}||^\nu}dF(\lambda,\lambda') \tag{10}$$

where $J_\nu(\cdot)$ is the Bessel function (of the first kind) of order $\nu = \frac{n-2}{2}$ and $F(\cdot,\cdot)$ is a complex function of bounded Fréchet variation, with $||\cdot||$ denoting the vector norm.

Isotropic covariances $r(\boldsymbol{s},\boldsymbol{t})$ are functions of the lengths $||\boldsymbol{s}||$, $||\boldsymbol{t}||$ of the vectors $\boldsymbol{s},\boldsymbol{t}$ and of the angle θ between $\boldsymbol{s}$ and $\boldsymbol{t}$. Detailed studies of harmonizable isotropic random fields can be found in the papers of Swift cited above.

Using the ideas presented above, Swift [14], obtained the representation of a generalized random field with strongly harmonizable isotropic increments of order M as

$$\begin{aligned}\widetilde{X}(h) =& \alpha_n \int_{\mathbb{R}^n} h(\boldsymbol{t}) \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} S_m^l(\boldsymbol{u}) \\ &\times \int_{+0}^{\infty} \frac{J_{m+\nu}(\lambda||\boldsymbol{t}||)}{(\lambda||\boldsymbol{t}||)^\nu} dZ_m^l(\lambda)\, d\boldsymbol{t} \qquad (11) \\ &+ \boldsymbol{X}_M \cdot \sum_{|j|=M} \mu_j, \qquad (12)\end{aligned}$$

where $E(Z_m^l(B_1)Z_{m'}^{l'}(B_2)) = \delta_{mm'}\delta_{ll'}F(B_1, B_2)$, with $F(\cdot,\cdot)$ as a tempered function of bounded Vitali variation, and $\boldsymbol{u} = \frac{\boldsymbol{s}}{||\boldsymbol{t}||}$ a unit vector. Further,

$$S_m^l(\cdot), \quad 1 \leq l \leq h(m,n) = \frac{(2m+2\nu)(m+2\nu-1)!}{(2\nu)!m!}, \quad m \geq 1, \quad S_0^l(\boldsymbol{u}) = 1$$

are the spherical harmonics on the unit n-sphere of order m with $\alpha_n > 0, \alpha_n^2 = 2^{2\nu+1}\Gamma(\frac{n}{2})\pi^{\frac{n}{2}}$ and $\boldsymbol{X}_M = (X_{M1}, X_{M2}, \ldots, X_{Mn})$ is a random vector which satisfies

$$E(X_{Mk}Z_m^l(B)) = 0, \quad k = 1, \ldots, n$$

and

$$E(X_{Mk}\overline{X_{1j}}) = \begin{cases} 0, & \text{for } k \neq j, \\ b, & \text{for } k = j, \end{cases}$$

where the μ_j denotes the moments of h.

4. Harmonizable locally spatially isotropic fields

An addition to the theory given above is obtained by considering a random field $X : \mathbb{R} \times \mathbb{R}^n \to L_0^2(P)$. The random field $X(t, \boldsymbol{x})$ is both a function of a spatial variable $\boldsymbol{x}$ and a time variable t. These fields are often useful in applications such as the theory of turbulence, cf. Yaglom [17], and meteorology, cf. Jones [5]. These fields have been previously considered in the stationary isotropic case, and some results for these fields may be found in papers by Jones, [5] and Roy [8], as well as the texts of Adler [1], Yadrenko [16], and Yaglom [17]. Recently, Swift [11], considered these fields in the harmonizable case. The spectral representation of these fields, termed *weakly harmonizable spatially isotropic*, is

$$X(t, \boldsymbol{x}) = \alpha_n \sum_{m=0}^{\infty} \sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u}) \int_{\mathbb{R}^k} \int_0^{\infty} e^{i\omega t} \frac{J_{\nu+k}(\lambda\|\boldsymbol{x}\|)}{(\lambda\|\boldsymbol{x}\|)^\nu} dZ_m^l(\omega, \lambda).$$

A natural problem, in light of the previous sections, is the spectral representation of a field $X(t,\boldsymbol{x})$ that has harmonizable spatially isotropic increments in the spatial variable $\boldsymbol{x}$. The representation of a field $X(t,\boldsymbol{x})$ that has harmonizable increments in the time variable t was recently obtained by Swift [15].

The fields under consideration are mappings on $\mathbb{R}\times\mathbb{R}^n$ and can thus be regarded as mappings on $\mathbb{R}^{n+1}$ so that the above outlined theory of generalized fields may be applied with minor modifications detailed below. In light of this theory, it is natural to consider an appropriate derivative field. More precisely,

Definition 4.1. A mapping $X:\mathbb{R}\times\mathbb{R}^n\to L^2(P)$ is a strongly harmonizable spatially isotropic random field with strongly harmonizable spatially isotropic increments of order M if its generalized partial derivatives

$$\frac{\partial^{(m_1,m_2,\dots,m_n)}\widetilde{X}(h(t,\boldsymbol{x}))}{\partial \boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}$$

where $m_1+m_2+\ldots+m_n=M$ are strongly harmonizable spatially isotropic.

A spectral representation for such a field is given in the following theorem.

Theorem 4.1. *A generalized strongly harmonizable spatially isotropic random field $\widetilde{X}(h)$ with strongly harmonizable spatially isotropic increments of order M has spectral representation:*

$$\widetilde{X}(h(t,\boldsymbol{x}))\int_{\mathbb{R}}\int_{\mathbb{R}^n}\alpha_n=$$

$$\sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)}S_m^l(\boldsymbol{u})\times\int_{\mathbb{R}}\int_{0+}^{\infty}e^{i\omega t}\frac{J_{m+\nu}(\lambda\|\boldsymbol{x}\|)}{(\lambda\|\boldsymbol{x}\|)^{\nu}}h(t,\boldsymbol{x})dZ_m^l(\boldsymbol{\omega},\lambda)\,d\boldsymbol{x}dt$$

$$+\alpha_n\sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)}S_m^l(\boldsymbol{u})\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\|\boldsymbol{x}\|^M h(t,\boldsymbol{x})e^{i\omega t}\,d\boldsymbol{x}dt\,dW_m^l(\omega)$$

where $Z_m^l(\cdot,\cdot)$ is the spectral measure associated with its strongly harmonizable spatial isotropic partial derivative random field

$$Y(t,\boldsymbol{x})=\frac{\partial^{(m_1,m_2,\dots,m_n)}\widetilde{X}(h(t,\boldsymbol{x}))}{\partial \boldsymbol{x}^{(m_1,m_2,\dots,m_n)}},$$

and

$$\widehat{h}_m^l(\omega,\lambda)=\int_{\mathbb{R}}\int_{\mathbb{R}^n}h(t,\boldsymbol{x})e^{i\omega t}\frac{J_{\nu+m}(\lambda\|\boldsymbol{x}\|)}{(\lambda\|\boldsymbol{x}\|)^{\nu}}d\boldsymbol{x}dt \tag{13}$$

is the Fourier-Bessel transform of $h(t,\boldsymbol{x})$. More specifically,

$$Z_m^l:\mathcal{B}(\mathbb{R}\times(0,\infty))\to L_0^2(P)$$

is a measure such that $F(\cdot,\cdot,\cdot,\cdot)$ is of finite Vitali variation, where $\mathcal{B}(\mathbb{R}\times(0,\infty))$ is the Borel σ-algebra of $(\mathbb{R}\times(0,\infty))$. Further, for each $m=0,1,\dots,\infty$ and $l=0,\dots,h(m,n)$ the stochastic measure $W_m^l(\cdot)$ is defined by

$$W_m^l(\omega)=\lim_{\varepsilon\to 0}\frac{Z_m^l(\omega,\varepsilon)-Z_m^l(\omega,-\varepsilon)}{k!}.$$

Proof. Using the relationship between $\widetilde{X}$ and the partial derivative

$$\partial^{(m_1,m_2,\dots,m_n)}\widetilde{X}(h(t,\boldsymbol{x}))/\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)},$$

(cf. Yaglom [17]), it follows that since the measure F is tempered,

$$\begin{aligned}\widetilde{X}\left(\frac{\partial^{(m_1,m_2,\dots,m_n)}h(t,\boldsymbol{x})}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}\right)&=(-1)^M\frac{\partial^{(m_1,m_2,\dots,m_n)}}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}\widetilde{X}(h(t,\boldsymbol{x}))\\&=(-1)^M\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\frac{\partial^{(m_1,m_2,\dots,m_n)}}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}\widetilde{X}(t,\boldsymbol{x})h(t,\boldsymbol{x})d\boldsymbol{x}dt.\end{aligned}$$

Since the partial derivative

$$\partial^{(m_1,m_2,\dots,m_n)}\widetilde{X}(h(t,\boldsymbol{x}))/\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)},$$

is a strongly harmonizable spatially isotropic field with spectral representation

$$\begin{aligned}Y(t,\boldsymbol{x})&=\frac{\partial^{(m_1,m_2,\dots,m_n)}}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}\widetilde{X}(h(t,\boldsymbol{x}))\\&=\alpha_n\sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)}S_m^l(\boldsymbol{u})\int_{\mathbb{R}}\int_0^{\infty}e^{i\omega t}\frac{J_{\nu+m}(\lambda\|\boldsymbol{x}\|)}{(\lambda\|\boldsymbol{x}\|)^{\nu}}dZ_m^l(\omega,\lambda),\end{aligned}$$

then

$$\widetilde{X}\left(\frac{\partial^{(m_1,m_2,\dots,m_n)}h(t,\boldsymbol{x})}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}\right)=(-1)^M\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}Y(t,\boldsymbol{x})h(t,\boldsymbol{x})d\boldsymbol{x}dt.$$

Integrating by parts repeatedly and noting that the various partial derivatives of $h(\cdot,\cdot)$ have compact supports one has

$$\begin{aligned}\widetilde{X}\left(\frac{\partial^{(m_1,m_2,\dots,m_n)}h(t,\boldsymbol{x})}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}\right)&=\int_{\mathbb{R}}\int_{\mathbb{R}^n}\alpha_n\sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)}S_m^l(\boldsymbol{u})\\&\times\int_{\mathbb{R}}\int_{0+}^{\infty}e^{i\omega t}\frac{J_{m+\nu}(\lambda\|\boldsymbol{x}\|)}{(\lambda\|\boldsymbol{x}\|)^{\nu}}\frac{\partial^{(m_1,m_2,\dots,m_n)}h(t,\boldsymbol{x})}{\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}}dZ_m^l(\boldsymbol{\omega},\lambda)\,d\boldsymbol{x}dt\\&\quad+\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\widehat{\nabla}_M\widehat{h}_m^l(\omega,\lambda)\,dZ_m^l(\boldsymbol{\omega},\lambda),\end{aligned}$$

where $\widehat{\nabla}_M$ is the Mth order gradient.

The partial derivative $\partial^{(m_1,m_2,\dots,m_n)}h(t,\boldsymbol{x})/\partial\boldsymbol{x}^{(m_1,m_2,\dots,m_n)}$ can be replaced by $h(t,\boldsymbol{x})$ since the set of partial derivatives of functions γ in $\mathcal{K}$ coincides with the subspace of $\mathcal{K}$, consisting of functions satisfying

$$\gamma_0(h) = \cdots = \gamma_{m-1}(h) = 0.$$

Thus

$$\begin{aligned}
\widetilde{X}(h(t,\boldsymbol{x})) &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \alpha_n \sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u}) \\
&\quad \times \int_{\mathbb{R}}\int_{0+}^{\infty} e^{i\omega t}\frac{J_{m+\nu}(\lambda||\boldsymbol{x}||)}{(\lambda||\boldsymbol{x}||)^{\nu}} h(t,\boldsymbol{x})dZ_m^l(\boldsymbol{\omega},\lambda)\,d\boldsymbol{x}dt \\
&\quad + \int_{\mathbb{R}} \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \widehat{\nabla}_M \widehat{h}_m^l(\omega,\lambda)\,dZ_m^l(\boldsymbol{\omega},\lambda) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \alpha_n \sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u}) \\
&\quad \times \int_{\mathbb{R}}\int_{0+}^{\infty} e^{i\omega t}\frac{J_{m+\nu}(\lambda||\boldsymbol{x}||)}{(\lambda||\boldsymbol{x}||)^{\nu}} h(t,\boldsymbol{x})dZ_m^l(\boldsymbol{\omega},\lambda)\,d\boldsymbol{x}dt \\
&\quad + \alpha_n \sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u}) \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} ||\boldsymbol{x}||^M h(t,\boldsymbol{x})e^{i\omega t}\,d\boldsymbol{x}dt\,dW_m^l(\omega),
\end{aligned}$$

where for $m = 0,1,\dots,\infty$ and $l = 0,\dots,h(m,n)$, the stochastic measure $W_m^l(\cdot)$ is defined by

$$W_m^l(\omega) = \lim_{\varepsilon\to 0}\frac{Z_m^l(\omega,\varepsilon) - Z_m^l(\omega,-\varepsilon)}{k!}.$$

This gives the desired spectral representation. ☑

Using the relationship

$$\widetilde{X}(h) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} h(t,\boldsymbol{x})\,X(t,\boldsymbol{x})\,dtd\boldsymbol{x}$$

with the spectral representation of the previous theorem, (since $X(\cdot)$ is point valued), the spectral representation of the ordinary field $X(\cdot,\cdot)$ can be obtained as

$$\begin{aligned}
X(t,\boldsymbol{x}) = \alpha_n \sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u}) \int_{\mathbb{R}}\int_{0+}^{\infty} e^{i\omega t}\frac{J_{m+\nu}(\lambda||\boldsymbol{x}||)}{(\lambda||\boldsymbol{x}||)^{\nu}} dZ_m^l(\boldsymbol{\omega},\lambda) \\
+ \alpha_n \sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u}) \int_{\mathbb{R}} ||\boldsymbol{x}||^M e^{i\omega t}\,dW_m^l(\omega).
\end{aligned} \tag{14}$$

This result is summarized in the following proposition.

Proposition 4.1. *A strongly harmonizable spatially isotropic random field with strongly harmonizable spatially isotropic increments of order M has a spectral representation given by (14) where $Z_m^l(\cdot,\cdot)$ is the spectral measure associated with its strongly harmonizable spatial isotropic partial derivative random field*

$$Y(t,\boldsymbol{x}) = \frac{\partial^{(m_1,m_2,\dots,m_n)}\widetilde{X}(h(t,\boldsymbol{x}))}{\partial \boldsymbol{x}^{(m_1,m_2,\dots,m_n)}},$$

and for each $m = 0,1,\dots,\infty$ and $l = 0,\dots,h(m,n)$ the stochastic measure $W_m^l(\cdot)$ is defined by

$$W_m^l(\omega) = \lim_{\varepsilon\to 0}\frac{Z_m^l(\omega,\varepsilon) - Z_m^l(\omega,-\varepsilon)}{k!}.$$

Now letting

$$\Psi_m^l(t,\|\boldsymbol{x}\|) = \alpha_n \int_{\mathbb{R}}\int_{0+}^{\infty} e^{i\omega t}\frac{J_{m+\nu}(\lambda\|\boldsymbol{x}\|)}{(\lambda\|\boldsymbol{x}\|)^{\nu}}\,dZ_m^l(\boldsymbol{\omega},\lambda)$$

one has

$$E(\Psi_m^l(t,\|\boldsymbol{x}\|)) = 0$$

and

$$E(\Psi_m^l(s,\|\boldsymbol{x}\|)\overline{\Psi_{m'}^l(t,\|\boldsymbol{y}\|)}) = \delta_{mm'}\delta_{ll'}F(s,t,\|\boldsymbol{x}\|,\|\boldsymbol{y}\|)$$

using a form of Fubini's theorem. More specifically, first apply $x^* \in (L_0^2(P))^*$ to both sides, then taking x^* inside the integral, which is permissible, (cf. [3], IV.9), since $x^* Z_m^l(\cdot,\cdot)$ is a scalar measure, the classical Fubini theorem applies, Dunford and Schwartz, [3]. Hence, the above representation can be extended for all time-varying random fields with Mth order spatially isotropic increments which need not be harmonizable. These facts are summarized in the following theorem.

Theorem 4.2. *A random field $X : \mathbb{R}\times\mathbb{R}^n \to L_0^2(P)$ is time-varying with Mth order spatially isotropic increments iff it admits the spectral representation*

$$X(t,\boldsymbol{x}) = \sum_{m=0}^{\infty}\sum_{l=0}^{p(m,n)} S_m^l(\boldsymbol{u})\Psi_m^l(t,\|\boldsymbol{x}\|) + \alpha_n \sum_{m=0}^{\infty}\sum_{l=0}^{h(m,n)} S_m^l(\boldsymbol{u})\int_{\mathbb{R}} \|\boldsymbol{x}\|^M e^{i\omega t}\,dW_m^l(\omega)$$

where

$$\Psi_m^l(\cdot,\cdot), m = 0,1,\dots,\ l = 1,\dots,p(m,n)$$

are a sequence of random fields such that

$$E(\Psi^l_m(t, \|\boldsymbol{x}\|)\overline{\Psi^{l'}_{m'}(t, \|\boldsymbol{x}\|)}) = \delta_{mm'}\delta_{ll'}b_m(t, t, \|\boldsymbol{x}\|, \|\boldsymbol{x}\|)$$

and

$$\sum_{m=0}^{\infty} p(m, n) b_m(t, t, \|\boldsymbol{x}\|, \|\boldsymbol{x}\|) < \infty.$$

This result gives the representation of a time-varying field with Mth order spatially isotropic increments and for $M = 1$ reduces to the representation of a time varying field on a sphere, given by R. H. Jones [5].

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