A first passage problem for three-dimensional diffusion processes

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ABSTRACT. Let $dx(t) = [\phi(y(t)) + \psi(z(t))]dt$, where $y(t)$ and $z(t)$ are independent diffusion process. The problem of computing the moment generating function and the moments of $x[T(y,z)]$, where $T(y,z)$ is a first passage time for $(y(t), z(t))$, is considered and solved explicitly in particular instances.

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1. Introduction

We first consider the system of stochastic differential equations

$$dy(t) = f[y(t)]\,dt + [by^j(t)]^{1/2}\,dB(t), \tag{1}$$
$$dz(t) = g[z(t)]\,dt + [cz^k(t)]^{1/2}\,dW(t), \tag{2}$$

where $b > 0$, $c > 0$, $j \geq 0$ and $k \geq 0$ are constants, and $B(t)$ and $W(t)$ are independent standard Brownian motions. Let

$$T(y,z) := \inf\{t \geq 0 : (y(t), z(t)) \in D \ (\subset \mathbb{R}^2) | y(0) = y, z(0) = z\}.$$

In [6] the authors showed that if $j = k = 0$, then the (truly) two-dimensional problems for which we can write that $T(y,z) = T(\phi(y) + \psi(z))$ for some functions $\phi(\cdot)$ and $\psi(\cdot)$ are essentially those where D is a straight line or a circle. Actually, we could also have two boundaries.

Next, the author of the present paper (see [5]) generalized their results by considering the system (1), (2). In this case, the boundary of the stopping region D may be a parabola or an exponential function, for instance. More precisely, let

$$M(y, z; a) := E[e^{-aT(y,z)}],$$

where we assume that a is a real, positive parameter. Extending the results contained in [5] to the case of two boundaries, we may state the following results.

Proposition 1.1. *Let*

$$T(y, z) = \inf\{t \geq 0 : \phi(y(t)) + \psi(z(t)) = K_1 \text{ or } K_2 | y(0) = y, z(0) = z\}.$$

Assume that $\phi''(y)$ and $\psi''(z)$ exist and that $\phi'(y)$ and $\psi'(z)$ are different from zero for all couples in the continuation region

$$C := \{(y, z) \in \mathbb{R}^2 : K_1 < \phi(y) + \psi(z) < K_2\}. \tag{3}$$

Then a necessary condition to have the right to write that

$$M(y, z; a) = N(\phi(y) + \psi(z); a) \tag{4}$$

is that

$$\phi(y) = \frac{d}{(2-j)^2} y^{2-j} + \frac{e}{2-j} y^{1-\frac{j}{2}} \quad \text{if } j \neq 2$$

and

$$\phi(y) = \frac{d}{4} \ln^2(y) + \frac{e}{2} \ln(y) \quad \text{if } j = 2,$$

where d and e are constants (and we added the constant 2 under e in the last expression for simplicity in the sequel).

Similarly, we must have:

$$\psi(z) = \frac{\delta}{(2-k)^2} z^{2-k} + \frac{\varepsilon}{2-k} z^{1-\frac{k}{2}} \quad \text{if } k \neq 2$$

and

$$\psi(z) = \frac{\delta}{4} \ln^2(z) + \frac{\varepsilon}{2} \ln(z) \quad \text{if } k = 2,$$

where δ and ε are constants. Furthermore, we have the relationship $bd = c\delta$.

Remarks.

1. In the proof of Proposition 1.1, the author assumed that the functions N, N' $(= \frac{d}{dw} N(w; a)$, where $w := \phi(y) + \psi(z))$ and N'' are all differentiable with respect to the parameter a.

2. It was also assumed, without loss of generality, that the functions $\phi(\cdot)$ and $\psi(\cdot)$ do not contain constant terms.

Corollary 1.1. *A second necessary condition for the relation (4) to hold is that the functions $f(\cdot)$ and $g(\cdot)$ that appear in (1) and (2) must be of the form*

$$f(y) = \frac{1}{\phi'(y)}[\mu\phi(y) - \tfrac{1}{2}by^j\phi''(y) + \beta]$$

and

$$g(z) = \frac{1}{\psi'(z)}[\mu\psi(z) - \tfrac{1}{2}cz^k\psi''(z) + \gamma],$$

where μ, β and γ are constants. Furthermore, we assume that $\phi'(y)$ and $\psi'(z)$ are $\neq 0$ for all $(y, z) \in C$, where C is defined in (3).

Now, in the present paper we are interested in computing the distribution of the integral

$$\int_0^{T(y,z)} [\phi(y(t)) + \psi(z(t))]\, dt$$

for the cases when we can write that $T(y, z) = T(\phi(y) + \psi(z))$. In Section 2, the problem of obtaining the moment generating function of the random variable $x[T(y, z)]$, where

$$dx(t) = [\phi(y(t)) + \psi(z(t))]\, dt, \tag{5}$$

is treated. Next, Section 3 deals with the computation of the moments of $x[T(y, z)]$. Examples solved explicitly are presented in Section 4. Finally, concluding remarks are made in Section 5.

2. Moment generating function of $x[T(y, z)]$

We consider the three-dimensional diffusion process $(x(t), y(t), z(t))$ defined by (5), (1) and (2). We may write that

$$x[T(y, z)] = x(0) + \int_0^{T(y,z)} [\phi(y(t)) + \psi(z(t))]dt. \tag{6}$$

Let

$$L(x, y, z; s) := E_x[e^{-sx[T(y,z)]}], \tag{7}$$

where $x = x(0)$ and we assume that s is real and positive. The function $L(x, y, z; s)$ satisfies the Kolmogorov backward equation

$$\tfrac{1}{2}(by^j L_{yy} + cz^k L_{zz}) + fL_y + gL_z + (\phi(y) + \psi(z))L_x = 0, \tag{8}$$

where $L_x := \partial L/\partial x$, etc. The equation is valid for $x \in \mathbb{R}$ and for (y, z) in the continuation region C defined in (3). We also have the boundary condition

$$L(x, y, z; s) = e^{-sx} \quad \text{if } \phi(y) + \psi(z) = K_1 \text{ or } K_2. \tag{9}$$

Next, we deduce from (6) and (7) that

$$L(x, y, z; s) = e^{-sx} P(y, z; s) \tag{10}$$

for a certain function P, so that $L_x = -sL$. Furthermore, if the functions ϕ, ψ, f and g are as in Proposition 1.1 and Corollary 1.1, we may write that

$$P(y, z; s) = Q(w; s), \tag{11}$$

where

$$w := \phi(y) + \psi(z). \tag{12}$$

It is now a simple matter to prove the following proposition.

Proposition 2.1. *The function $Q(w; s)$ satisfies the ordinary differential equation*

$$\tfrac{1}{2}(Aw + B)Q''(w; s) + (\mu w + \Delta)Q'(w; s) - swQ(w; s) = 0, \tag{13}$$

where

$$A := bd \ (= c\delta) \tag{14}$$

$$B := \frac{be^2 + c\varepsilon^2}{4} \tag{15}$$

$$\Delta := \beta + \gamma. \tag{16}$$

The equation is valid for $K_1 < w < K_2$ and is subject to the boundary conditions

$$Q(K_1; s) = Q(K_2; s) = 1. \tag{17}$$

Proof. Substituting (10) into Eq. (8) and making use of (11) and (12), we find that the partial differential equation (8) indeed reduces to the ordinary differential equation (13). Moreover, the boundary conditions (17) follow at once from (9), (10) and (11).   ☑

Remark. We can have $A = 0$ or $B = 0$, but A and B cannot be equal to zero at the same time. We can also have $\mu = 0$ and/or $\Delta = 0$.

Corollary 2.1. *If $A > 0$ in (13), the moment generating function of the random variable $x[T(y, z)]$ defined in (6) is given by*

$$L(x, y, z; s) := E_x\left[e^{-sx[T(y,z)]}\right] =$$

$$e^{-sx}e^{-\mu w/A}\exp\left\{-\left[\frac{\mu^2 + 2As}{A^2}\right]^{1/2}\left(\frac{B}{A} + w\right)\right\}$$

$$\times \left(\frac{B}{A} + w\right)^{\left(\frac{1}{2} + \frac{1}{2A^2}|A^2 - 2A\Delta + 2B\mu| - \frac{\Delta}{A} + \frac{B\mu}{A^2}\right)} A^{\frac{B\mu}{A^2} - \frac{\Delta}{A}}$$

$$\times \left\{ c_1 M\left(\frac{-ABs + A\Delta\mu - B\mu^2}{A^2(\mu^2 + 2As)^{1/2}} + \frac{1}{2} + \frac{1}{2A^2}|A^2 - 2A\Delta + 2B\mu|, \right.\right.$$

$$\left. 1 + \frac{1}{A^2}|A^2 - 2A\Delta + 2B\mu|, \frac{2}{A}(\mu^2 + 2As)^{1/2}\left(\frac{B}{A} + w\right)\right)$$

$$\left. + c_2 U\left(\frac{-ABs + A\Delta\mu - B\mu^2}{A^2(\mu^2 + 2As)^{1/2}} + \frac{1}{2} + \frac{1}{2A^2}|A^2 - 2A\Delta + 2B\mu|, \right.\right.$$

$$\left.\left. 1 + \frac{1}{A^2}|A^2 - 2A\Delta + 2B\mu|, \frac{2}{A}(\mu^2 + 2As)^{1/2}\left(\frac{B}{A} + w\right)\right)\right\} \quad (18)$$

for $K_1 \leq w \leq K_2$, where $w := \phi(y) + \psi(z)$, $M(\cdot, \cdot, \cdot)$ and $U(\cdot, \cdot, \cdot)$ are confluent hypergeometric functions (see [1, p. 504]), and the constants c_1 and c_2 are uniquely determined by the boundary conditions (17).

Proof. The general solution of Eq. (13) can be found in many books or by using a computer software. ☑

Remarks.

1. The general solution of Eq. (13) can have many forms, according to the values taken by the various parameters it contains. We have given in Corollary 2.1 the solution in the most important case, namely when A is different from zero. We have assumed, without any (real) loss of generality, that A is positive. If A is negative, we must also have $\mu^2 \neq -2As$. The solution in the case when $A \neq 0$ but $\mu^2 = -2As$ involves Bessel functions instead of confluent hypergeometric functions (see [7, p. 143]). The other possible cases are treated in [7, p. 143] as well.

2. The constants K_1 and K_2 cannot take any values. Indeed if $\phi(y) + \psi(z) = y^2 + z^2$, for instance, we must obviously have $K_1 > 0$.

3. When we want to solve a problem with only one boundary, we must use a mathematical argument to discard one of the confluent hypergeometric functions in (18) by choosing c_1 or c_2 equal to zero. For example, if we have $M(\cdot, \cdot, w)$ and $U(\cdot, \cdot, w)$ in the general solution of Eq. (13) and if the variable w can take any value in the interval $[K_1, \infty)$, then we must get rid

of the function $M(\cdot, \cdot, w)$ by setting $c_1 = 0$ because the function $M(\cdot, \cdot, w)$ diverges as $w \to \infty$ (whereas $U(\cdot, \cdot, w)$ tends to zero with $w \to \infty$; see [1, p. 508]) and we deduce from the definition of the function $L(x, y, z; s)$ that

$$0 < L(x, y, z; s) \le e^{-sx}.$$

Conversely, if $w \in [0, K_2]$, then we must (generally) choose $c_2 = 0$ and keep only the function M in (18) (see again [1, p. 508]).

In the next section, the problem of computing the moments of $x[T(y, z)]$ is discussed.

3. Expected value of $x[T(y, z)]$

In theory, assuming that it exists, we can obtain the moment of order n of $x[T(y, z)]$ with respect to the origin, that is $m^n(x, y, z) := E_x[x^n[T(y, z)]]$, from the derivative of the function $L(x, y, z; s)$ with respect to s (n times) and its limit as $s \downarrow 0$. However, because the function L involves the parameter s in the arguments of the confluent hypergeometric functions, the computation of this derivative, and the limit as $s \downarrow 0$, would prove to be quite tedious (all the more tedious because the constants c_1 and c_2 also will involve the parameter s).

Instead of using this technique, we will rather consider the Kolmogorov backward equation satisfied by the function $m^n(x, y, z)$, namely

$$\tfrac{1}{2}(by^j m_{yy}^n + cz^k m_{zz}^n) + fm_y^n + gm_z^n + (\phi(y) + \psi(z))m_x^n = 0. \qquad (19)$$

We have:

$$m^n(x, y, z) = E\left\{\left[x + \int_0^{T(y,z)} [\phi(y(t)) + \psi(z(t))]dt\right]^n\right\} \qquad (20)$$

for $n = 0, 1, \ldots$ Using the fact that $T(y, z) = T(w)$, where $w := \phi(y) + \psi(z)$, we find that Eq. (19) reduces to

$$\tfrac{1}{2}(Aw + B)q_{ww}^n(x, w) + (\mu w + \Delta)q_w^n(x, w) + wq_x^n(x, w) = 0, \qquad (21)$$

where

$$q^n(x, w) = m^n(x, y, z) \qquad (22)$$

and the constants A, B and Δ are defined in (14), (15) and (16).

We may now state the following result.

Proposition 3.1. *The mean of the random variable $x[T(y, z)]$ can be obtained by solving the ordinary differential equation*

$$\tfrac{1}{2}(Aw + B)q_1''(w) + (\mu w + \Delta)q_1'(w) + w = 0, \tag{23}$$

subject to the boundary condition

$$q_1(K_1) = q_1(K_2) = 0, \tag{24}$$

where

$$q_1(w) + x = q^1(x, w) := E_x[x[T(y, z)]].$$

Proof. Eq. (23) and the boundary condition (24) follow directly from (20), (21) and (22) with $n = 1$. ☑

Corollary 3.1. *If we assume that $q^2(x, w = \phi(y) + \psi(z)) := E_x[x^2[T(y, z)]]$ exists, then the function $q^2(x, w)$ satisfies the ordinary differential equation*

$$\tfrac{1}{2}(Aw + B)q_{ww}^2(x, w) + (\mu w + \Delta)q_w^2(x, w) + 2wq^1(x, w) = 0, \tag{25}$$

subject to

$$q^2(x, K_1) = q^2(x, K_2) = x^2.$$

Remarks.

1. We could of course use the functions $q^2(x, w)$ and $q^1(x, w)$ to compute the variance of $x[T(y, z)]$. This variance should be independent of x.
2. In general, if it exists, the function $q^n(x, w)$ satisfies the ordinary differential equation

   $$\tfrac{1}{2}(Aw + B)q_{ww}^n(x, w) + (\mu w + \Delta)q_w^n(x, w) + nwq^{n-1}(x, w) = 0,$$

   with the boundary conditions

   $$q^n(x, K_1) = q^n(x, K_2) = x^n.$$

3. Solving Eq. (23) (and/or Eq. (25)) is, in theory, straightforward. We have in fact a first order linear equation in $h(w) := q_1'(w)$, the solution of which is easy to obtain. However, we are then left with the problem of integrating the function $h(w)$, which, in the general case, is not obvious. Instead of attempting to find an explicit solution (that is, without any integral sign), we will give a couple of examples in the next section. Furthermore, in the special case when $\mu = 0$, we find that

   $$q^1(x, w) := E_x[x[T(w)]]$$

   $$= \frac{Bw}{\Delta(A + 2\Delta)} - \frac{w^2}{A + 2\Delta} + c_1 \frac{\frac{B}{2\Delta - A} + \frac{Aw}{A - 2\Delta}}{(B + Aw)^{2\Delta/A}} + c_2,$$

   where the constants c_1 and c_2 are such that $q^1(x, K_1) = q^1(x, K_2) = x$.

4. Examples

The first particular case that we consider is the one where the stochastic differential equations (1), (2) take the form (see [5])

$$dy(t) = -\beta_0 y(t)\, dt + \frac{\alpha_1 - 1}{y(t)}\, dt + dB(t), \tag{26}$$

$$dz(t) = -\beta_0 z(t)\, dt + \frac{\alpha_2 - 1}{z(t)}\, dt + dW(t), \tag{27}$$

where β_0, α_1 and α_2 are non-negative constants. If β_0 is equal to zero, then the process $(y(t), z(t))$ is a two-dimensional Bessel process, whereas when $\alpha_1 = \alpha_2 = 0$, $(y(t), z(t))$ is a two-dimensional Ornstein-Uhlenbeck process (with the same parameter β_0). In the notation of Section 1, we have $b = c = 1$ and $j = k = 0$.

Next, let

$$T(y, z) := \inf\{t \geq 0 : y^2(t) + z^2(t) = r^2 | y(0) = y, z(0) = z\}.$$

That is, we have $\phi(y) = y^2$ and $\psi(z) = z^2$, so that $d = \delta = 4$ and $e = \varepsilon = 0$. Furthermore (see Corollary 1.1)

$$f(y) = -\beta_0 y + \frac{\alpha_1 - 1}{2y} = \frac{1}{2y}\left(\mu y^2 - \frac{1}{2} \cdot 1 \cdot 2 + \beta\right)$$

implies that $\mu = -2\beta_0$ and $\beta = \alpha_1$. Similarly, we find that $\gamma = \alpha_2$. Using these values, we deduce from (13) that we must solve the ordinary differential equation

$$\tfrac{1}{2}(4w)Q''(w; s) + (-2\beta_0 w + \alpha_1 + \alpha_2)Q'(w; s) - swQ(w; s) = 0.$$

Let us choose $\beta_0 = 1$ and $\alpha_1 = \alpha_2 = 3/2$. We can then show that the diffusion processes $y(t)$ and $z(t)$ have an inaccessible boundary at the origin (see [5]). Hence, if we assume that the starting point (y, z) is situated in the first quadrant, then the two-dimensional process $(y(t), z(t))$ cannot leave this quadrant.

Now, the general solution of the differential equation

$$2wQ''(w; s) + (3 - 2w)Q'(w; s) = swQ(w; s)$$

can be written as

$$Q(w; s) = (1/2)^{3/4} \exp\left[\frac{w}{2}(1 - (1 + 2s)^{1/2})\right]$$

$$\times \left\{ c_1 M\left(\frac{3}{4}\left(1 - \frac{1}{(1 + 2s)^{1/2}}\right), \frac{3}{2}, (1 + 2s)^{1/2} w\right)\right.$$

$$\left. + c_2 U\left(\frac{3}{4}\left(1 - \frac{1}{(1 + 2s)^{1/2}}\right), \frac{3}{2}, (1 + 2s)^{1/2} w\right)\right\}. \quad (28)$$

If we assume that $0 < y^2 + z^2 \leq r^2$, we must set c_2 equal to zero in (28) because (as mentioned above) the function $U(\cdot, \cdot, w)$ diverges as $w \downarrow 0$. It follows that

$$E_x[e^{-sx[T(y,z)]}] =$$

$$e^{-sx} \exp\left[\left(\frac{w - r^2}{2}\right)[1 - (1 + 2s)^{1/2}]\right] \frac{M\left(\frac{3}{4}\left(1 - \frac{1}{(1+2s)^{1/2}}\right), \frac{3}{2}, (1 + 2s)^{1/2} w\right)}{M\left(\frac{3}{4}\left(1 - \frac{1}{(1+2s)^{1/2}}\right), \frac{3}{2}, (1 + 2s)^{1/2} r^2\right)}$$

for $0 < w \leq r^2$.

To obtain the mean of $x[T(y, z)]$, we must solve the differential equation

$$2w q_1''(w) + (3 - 2w) q_1'(w; s) + w = 0.$$

In theory, this is an easy task. However, the solution involves special functions and will not be given explicitly here.

To conclude, we consider another particular case of the system (1), (2). We take

$$dy(t) = (1/2)^{1/2} dB(t), \quad (29)$$

$$dz(t) = 3/2\, dt + [2z(t)]^{1/2} dW(t). \quad (30)$$

The author has solved the problem of computing the moment generating function of the first passage time

$$T(y, z) := \inf\{t \geq 0 : y^2(t) + z(t) = r \ (> 0) | y(0) = y, z(0) = z\}$$

(see [5]). We can show that the diffusion process $z(t)$ cannot cross the origin. We assume that $y^2 + z \leq r$. Then the continuation region is bounded.

In the notation of Section 1, we have $b = 1/2$, $c = 2$, $d = 4$, $e = 0$, $\delta = 1$, $\varepsilon = 0$, $\mu = 0$, $\gamma = 3/2$ and $\beta = 1/2$. It follows that $A = 2$, $B = 0$ and $\Delta = 2$. Therefore, we must solve the following equation:

$$wQ''(w; s) + 2Q'(w; s) = swQ(w; s).$$

Its general solution can be written as

$$Q(w; s) = \frac{e^{-s^{1/2}w}}{2s^{1/2}w}\left[c_1 + c_2(e^{2s^{1/2}w} - 1)\right],$$

so that

$$E_x[e^{-sx[T(y,z)]}] = \frac{e^{-sx}e^{-s^{1/2}(y^2+z)}}{2s^{1/2}(y^2 + z)}\left[c_1 + c_2(e^{2s^{1/2}(y^2+z)} - 1)\right] \quad (31)$$

for $0 < y^2 + z \le r$.

Because the moment generating function of $x[T(y, z)]$ is bounded and the function in the right-hand member of Eq. (31) diverges as $y^2 + z \downarrow 0$ if $c_1 \ne 0$, we deduce that we must choose $c_1 = 0$. Then, using the boundary condition

$$E_x[e^{-sx[T(y,z)]}] = e^{-sx} \quad \text{if } y^2 + z = r,$$

we can state that

$$E_x[e^{-sx[T(y,z)]}] = e^{-sx}e^{-s^{1/2}(y^2+z-r)}\left(\frac{r}{y^2 + z}\right)\frac{(e^{2s^{1/2}(y^2+z)} - 1)}{(e^{2s^{1/2}r} - 1)}$$

for $0 < y^2 + z \le r$.

Finally, to obtain the mean of $x[T(y, z)]$, we solve the ordinary differential equation

$$wq_1''(w) + 2q_1'(w) + w = 0,$$

subject to the boundary condition

$$q_1(r) = 0.$$

We find that

$$q_1(w) = -\frac{w^2}{6} - \frac{c_1}{w} + c_2.$$

We can show that we must choose $c_1 = 0$ in the preceding equation. The boundary condition $q_1(r) = 0$ then yields that

$$q_1(w) = \frac{r^2 - w^2}{6},$$

so that

$$E_x[x[T(y, z)]] = x + \frac{r^2 - (y^2 + z)^2}{6}$$

for $0 < y^2 + z \le r$.

5. Conclusion

The problem of computing the moment generating function of the random variable $x[\tau(y)]$, where $\tau(y)$ is a first passage time for a one-dimensional diffusion process $y(t)$ and $x(t)$ is defined by $dx(t) = y(t)dt$, has already been considered by the author (see [3], [4]) and in [2], in particular. In the present note, a similar problem was treated, namely that of obtaining the moment generating function of an integrated process evaluated at a first passage time $T(y, z)$ for a two-dimensional diffusion process. We considered the case when the random variable $T(y, z)$ can be expressed as $T(\phi(y) + \psi(z))$ for some functions $\phi(\cdot)$ and $\psi(\cdot)$, thus reducing the level of difficulty of the problem.

A problem that has not been discussed in this note is that of computing explicitly the probability density function of the random variable $x[T(y, z)]$. Although this problem is very difficult in general, because it implies inverting a Laplace transform with the parameter s appearing as argument of special functions, it must surely be possible to obtain such a probability density function in some special cases.

Finally, if it is not possible to write that $T(y, z) = T(\phi(y) + \psi(z))$, then we could try to use other techniques to solve the appropriate Kolmogorov backward equation.

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