Some results on the geometry of full flag manifolds and harmonic maps

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ABSTRACT. In this note we study, for \( n = 5, 6, 7 \), the geometry of the full flag manifolds, \( F(n) = \frac{U(n)}{U(1)^{n}} \). By using tournaments we characterize all of the \((1,2)\)-symplectic invariant metrics on \( F(n) \), for \( n = 5, 6, 7 \), corresponding to different classes of non-integrable invariant almost complex structure.

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1. Introduction

Eells and Sampson [ES], proved that if \( \phi: M \to N \) is a holomorphic map between Kähler manifolds then \( \phi \) is harmonic. This result was generalized by Lichnerowicz (see [L] or [Sa]) as follows: Let \((M, g, J_1)\) and \((N, h, J_2)\) be almost Hermitian manifolds with \( M \) cosymplectic and \( N \) \((1,2)\)-symplectic. Then any \( \pm \) holomorphic map \( \phi: (M, J_1) \to (N, J_2) \) is harmonic.

We are interested to study harmonic maps, \( \phi: M^2 \to F(n) \), from a closed Riemannian surface \( M^2 \) to a full flag manifold \( F(n) \). Then by the Lichnerowicz theorem, we must study \((1,2)\)-symplectic metrics on \( F(n) \), because a Riemannian surface is a Kähler manifold and a Kähler manifold is a cosymplectic manifold (see [Sa] or [GH]).

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The study of invariant metrics on $F(n)$ involves almost complex structures on $F(n)$. Borel and Hirzebruch [BH], proved that there are $2(\binom{n}{2}) U(n)$-invariant almost complex structures on $F(n)$. This number is the same number of tournaments with $n$ players or nodes. A tournament is a digraph in which any two nodes are joined by exactly one oriented edge (see [M] or [BS]). There is a natural identification between almost complex structures on $F(n)$ and tournaments with $n$ players, see [MN3] or [BS].

The tournaments can be classified in isomorphism classes. In that classification, one of this classes corresponds to the integrable structures and the another ones correspond to non-integrable structures. Burstall and Salamon [BS], proved that a almost complex structure $J$ on $F(n)$ is integrable if and only if the associated tournament to $J$ is isomorphic to the canonical tournament (the canonical tournament with $n$ players, $\{1, 2, \ldots, n\}$, is defined by $i \rightarrow j$ if and only if $i < j$). In that paper the identification between almost complex structures and tournaments plays a very important role.

Borel [Bo], proved that exits a $(n - 1)$-dimensional family of invariant Kähler metrics on $F(n)$ for each invariant complex structure on $F(n)$. Eells and Salamon [ESa], proved that any parabolic structure on $F(n)$ admits a $(1,2)$-symplectic metric. Mo and Negreiros [MN2], showed explicitly that there is a $n$-dimensional family of invariant $(1,2)$-symplectic metrics for each parabolic structure on $F(n)$, the identification between almost complex structures and tournaments is strongly used in that paper.

Mo and Negreiros ([MN1], [MN2]) studied the geometry of $F(3)$ and $F(4)$. In this paper we study the $F(5)$, $F(6)$ and $F(7)$ cases. We obtain the following families of $(1,2)$-symplectic invariant metrics, different to the Kähler and parabolic: On $F(5)$, two 5-parametric families; on $F(6)$, four 6-parametric families, two of them generalizing the two families on $F(5)$ and, on $F(7)$ we obtain eight 7-parametric families, four of them generalizing the four ones on $F(6)$.

These metrics are used to produce new examples of harmonic maps $\phi : M^2 \rightarrow F(n)$, applying the result of Lichnerowicz mentioned above.

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2. Preliminaries

A full flag manifold is defined by

\[ F(n) = \{(L_1, \ldots, L_n) : L_i \text{ is a subspace of } \mathbb{C}^n, \dim \mathbb{C}L_i = 1, \ L_i \perp L_j \}. \]
The unitary group $U(n)$ acts transitively on $F(n)$. Using this action we obtain an algebraic description for $F(n)$:

$$F(n) = \frac{U(n)}{T} = \frac{U(n)}{U(1) \times \cdots \times U(1)},$$

where $T = U(1) \times \cdots \times U(1)$ is a maximal torus in $U(n)$.

Let $p$ be the tangent space of $F(n)$ in $(T)$. The Lie algebra $u(n)$ is such that (see [ChE])

$$u(n) = \{X \in \text{Mat}(n, \mathbb{C}) : X + X^t = 0\}$$

$$= p \oplus u(1) \oplus \cdots \oplus u(1) \ (n\text{-times}) .$$

**Definition 2.1.** An invariant almost complex structure on $F(n)$ is a linear map $J : p \to p$ such that $J^2 = -I$.

**Example 2.1.** If we consider

$$F(3) = \frac{U(3)}{U(1) \times U(1) \times U(1)} = \frac{U(3)}{T},$$

in this case

$$p = T(F(3))(T) = \left\{ \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} : a, b, c, \in \mathbb{C} \right\} .$$

The following linear map is an example of a almost complex structure on $F(3)$

$$\begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & (-\sqrt{-1})a & (-\sqrt{-1})b \\ (\sqrt{-1})\bar{a} & 0 & (\sqrt{-1})\bar{c} \\ (-\sqrt{-1})\bar{b} & (\sqrt{-1})\bar{c} & 0 \end{pmatrix} .$$

There is a natural identification between almost complex structures on $F(n)$ and tournaments with $n$ players.

**Definition 2.2.** A tournament or $n$-tournament $\mathcal{T}$, consists of a finite set $T = \{p_1, p_2, \ldots, p_n\}$ of $n$ players, together with a dominance relation, $\to$, that assigns to every pair of players a winner, i.e. $p_i \to p_j$ or $p_j \to p_i$. If $p_i \to p_j$ then we say that $p_i$ beats $p_j$.

A tournament $\mathcal{T}$ may be represented by a directed graph in which $T$ is the set of vertices and any two vertices are joined by an oriented edge.

Let $\mathcal{T}_1$ be a tournament with $n$ players $\{1, \ldots, n\}$ and $\mathcal{T}_2$ another tournament with $m$ players $\{1, \ldots, m\}$. A homomorphism between $\mathcal{T}_1$ and $\mathcal{T}_2$ is a mapping $\phi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that

$$s \overset{\mathcal{T}_1}{\rightarrow} t \implies \phi(s) \overset{\mathcal{T}_2}{\rightarrow} \phi(t) \quad \text{or} \quad \phi(s) = \phi(t).$$
When \( \phi \) is bijective we said that \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are isomorphic.

An \( n \)-tournament determines a score vector

\[
(s_1, \ldots, s_n), \quad \text{such that} \quad \sum_{i=1}^{n} s_i = \binom{n}{2},
\]

with components equal the number of games won by each player. Isomorphic tournaments have identical score vectors. Figure 1 shows the isomorphism classes of \( n \)-tournaments for \( n = 2, 3, 4 \), together with their score vectors. For \( n \geq 5 \), there exist non-isomorphic \( n \)-tournaments with identical score vectors, see Figure 2. The canonical \( n \)-tournament \( \mathcal{T}_n \) is defined by setting \( i \to j \) if

\[
\text{FIGURE 1. Isomorphism classes of } n \text{-tournaments to } n = 2, 3, 4.
\]

and only if \( i < j \). Up to isomorphism, \( \mathcal{T}_n \) is the unique \( n \)-tournament satisfying the following equivalent conditions:

- the dominance relation is transitive, i.e. if \( i \to j \) and \( j \to k \) then \( i \to k \),
- there are no 3-cycles, i.e. closed paths \( i_1 \to i_2 \to i_3 \to i_1 \), see [M],
- the score vector is \( (0, 1, 2, \ldots, n - 1) \).

For each invariant almost complex structure \( J \) on \( F(n) \), we can associate a \( n \)-tournament \( \mathcal{T}(J) \) in the following way: If \( J(a_{ij}) = (a'_{ij}) \) then \( \mathcal{T}(J) \) is such that for \( i < j \)

\[
(2.6) \quad (i \to j \iff a'_{ij} = \sqrt{-1}a_{ij}) \quad \text{or} \quad (i \leftarrow j \iff a'_{ij} = -\sqrt{-1}a_{ij}),
\]

see [MN3].

**Example 2.2.** The tournament in the Figure 3 corresponds to the almost complex structure in the example 2.1
An almost complex structure $J$ on $F(n)$ is said to be integrable if $(F(n), J)$ is a complex manifold. An equivalent condition is the famous Newlander-Nirenberg equation (see [NN]):


**Figure 2.** Isomorphism classes of 5-tournaments.

**Figure 3.** Tournament of the example 2.2
for all tangent vectors $X, Y$.

Burstall and Salamon [BS] proved the following result:

**Theorem 2.1.** An almost complex structure $J$ on $F(n)$ is integrable if and only if $T(J)$ is isomorphic to the canonical tournament $T_n$.

Thus, if $T(J)$ contains a 3-cycle then $J$ is not integrable. The almost complex structure of example 2.1 is integrable.

An invariant almost complex structure $J$ on $F(n)$ is called parabolic if there is a permutation $\tau$ of $n$ elements such that the associate tournament $T(J)$ is given, for $i < j$, by

$$ (\tau(j) \rightarrow \tau(i), \quad \text{if } j - i \text{ is even} ) \quad \text{or} \quad (\tau(i) \rightarrow \tau(j), \quad \text{if } j - i \text{ is odd} ) $$

Classes (3) and (7) in Figure 1 and (12) in Figure 2 represent the parabolic structures on $F(3)$, $F(4)$ and $F(5)$ respectively.

A $n$-tournament $T$, for $n \geq 3$, is called irreducible or Hamiltonian if it contains a $n$-cycle, i.e. a path

$$ \pi(n) \rightarrow \pi(1) \rightarrow \pi(2) \rightarrow \cdots \rightarrow \pi(n-1) \rightarrow \pi(n), $$

where $\pi$ is a permutation of $n$ elements.

A $n$-tournament $T$ is transitive if given three nodes $i, j, k$ of $T$ then

$$ i \rightarrow j \quad \text{and} \quad j \rightarrow k \implies i \rightarrow k. $$

The canonical tournament is the only one transitive tournament up to isomorphisms.

We consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product, i.e. for $V = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we have

$$ \langle V, W \rangle = \sum_{i=1}^{n} v_i \overline{w_i}. $$

We use the convention

$$ \overline{v_i} = v_i \quad \text{and} \quad \overline{f_{ij}} = f_{ij}. $$

A frame consists of an ordered set of $n$ vectors $(Z_1, \ldots, Z_n)$, such that $Z_1 \wedge \cdots \wedge Z_n \neq 0$, and it is called unitary, if $\langle Z_i, Z_j \rangle = \delta_{ij}$. The set of unitary frames can be identified with the unitary group.

If we write

$$ dZ_i = \sum_j \omega_{ij} Z_j, $$

the coefficients $\omega_{ij}$ are the Maurer-Cartan forms of the unitary group $U(n)$. They are skew-Hermitian, i.e.

$$ \omega_{ij} + \omega_{ji} = 0, $$
and satisfy the equation
\begin{equation}
\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.
\end{equation}

For more details see [ChW].

We may define all left invariant metrics on \((F(n), J)\) by (see [Bl] or [N1])
\begin{equation}
ds_A^2 = \sum_{i,j} \lambda_{ij} \omega_{ij} \otimes \omega_{ij},
\end{equation}
where \(\Lambda = (\lambda_{ij})\) is a real matrix such that:
\begin{equation}
\begin{cases}
\lambda_{ij} > 0, & \text{if } i \neq j \\
\lambda_{ij} = 0, & \text{if } i = j
\end{cases}
\end{equation}
and the Maurer-Cartan forms \(\omega_{ij}\) are such that
\begin{equation}
\omega_{ij} \in C^{1,0} \text{ ((1,0) type forms)} \iff i \Rightarrow T(j) \Rightarrow j.
\end{equation}
Note that, if \(\lambda_{ij} = 1\) for all \(i, j\) in (2.13), then we obtain the normal metric (see [ChE]) induced by the Cartan-Killing form of \(U(n)\).

The metrics (2.13) are called Borel type and they are almost Hermitian for every invariant almost complex structure \(J\), i.e. \(ds_A^2(JX, JY) = ds_A^2(X, Y)\), for all tangent vectors \(X, Y\). When \(J\) is integrable \(ds_A^2\) is said to be Hermitian.

**Definition 2.3.** Let \(J\) be an invariant almost complex structure on \(F(n)\), \(T(J)\) the associated tournament, and \(ds_A^2\) an invariant metric. The Kähler form with respect to \(J\) and \(ds_A^2\) is defined by
\begin{equation}
\Omega(X, Y) = ds_A^2(X, JY),
\end{equation}
for any tangent vectors \(X, Y\).

For each permutation \(\tau\) of \(n\) elements, the Kähler form can be write in the following way (see [MN2])
\begin{equation}
\Omega = -2\sqrt{-1} \sum_{i<j} \mu_{\tau(i)\tau(j)} \omega_{\tau(i)\tau(j)} \wedge \omega_{\tau(i)\tau(j)},
\end{equation}
where
\begin{equation}
\mu_{\tau(i)\tau(j)} = \varepsilon_{\tau(i)\tau(j)} \lambda_{\tau(i)\tau(j)},
\end{equation}
and
\begin{equation}
\varepsilon_{ij} = \begin{cases}
1 & \text{if } i \rightarrow j \\
-1 & \text{if } j \rightarrow i \\
0 & \text{if } i = j
\end{cases}
\end{equation}

**Definition 2.4.** Let \(J\) be an invariant almost complex structure on \(F(n)\). Then \(F(n)\) is said to be almost Kähler if and only if \(\Omega\) is closed, i.e. \(d\Omega = 0\). If \(J\) is integrable and \(\Omega\) is closed then \(F(n)\) is said to be a Kähler manifold.

The following result was proved by Mo and Negreiros in [MN2].
Theorem 2.2.

\[(2.20) \quad d \Omega = 4 \sum_{i<j<k} C_{\tau(i) \tau(j) \tau(k)} \Psi_{\tau(i) \tau(j) \tau(k)}, \]

where

\[(2.21) \quad C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk}, \]

and

\[(2.22) \quad \Psi_{ijk} = \text{Im}(\omega_{ij} \wedge \omega_{ik} \wedge \omega_{jk}). \]

We denote by $\mathbb{C}^{p,q}$ the space of complex forms with degree $(p,q)$ on $F(n)$. Then, for any $i, j, k$, we have either

\[(2.23) \quad \Psi_{ijk} \in \mathbb{C}^{0,3} \oplus \mathbb{C}^{3,0} \quad \text{or} \quad \Psi_{ijk} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1}. \]

Definition 2.5. An invariant almost Hermitian metric $ds^2_\Lambda$ is said to be $(1,2)$-symplectic if and only if $(d\Omega)^{1,2} = 0$. If $d^*\Omega = 0$ then the metric is said to be cosymplectic.

Figure 4 is included in the known Salamon's paper [Sa] and it contains a classification of the almost Hermitian structures. This figure provides the following implications

\[
\text{Kähler} \quad \implies \quad (1,2)\text{-symplectic} \quad \implies \quad \text{cosymplectic}.
\]

For a complete classification see [GH].

The following result due to Mo and Negreiros [MN2], is very useful to study $(1,2)$-symplectic metrics on $F(n)$:

Theorem 2.3. If $J$ is a $U(n)$-invariant almost complex structure on $F(n)$, $n \geq 4$, such that $T(J)$ contains one of 4-tournaments in the Figure 5 then $J$ does not admit any invariant $(1,2)$-symplectic metric.

A smooth map $\phi: (M,g) \to (N,h)$ between two Riemannian manifolds is said to be harmonic if and only if it is a critical point of the energy functional

\[(2.24) \quad E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g, \]

where $|d\phi|$ is the Hilbert–Schmidt norm of the linear map $d\phi$, i.e. $\phi$ is harmonic if and only if it satisfies the Euler–Lagrange equations

\[(2.25) \quad \delta E(\phi) = \left. \frac{d}{dt} \right|_{t=0} E(\phi_t) = 0 \]

for all variation $(\phi_t)$ of $\phi$ and $t \in (-\varepsilon, \varepsilon)$ (see [EL]).
3. \((1, 2)\)-Symplectic Structures on \(F(3)\) and \(F(4)\)

It is known that, on \(F(3)\) there is a 2-parametric family of Kähler metrics and a 3-parametric family of \((1,2)\)-symplectic metrics corresponding to the non-integrable almost complex structures class. Then each invariant almost complex structure on \(F(3)\) admits a \((1,2)\)-symplectic metric, see [ESa], [Bo].
On \( F(4) \) there are four isomorphism classes of 4-tournaments or equivalently almost complex structures and the Theorem 2.3 shows that two of them do not admit (1,2)-symplectic metric. The another two classes corresponding to the Kähler and parabolic cases. \( F(4) \) has a 3-parametric family of Kähler metrics and a 4-parametric family of (1,2)-symplectic metrics which is not Kähler, see [MN2].

4. \((1,2)\)-Symplectic Structures on \( F(5) \)

Figure 2 shows the twelve isomorphism classes of 5-tournaments. The class (1) corresponds to the integrable complex structures and it contains the Kähler metrics. The other classes correspond to non-integrable almost complex structures, in particular the class (11) corresponds to the parabolic structure.

To the remain classes we have the following result:

**Theorem 4.1.** Between the classes of 5-tournaments (Figure 2), the only ones that admit \((1,2)\)-symplectic metrics, different to the Kähler and parabolic, are (7) and (9).

**Proof.** We use the Theorem 2.3 to prove that (2), (3), (4), (5), (6), (8), (10) and (11) do not admit \((1,2)\)-symplectic metric. It is easy to see that: (2) contains \( T_1 \) formed by the vertices 1,2,3,4; (3) contains \( T_1 \) formed by the vertices 2,3,4,5; (4) contains \( T_2 \) formed by the vertices 1,2,3,4; (5) contains \( T_2 \) formed by the vertices 2,3,4,5; (6) contains \( T_2 \) formed by the vertices 1,3,4,5; (8) contains \( T_2 \) formed by the vertices 2,3,4,5; (10) contains \( T_1 \) formed by the vertices 1,2,3,4 and (11) contains \( T_2 \) formed by the vertices 1,2,3,4. Then neither of them admit \((1,2)\)-symplectic metric.

Using formulas (2.20)-(2.23), we obtain that (7) admits \((1,2)\)-symplectic metric if and only if \( \Lambda = (\lambda_{ij}) \) satisfies the linear system

\[
\begin{align*}
\lambda_{12} - \lambda_{13} + \lambda_{23} & = 0 \\
\lambda_{12} - \lambda_{14} + \lambda_{24} & = 0 \\
\lambda_{13} - \lambda_{14} + \lambda_{34} & = 0 \\
\lambda_{23} - \lambda_{24} + \lambda_{34} & = 0 \\
\lambda_{23} - \lambda_{25} + \lambda_{35} & = 0 \\
\lambda_{24} - \lambda_{25} + \lambda_{45} & = 0 \\
\lambda_{34} - \lambda_{35} + \lambda_{45} & = 0
\end{align*}
\]

Then (7) admits \((1,2)\)-symplectic metric if and only if \( \Lambda = (\lambda_{ij}) \) satisfies

\[
\begin{align*}
\lambda_{13} & = \lambda_{12} + \lambda_{23} \\
\lambda_{14} & = \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{24} & = \lambda_{23} + \lambda_{34} \\
\lambda_{25} & = \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{35} & = \lambda_{34} + \lambda_{45}
\end{align*}
\]
Similarly, we obtain that (9) admit (1,2)-symplectic metric if and only if $\Lambda = (\Lambda_{ij})$ satisfies

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} \\
\lambda_{25} &= \lambda_{12} + \lambda_{15} \\
\lambda_{35} &= \lambda_{34} + \lambda_{45}
\end{align*}
\]

Now we can write the respective matrices

\[
\Lambda(7) = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{15} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} \\
\lambda_{15} & \lambda_{12} + \lambda_{15} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0
\end{pmatrix}
\]

\[
\Lambda(9) = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{15} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{15} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} \\
\lambda_{15} & \lambda_{12} + \lambda_{15} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0
\end{pmatrix}
\]

The Theorem 4.1 says that $F(n)$ admits (1,2)-symplectic metrics, different to the Kähler and parabolic, if and only if $n \geq 5$.

5. (1, 2)-Symplectic Structures on $F(6)$

There are 56 isomorphism classes of 6-tournaments (see [MI]), which are presented in Figures 6, 7 and 8. Again, the class (1) corresponds to the integrable complex structures. The other classes correspond to non-integrable almost complex structures, and the class (52) corresponds to the parabolic structure.

In this case we have the following result

**Theorem 5.1.** Between the classes of 6-tournaments (Figure 6, 7 and 8), the only ones that admit (1, 2)-symplectic metrics, different to the Kähler and parabolic, are (19), (31), (37) and (55).

**Proof.** We use the Theorem 2.3 to prove that each of the classes of 6-tournaments different to the (1), (19), (31), (37), (52) and (55) does not admit (1,2)-symplectic metrics:

- (2) contains $T_1$ formed by the vertices 1,2,3,4.
- (3) contains $T_2$ formed by the vertices 1,2,3,4.
- (4) contains $T_1$ formed by the vertices 1,2,3,5.
- (5) contains $T_2$ formed by the vertices 2,3,4,5.
Figure 6. Isomorphism classes of 6-tournaments

- (6) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (7) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
- (8) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
Figure 7. Isomorphism classes of 6-tournaments

- (9) contains $T_1$ formed by the vertices 1,2,3,4.
- (10) contains $T_1$ formed by the vertices 1,2,3,4.
- (11) contains $T_2$ formed by the vertices 1,2,3,4.
• (12) contains $\mathcal{T}_1$ formed by the vertices 2,3,5,6.
• (13) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
• (14) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
• (15) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
• (16) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
• (17) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
• (18) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
• (20) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
• (21) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
• (22) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,5.
• (23) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,5.
(24) contains \( T_2 \) formed by the vertices 1,2,3,4.
(25) contains \( T_2 \) formed by the vertices 1,2,3,4.
(26) contains \( T_2 \) formed by the vertices 3,4,5,6.
(27) contains \( T_2 \) formed by the vertices 2,3,4,5.
(28) contains \( T_2 \) formed by the vertices 3,4,5,6.
(29) contains \( T_2 \) formed by the vertices 2,3,4,5.
(30) contains \( T_2 \) formed by the vertices 2,3,4,5.
(32) contains \( T_1 \) formed by the vertices 1,2,3,4.
(33) contains \( T_2 \) formed by the vertices 3,4,5,6.
(34) contains \( T_2 \) formed by the vertices 3,4,5,6.
(35) contains \( T_2 \) formed by the vertices 2,3,4,5.
(36) contains \( T_2 \) formed by the vertices 1,2,3,4.
(38) contains \( T_1 \) formed by the vertices 3,4,5,6.
(39) contains \( T_2 \) formed by the vertices 1,2,3,4.
(40) contains \( T_1 \) formed by the vertices 3,4,5,6.
(41) contains \( T_1 \) formed by the vertices 3,4,5,6.
(42) contains \( T_2 \) formed by the vertices 1,2,3,6.
(43) contains \( T_1 \) formed by the vertices 3,4,5,6.
(44) contains \( T_1 \) formed by the vertices 3,4,5,6.
(45) contains \( T_2 \) formed by the vertices 1,2,3,4.
(46) contains \( T_1 \) formed by the vertices 2,3,5,6.
(47) contains \( T_2 \) formed by the vertices 1,3,4,6.
(48) contains \( T_2 \) formed by the vertices 2,3,4,5.
(49) contains \( T_2 \) formed by the vertices 1,2,3,4.
(50) contains \( T_2 \) formed by the vertices 1,2,3,4.
(51) contains \( T_2 \) formed by the vertices 1,3,5,6.
(53) contains \( T_1 \) formed by the vertices 1,2,4,6.
(54) contains \( T_2 \) formed by the vertices 1,2,4,5.
(56) contains \( T_1 \) formed by the vertices 1,2,4,6.

By making similar computations to we made in the proof of Theorem 4.1 we obtain:

- The class (19) admits (1,2)-symplectic metric if and only if the elements of corresponding matrix \( \Lambda_{(19)} = (\lambda_{ij}) \) satisfy the following system of linear equations:

\[
\begin{align*}
\lambda_{12} - \lambda_{13} + \lambda_{23} &= 0 \\
\lambda_{12} - \lambda_{15} + \lambda_{25} &= 0 \\
\lambda_{13} - \lambda_{15} + \lambda_{35} &= 0 \\
\lambda_{23} - \lambda_{24} + \lambda_{34} &= 0 \\
\lambda_{23} - \lambda_{26} + \lambda_{36} &= 0 \\
\lambda_{34} - \lambda_{35} + \lambda_{45} &= 0 \\
\lambda_{35} - \lambda_{36} + \lambda_{56} &= 0
\end{align*}
\]

\[
\begin{align*}
\lambda_{12} - \lambda_{14} + \lambda_{24} &= 0 \\
\lambda_{13} - \lambda_{14} + \lambda_{34} &= 0 \\
\lambda_{14} - \lambda_{15} + \lambda_{45} &= 0 \\
\lambda_{23} - \lambda_{25} + \lambda_{35} &= 0 \\
\lambda_{24} - \lambda_{25} + \lambda_{45} &= 0 \\
\lambda_{25} - \lambda_{26} + \lambda_{56} &= 0 \\
\lambda_{34} - \lambda_{36} + \lambda_{46} &= 0 \\
\lambda_{45} - \lambda_{46} + \lambda_{56} &= 0.
\end{align*}
\]
Then the metric $ds^2_{\Lambda_{(19)}}$ is (1,2)-symplectic if and only if

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} & \lambda_{26} &= \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{35} &= \lambda_{34} + \lambda_{45} \\
\lambda_{15} &= \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{36} &= \lambda_{34} + \lambda_{45} + \lambda_{56} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} & \lambda_{46} &= \lambda_{45} + \lambda_{56} \\
\lambda_{25} &= \lambda_{23} + \lambda_{34} + \lambda_{45}.
\end{align*}
\]

- In similar way the class (31) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(31)} = (\lambda_{ij})$ satisfy the following relations

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{23} & \lambda_{26} &= \lambda_{12} + \lambda_{16} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{35} &= \lambda_{34} + \lambda_{45} \\
\lambda_{15} &= \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{36} &= \lambda_{34} + \lambda_{45} + \lambda_{56} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} & \lambda_{46} &= \lambda_{45} + \lambda_{56} \\
\lambda_{25} &= \lambda_{23} + \lambda_{34} + \lambda_{45}.
\end{align*}
\]

- Similarly, the class (37) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(37)} = (\lambda_{ij})$ satisfy the following relations

\[
\begin{align*}
\lambda_{14} &= \lambda_{12} + \lambda_{25} + \lambda_{45} & \lambda_{26} &= \lambda_{25} + \lambda_{45} + \lambda_{46} \\
\lambda_{15} &= \lambda_{12} + \lambda_{25} & \lambda_{34} &= \lambda_{36} + \lambda_{46} \\
\lambda_{16} &= \lambda_{12} + \lambda_{25} + \lambda_{45} + \lambda_{46} & \lambda_{35} &= \lambda_{12} + \lambda_{13} + \lambda_{25} \\
\lambda_{23} &= \lambda_{12} + \lambda_{13} & \lambda_{56} &= \lambda_{45} + \lambda_{46} \\
\lambda_{24} &= \lambda_{25} + \lambda_{45}.
\end{align*}
\]

- Finally, the class (55) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(55)} = (\lambda_{ij})$ satisfy the following relations

\[
\begin{align*}
\lambda_{13} &= \lambda_{12} + \lambda_{25} + \lambda_{35} & \lambda_{26} &= \lambda_{12} + \lambda_{14} + \lambda_{46} \\
\lambda_{15} &= \lambda_{12} + \lambda_{25} & \lambda_{34} &= \lambda_{36} + \lambda_{46} \\
\lambda_{16} &= \lambda_{14} + \lambda_{46} & \lambda_{45} &= \lambda_{35} + \lambda_{36} + \lambda_{46} \\
\lambda_{23} &= \lambda_{25} + \lambda_{35} & \lambda_{56} &= \lambda_{35} + \lambda_{36} \\
\lambda_{24} &= \lambda_{12} + \lambda_{14}.
\end{align*}
\]

The matrices $\Lambda_{(19)}, \Lambda_{(31)}, \Lambda_{(37)}$ and $\Lambda_{(55)}$ corresponding to the classes (19), (31), (37) and (55) are presented on the end of this paper.

6. (1, 2)-Symplectic Structures on $F(7)$

This case has a problem because it is not known any collection of tournament drawings for $n \geq 7$. The collection of tournaments drawings of $n = 2, 3, 4, 5, 6$, is contained in the Moon's book [M].
There are 456 isomorphism classes of 7-tournaments. In the Dias's M. Sc. Thesis [D] was obtained a representant matrix of each class of 7-tournament. The matrix $M(T) = (a_{ij})$ of the tournament $T$ is defined by

$$a_{ij} = \begin{cases} 0, & \text{if } j \not\rightarrow i \\ 1, & \text{if } i \not\rightarrow j. \end{cases}$$

Obviously, it has the matrix is equivalent to have the tournament drawing.

We used the matrices generated in [D] together with the Digraph computer program, created by Professor Davide Carlo Demaria, in order to know which 7-tournaments contain the tournaments in Figure 5. Table 1 shows the matrices of the 7-tournaments which admit $(1,2)$-symplectic metric. Using the matrices in the Table 1 we construct the 7-tournament drawings which admit $(1,2)$-symplectic metric. Figures 9 and 10 show this 7-tournaments. Class (1) in the Figure 9 represents the integrable structures and the class (10) in Figure 10 corresponds to the parabolic structures. To the remain classes we have the following result.

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**Table 1.** Matrices of the 7-tournaments which admit $(1,2)$-symplectic metric
Theorem 6.1. The classes of 7-tournaments (2) through (9) in the Figures 9 and 10 admit (1,2)-symplectic metrics, different to the Kähler and parabolic.
\textbf{Proof.} The proof is made through a long calculation similar to the proof of Theorem 4.1.

The matrices $\Lambda_{(2)}$ through $\Lambda_{(9)}$ corresponding to the classes (2) through (9) are presented on the end of this paper.

Wolf and Gray [WG] proved that the normal metric on $F(n)$ is not (1,2)-symplectic for $n \geq 4$. Our results give a simple proof of this fact to $n = 5, 6, 7$.

\section{7. Harmonic Maps}

In this section we construct new examples of harmonic maps using the following result due to Lichnerowicz [L]:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Isomorphism classes of 7-tournaments which admit (1,2)-symplectic metric}
\end{figure}
Theorem 7.1. Let \( \phi: (M, g, J_1) \rightarrow (N, h, J_2) \) be a \( \pm \) holomorphic map between almost Hermitian manifolds where \( M \) is cosymplectic and \( N \) is \((1,2)\)-symplectic. Then \( \phi \) is harmonic. (\( \phi \) is \( \pm \) holomorphic if and only if \( d\phi \circ J_1 = \pm J_2 \circ d\phi \)).

In order to construct harmonic maps \( \phi: M^2 \rightarrow F(n) \) using the theorem above, we need to know examples of holomorphic maps. Then we use the following construction due to Eells and Wood [EW].

Let \( h: M^2 \rightarrow \mathbb{CP}^{n-1} \) be a full holomorphic map (\( h \) is full if \( h(M) \) is not contained in none \( \mathbb{CP}^k \), for all \( k < n-1 \)). We can lift \( h \) to \( \mathbb{C}^n \), i.e. for every \( p \in M \) we can find a neighborhood of \( p, U \subset M \), such that \( h_U = (u_0, \ldots, u_{n-1}) : M^2 \cup U \rightarrow \mathbb{C}^n - 0 \) satisfies \( h(z) = [h_U(z)] = [(u_0(z), \ldots, u_{n-1}(z))] \).

We define the \( k \)-th associate curve of \( h \) by

\[
O_k : M^2 \rightarrow \mathbb{G}_{k+1}(\mathbb{C}^n)
\]

\[
z \mapsto h_U(z) \wedge \partial h_U(z) \wedge \cdots \wedge \partial^k h_U(z),
\]

for \( 0 \leq k \leq n - 1 \). And we consider

\[
h_k : M^2 \rightarrow \mathbb{CP}^{n-1}
\]

\[
z \mapsto O_k^\perp(z) \cap O_{k+1}(z),
\]

for \( 0 \leq k \leq n - 1 \).

The following theorem, due to Eells and Wood ([EW]), is very important because it gives the classification of the harmonic maps from \( S^2 \sim \mathbb{CP}^1 \) into a projective space \( \mathbb{CP}^{n-1} \).

Theorem 7.2. For each \( k \in \mathbb{N}, 0 \leq k \leq n - 1 \), \( h_k \) is harmonic. Furthermore, given \( \phi: (\mathbb{CP}^1, g) \rightarrow (\mathbb{CP}^{n-1}, \text{Killing metric}) \) a full harmonic map, then there are unique \( k \) and \( h \) such that \( \phi = h_k \).

This theorem provides in a natural way the following holomorphic maps

\[
\Psi : M^2 \rightarrow F(n)
\]

\[
z \mapsto (h_0(z), \ldots, h_{n-1}(z)),
\]
called by Eells–Wood’s map (see [N2]).

We called \( \mathfrak{M}_n \) the set of \((1,2)\)-symplectic metrics on \( F(n) \), for \( n = 5, 6 \) and \( 7 \) characterized in the sections above. Using Theorem 7.1 we obtain the following result

Theorem 7.3. Let \( \phi: M^2 \rightarrow (F(n), g), g \in \mathfrak{M} \) a holomorphic map. Then \( \phi \) is harmonic.

In addition for maps from a flag manifold into a flag manifold we obtain the following result

Proposition 7.1. Let \( \phi: (F(l), g) \rightarrow (F(k), h) \) a holomorphic map, with \( g \in \mathfrak{M}_l \) and \( h \in \mathfrak{M}_k \). Then \( \phi \) is harmonic.
\[ \begin{array}{cccc}
\lambda_{16} & \lambda_{23} + \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{45} + \lambda_{56} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{45} + \lambda_{56} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & \lambda_{34} + \lambda_{45} & \lambda_{45} + \lambda_{56} \\
0 & \lambda_{12} + \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{34} + \lambda_{45} + \lambda_{56} \\
0 & \lambda_{12} + \lambda_{23} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{45} + \lambda_{56} \\
0 & \lambda_{12} + \lambda_{23} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} \\
\end{array} \]

\[ \Lambda(19) \]
\[
\begin{array}{cccccccc}
\lambda_{16} & & & & & & & \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{45} & \lambda_{45} + \lambda_{56} & 0 & \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & 0 & \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & & \\
0 & \lambda_{12} + \lambda_{23} & 0 & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{16} & \\
\end{array}
\]
\[
\begin{array}{cccccc}
\lambda_{12} + \lambda_{25} & \lambda_{25} + \lambda_{46} & \lambda_{35} + \lambda_{46} & \lambda_{56} + \lambda_{46} & 0 \\
\lambda_{12} + \lambda_{25} & \lambda_{25} + \lambda_{13} & \lambda_{25} + \lambda_{46} & \lambda_{45} & 0 \\
\lambda_{12} & 0 & \lambda_{36} + \lambda_{46} & \lambda_{12} + \lambda_{13} & \lambda_{12} + \lambda_{25} \\
\lambda_{12} & \lambda_{25} + \lambda_{46} & \lambda_{25} + \lambda_{13} & \lambda_{12} + \lambda_{25} & \lambda_{12} + \lambda_{25} + \lambda_{46} \\
0 & \lambda_{12} + \lambda_{25} & \lambda_{12} + \lambda_{25} & \lambda_{12} + \lambda_{25} + \lambda_{46} & \lambda_{12} + \lambda_{25} + \lambda_{46} \\
\lambda_{(37)} & = & & & & \\
\end{array}
\]
\[
\begin{array}{cccccccc}
0 & \lambda_{12} & \lambda_{12} + \lambda_{25} & \lambda_{25} + \lambda_{35} & \lambda_{35} + \lambda_{46} & \lambda_{35} + \lambda_{36} & 0 & \lambda_{35} + \lambda_{36} \\
\lambda_{14} & 0 & \lambda_{25} + \lambda_{35} & \lambda_{36} + \lambda_{46} & \lambda_{35} + \lambda_{36} & \lambda_{35} + \lambda_{36} & 0 & \lambda_{36} \\
\lambda_{14} + \lambda_{46} & \lambda_{12} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{36} + \lambda_{46} & 0 & \lambda_{35} + \lambda_{36} + \lambda_{46} & \lambda_{46} \\
\lambda_{14} + \lambda_{46} & \lambda_{12} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{12} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{36} + \lambda_{46} & 0 & \lambda_{35} + \lambda_{36} + \lambda_{46} \\
\lambda_{14} + \lambda_{46} & 0 & \lambda_{25} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{36} + \lambda_{46} & 0 & \lambda_{35} + \lambda_{36} + \lambda_{46} & \lambda_{46} \\
\lambda_{14} + \lambda_{46} & \lambda_{12} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{12} + \lambda_{14} & \lambda_{25} + \lambda_{14} & \lambda_{36} + \lambda_{46} & \lambda_{35} + \lambda_{36} + \lambda_{46} & 0 \\
\end{array}
\]
\[
\Lambda_{(2)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} + \lambda_{67} \\
\lambda_{17} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix}
\]
\[
\Lambda_{(3)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{34} + \lambda_{45} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & 0 \\
\end{pmatrix}
\]
\[ \Lambda_{(4)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{17} + \lambda_{23} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{56} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{56} + \lambda_{67} & \lambda_{17} + \lambda_{67}
\end{pmatrix} \]
\[
\begin{array}{cccccc}
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} & 0 & 0 & 0 \\
0 & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & 0 & \lambda_{23} + \lambda_{34} + \lambda_{45} \\
0 & 0 & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} \\
0 & 0 & 0 & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} \\
0 & 0 & 0 & 0 & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} \\
0 & 0 & 0 & 0 & 0 & \lambda_{45} + \lambda_{56} \\
\end{array}
\]
\[ \Lambda_{(6)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{67} \\
\lambda_{17} + \lambda_{67} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix} \]
\[
\lambda_{(7)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{17} + \lambda_{56} & \lambda_{12} + \lambda_{17} + \lambda_{56} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{67} \\
\lambda_{17} + \lambda_{67} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{23} + \lambda_{67} & \lambda_{12} + \lambda_{17} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix}
\]
\[ \Lambda(8) = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12} + \lambda_{23} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} & \lambda_{17} + \lambda_{67} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{23} + \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} & \lambda_{12} + \lambda_{17} & \lambda_{12} + \lambda_{17} \\
\lambda_{12} + \lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} & \lambda_{23} + \lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{12} + \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{23} + \lambda_{34} + \lambda_{45} & \lambda_{34} + \lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} + \lambda_{67} \\
\lambda_{17} + \lambda_{67} & \lambda_{12} + \lambda_{17} & \lambda_{34} + \lambda_{45} + \lambda_{56} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} & 0 & \lambda_{67} & \lambda_{67} \\
\lambda_{17} & \lambda_{12} + \lambda_{17} + \lambda_{56} + \lambda_{67} & \lambda_{34} + \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{45} + \lambda_{56} + \lambda_{67} & \lambda_{56} + \lambda_{67} + \lambda_{67} & \lambda_{67} & 0
\end{pmatrix} \]
References


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