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Some results on the geometry of full flag manifolds and harmonic maps

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ABSTRACT. In this note we study, for $n = 5, 6, 7$, the geometry of the full flag manifolds, $F(n) = \frac{U(n)}{U(1)\times\cdots\times U(1)}$. By using tournaments we characterize all of the (1,2)-symplectic invariant metrics on $F(n)$, for $n = 5, 6, 7$, corresponding to different classes of non-integrable invariant almost complex structure.

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1. Introduction

Eells and Sampson [ES], proved that if $\phi\colon M \to N$ is a holomorphic map between Kähler manifolds then ϕ is harmonic. This result was generalized by Lichnerowicz (see [L] or [Sa]) as follows: Let (M, g, J_1) and (N, h, J_2) be almost Hermitian manifolds with M cosymplectic and N (1,2)-symplectic. Then any $\pm$ holomorphic map $\phi\colon (M, J_1) \to (N, J_2)$ is harmonic.

We are interested to study harmonic maps, $\phi\colon M^2 \to F(n)$, from a closed Riemannian surface M^2 to a full flag manifold $F(n)$. Then by the Lichnerowicz theorem, we must study (1,2)-symplectic metrics on $F(n)$, because a Riemannian surface is a Kähler manifold and a Kähler manifold is a cosymplectic manifold (see [Sa] or [GH]).

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The study of invariant metrics on $F(n)$ involves almost complex structures on $F(n)$. Borel and Hirzebruch [BH], proved that there are $2^{\binom{n}{2}}$ $U(n)$-invariant almost complex structures on $F(n)$. This number is the same number of tournaments with n players or nodes. A tournament is a digraph in which any two nodes are joined by exactly one oriented edge (see [M] or [BS]). There is a natural identification between almost complex structures on $F(n)$ and tournaments with n players, see [MN3] or [BS].

The tournaments can be classified in isomorphism classes. In that classification, one of this classes corresponds to the integrable structures and the another ones correspond to non-integrable structures. Burstall and Salamon [BS], proved that a almost complex structure J on $F(n)$ is integrable if and only if the associated tournament to J is isomorphic to the canonical tournament (the canonical tournament with n players, $\{1, 2, \dots, n\}$, is defined by $i \to j$ if and only if $i < j$). In that paper the identification between almost complex structures and tournaments plays a very important role.

Borel [Bo], proved that exits a $(n-1)$-dimensional family of invariant Kähler metrics on $F(n)$ for each invariant complex structure on $F(n)$. Eells and Salamon [ESa], proved that any parabolic structure on $F(n)$ admits a (1,2)-symplectic metric. Mo and Negreiros [MN2], showed explicitly that there is a n-dimensional family of invariant (1,2)-symplectic metrics for each parabolic structure on $F(n)$, the identification between almost complex structures and tournaments is strongly used in that paper.

Mo and Negreiros ([MN1], [MN2]) studied the geometry of $F(3)$ and $F(4)$. In this paper we study the $F(5)$, $F(6)$ and $F(7)$ cases. We obtain the following families of (1,2)-symplectic invariant metrics, different to the Kähler and parabolic: On $F(5)$, two 5-parametric families; on $F(6)$, four 6-parametric families, two of them generalizing the two families on $F(5)$ and, on $F(7)$ we obtain eight 7-parametric families, four of them generalizing the four ones on $F(6)$.

These metrics are used to produce new examples of harmonic maps $\phi\colon M^2 \to F(n)$, applying the result of Lichnerowicz mentioned above.

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2. Preliminaries

A full flag manifold is defined by

$$F(n) = \{(L_1, \dots, L_n) : L_i \text{ is a subspace of } \mathbb{C}^n, \dim_{\mathbb{C}} L_i = 1, \ L_i \perp L_j\}. \tag{2.1}$$

The unitary group $U(n)$ acts transitively on $F(n)$. Using this action we obtain an algebraic description for $F(n)$:

$$F(n) = \frac{U(n)}{T} = \frac{U(n)}{\underbrace{U(1) \times \cdots \times U(1)}_{n-times}} , \tag{2.2}$$

where $T = \underbrace{U(1) \times \cdots \times U(1)}_{n-times}$ is a maximal torus in $U(n)$.

Let $\mathfrak{p}$ be the tangent space of $F(n)$ in (T). The Lie algebra $\mathfrak{u}(n)$ is such that (see [ChE])

$$\begin{aligned} \mathfrak{u}(n) &= \{X \in Mat(n, \mathbb{C}) : X + \overline{X}^t = 0\} \\ &= \mathfrak{p} \oplus \underbrace{\mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)}_{n-times} . \end{aligned} \tag{2.3}$$

Definition 2.1. *An invariant almost complex structure on $F(n)$ is a linear map $J \colon \mathfrak{p} \to \mathfrak{p}$ such that $J^2 = -I$.*

Example 2.1. *If we consider*

$$F(3) = \frac{U(3)}{U(1) \times U(1) \times U(1)} = \frac{U(3)}{T},$$

in this case

$$\mathfrak{p} = T(F(3))_{(T)} = \left\{ \begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} : a, b, c, \in \mathbb{C} \right\}.$$

The following linear map is an example of a almost complex structure on $F(3)$

$$\begin{pmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & (-\sqrt{-1})a & (-\sqrt{-1})b \\ (-\sqrt{-1})\bar{a} & 0 & (\sqrt{-1})c \\ (-\sqrt{-1})\bar{b} & (\sqrt{-1})\bar{c} & 0 \end{pmatrix}.$$

There is a natural identification between almost complex structures on $F(n)$ and tournaments with n players.

Definition 2.2. *A tournament or n-tournament $\mathcal{T}$, consists of a finite set $T = \{p_1, p_2, \ldots, p_n\}$ of n players, together with a dominance relation, $\to$, that assigns to every pair of players a winner, i.e. $p_i \to p_j$ or $p_j \to p_i$. If $p_i \to p_j$ then we say that p_i beats p_j.*

A tournament $\mathcal{T}$ may be represented by a directed graph in which T is the set of vertices and any two vertices are joined by an oriented edge.

Let $\mathcal{T}_1$ be a tournament with n players $\{1, \ldots, n\}$ and $\mathcal{T}_2$ another tournament with m players $\{1, \ldots, m\}$. A homomorphism between $\mathcal{T}_1$ and $\mathcal{T}_2$ is a mapping $\phi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that

$$s \xrightarrow{\mathcal{T}_1} t \quad \Longrightarrow \quad \phi(s) \xrightarrow{\mathcal{T}_2} \phi(t) \quad \text{or} \quad \phi(s) = \phi(t). \tag{2.4}$$

When ϕ is bijective we said that $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic.

An n-tournament determines a score vector

$$(2.5)\qquad (s_1, \ldots, s_n), \qquad \text{such that} \qquad \sum_{i=1}^{n} s_i = \binom{n}{2},$$

with components equal the number of games won by each player. Isomorphic tournaments have identical score vectors. Figure 1 shows the isomorphism classes of n-tournaments for $n = 2, 3, 4$, together with their score vectors. For $n \geq 5$, there exist non-isomorphic n-tournaments with identical score vectors, see Figure 2. The canonical n-tournament $\mathcal{T}_n$ is defined by setting $i \to j$ if

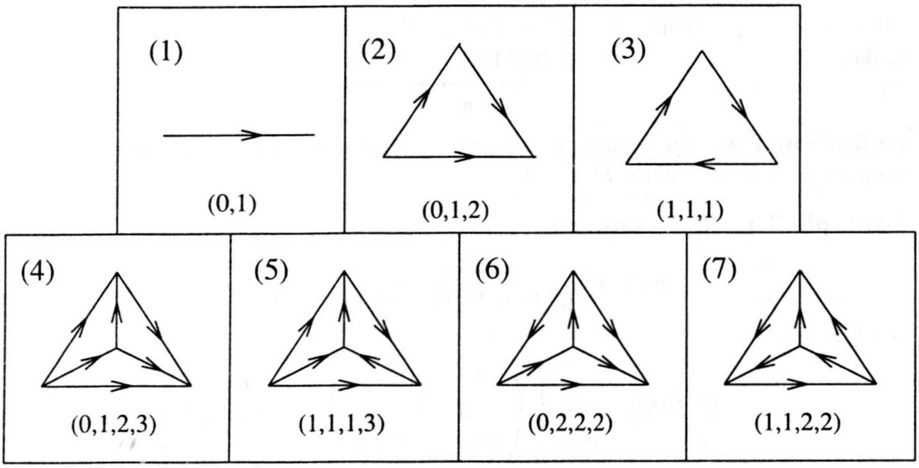

FIGURE 1. Isomorphism classes of n-tournaments to $n = 2, 3, 4$.

and only if $i < j$. Up to isomorphism, $\mathcal{T}_n$ is the unique n-tournament satisfying the following equivalent conditions:

- the dominance relation is transitive, i.e. if $i \to j$ and $j \to k$ then $i \to k$,
- there are no 3-cycles, i.e. closed paths $i_1 \to i_2 \to i_3 \to i_1$, see [M],
- the score vector is $(0, 1, 2, \ldots, n-1)$.

For each invariant almost complex structure J on $F(n)$, we can associate a n-tournament $\mathcal{T}(J)$ in the following way: If $J(a_{ij}) = (a'_{ij})$ then $\mathcal{T}(J)$ is such that for $i < j$

$$(2.6)\qquad \left(i \to j \;\Leftrightarrow\; a'_{ij} = \sqrt{-1}\, a_{ij}\right) \quad \text{or} \quad \left(i \leftarrow j \;\Leftrightarrow\; a'_{ij} = -\sqrt{-1}\, a_{ij}\right),$$

see [MN3].

Example 2.2. *The tournament in the Figure 3 corresponds to the almost complex structure in the example 2.1*

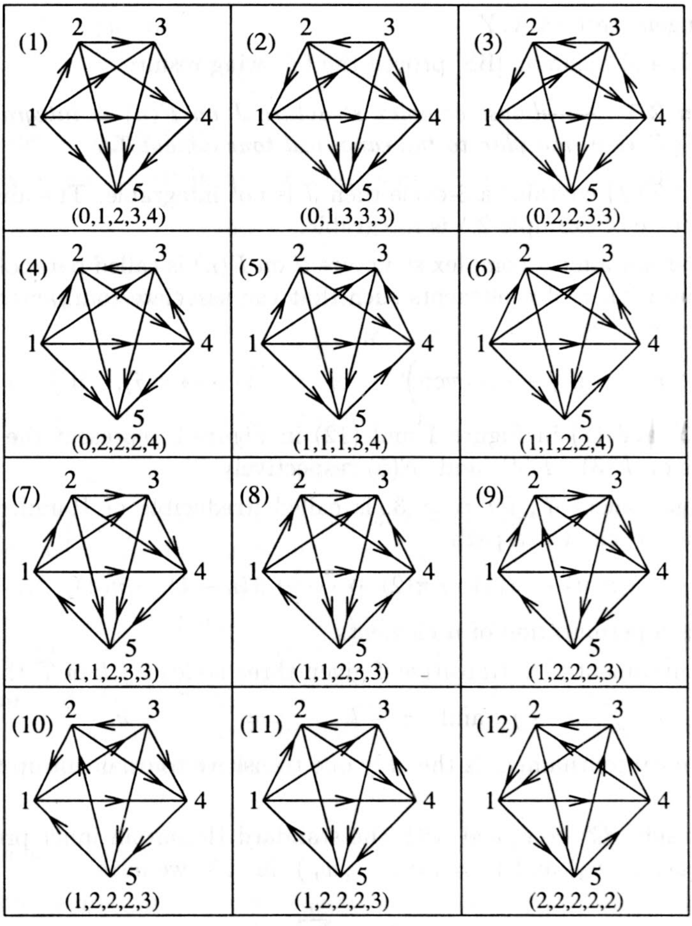

FIGURE 2. Isomorphism classes of 5-tournaments.

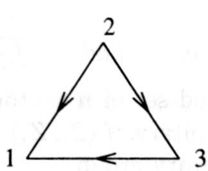

FIGURE 3. Tournament of the example 2.2

An almost complex structure J on $F(n)$ is said to be integrable if $(F(n), J)$ is a complex manifold. An equivalent condition is the famous Newlander-Nirenberg equation (see [NN]):

$$[JX, JY] = J[X, JY] + J[JX, Y] + [X, Y]. \tag{2.7}$$

for all tangent vectors X, Y.

Burstall and Salamon [BS] proved the following result:

Theorem 2.1. *An almost complex structure J on $F(n)$ is integrable if and only if $\mathcal{T}(J)$ is isomorphic to the canonical tournament $\mathcal{T}_n$.*

Thus, if $\mathcal{T}(J)$ contains a 3-cycle then J is not integrable. The almost complex structure of example 2.1 is integrable.

An invariant almost complex structure J on $F(n)$ is called parabolic if there is a permutation τ of n elements such that the associate tournament $\mathcal{T}(J)$ is given, for $i < j$, by

$$\Big(\tau(j) \to \tau(i), \quad \text{if } j - i \text{ is even}\Big) \qquad \text{or} \qquad \Big(\tau(i) \to \tau(j), \quad \text{if } j - i \text{ is odd}\Big)$$

Classes (3) and (7) in Figure 1 and (12) in Figure 2 represent the parabolic structures on $F(3)$, $F(4)$ and $F(5)$ respectively.

A n-tournament $\mathcal{T}$, for $n \geq 3$, is called irreducible or Hamiltonian if it contains a n-cycle, i.e. a path

$$\pi(n) \to \pi(1) \to \pi(2) \to \cdots \to \pi(n-1) \to \pi(n),$$

where π is a permutation of n elements.

A n-tournament $\mathcal{T}$ is transitive if given three nodes i, j, k of $\mathcal{T}$ then

$$i \to j \quad \text{and} \quad j \to k \qquad \Longrightarrow \qquad i \to k.$$

The canonical tournament is the only one transitive tournament up to isomorphisms.

We consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product, i.e. for $V = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we have

$$\langle V, W \rangle = \sum_{i=1}^{n} v_i \overline{w_i}. \tag{2.8}$$

We use the convention

$$\overline{v_i} = v_{\bar{\imath}} \qquad \text{and} \qquad \overline{f_{i\bar{\jmath}}} = f_{\bar{\imath}j}. \tag{2.9}$$

A frame consists of an ordered set of n vectors $(Z_1, \ldots, Z_n)$, such that $Z_1 \wedge \ldots \wedge Z_n \neq 0$, and it is called unitary, if $\langle Z_i, Z_j \rangle = \delta_{i\bar{\jmath}}$. The set of unitary frames can be identified with the unitary group.

If we write

$$dZ_i = \sum_j \omega_{i\bar{\jmath}} Z_j, \tag{2.10}$$

the coefficients $\omega_{i\bar{\jmath}}$ are the Maurer-Cartan forms of the unitary group $U(n)$. They are skew-Hermitian, i.e.

$$\omega_{i\bar{\jmath}} + \omega_{\bar{\jmath}i} = 0, \tag{2.11}$$

and satisfy the equation

$$d\omega_{i\bar{j}} = \sum_k \omega_{i\bar{k}} \wedge \omega_{k\bar{j}}. \tag{2.12}$$

For more details see [ChW].

We may define all left invariant metrics on $(F(n), J)$ by (see [Bl] or [N1])

$$ds^2_\Lambda = \sum_{i,j} \lambda_{ij}\omega_{i\bar{j}} \otimes \omega_{\bar{i}j}, \tag{2.13}$$

where $\Lambda = (\lambda_{ij})$ is a real matrix such that:

$$\begin{cases} \lambda_{ij} > 0, & \text{if } \ i \neq j \\ \lambda_{ij} = 0, & \text{if } \ i = j \end{cases}, \tag{2.14}$$

and the Maurer-Cartan forms $\omega_{i\bar{j}}$ are such that

$$\omega_{i\bar{j}} \in \mathbb{C}^{1,0} \ \text{((1,0) type forms)} \quad \Longleftrightarrow \quad i \overset{\mathcal{T}(J)}{\longrightarrow} j. \tag{2.15}$$

Note that, if $\lambda_{ij} = 1$ for all i, j in (2.13), then we obtain the normal metric (see [ChE]) induced by the Cartan-Killing form of $U(n)$.

The metrics (2.13) are called Borel type and they are almost Hermitian for every invariant almost complex structure J, i.e. $ds^2_\Lambda(JX, JY) = ds^2_\Lambda(X, Y)$, for all tangent vectors X, Y. When J is integrable ds^2_Λ is said to be Hermitian.

Definition 2.3. *Let J be an invariant almost complex structure on $F(n)$, $\mathcal{T}(J)$ the associated tournament, and ds^2_Λ an invariant metric. The Kähler form with respect to J and ds^2_Λ is defined by*

$$\Omega(X, Y) = ds^2_\Lambda(X, JY), \tag{2.16}$$

for any tangent vectors X, Y.

For each permutation τ, of n elements, the Kähler form can be write in the following way (see [MN2])

$$\Omega = -2\sqrt{-1} \sum_{i<j} \mu_{\tau(i)\tau(j)}\omega_{\tau(i)\overline{\tau(j)}} \wedge \omega_{\overline{\tau(i)}\tau(j)}, \tag{2.17}$$

where

$$\mu_{\tau(i)\tau(j)} = \varepsilon_{\tau(i)\tau(j)}\lambda_{\tau(i)\tau(j)}, \tag{2.18}$$

and

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } \ i \to j \\ -1 & \text{if } \ j \to i \\ 0 & \text{if } \ i = j \end{cases} \tag{2.19}$$

Definition 2.4. *Let J be an invariant almost complex structure on $F(n)$. Then $F(n)$ is said to be almost Kähler if and only if Ω is closed, i.e. $d\Omega = 0$. If J is integrable and Ω is closed then $F(n)$ is said to be a Kähler manifold.*

The following result was proved by Mo and Negreiros in [MN2].

Theorem 2.2.

$$d\Omega = 4 \sum_{i<j<k} C_{\tau(i)\tau(j)\tau(k)} \Psi_{\tau(i)\tau(j)\tau(k)}, \tag{2.20}$$

where

$$C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk}, \tag{2.21}$$

and

$$\Psi_{ijk} = Im(\omega_{i\bar{j}} \wedge \omega_{\bar{i}k} \wedge \omega_{j\bar{k}}). \tag{2.22}$$

We denote by $\mathbb{C}^{p,q}$ the space of complex forms with degree (p, q) on $F(n)$. Then, for any i, j, k, we have either

$$\Psi_{ijk} \in \mathbb{C}^{0,3} \oplus \mathbb{C}^{3,0} \quad \text{or} \quad \Psi_{ijk} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1} \tag{2.23}$$

Definition 2.5. *An invariant almost Hermitian metric ds^2_Λ is said to be $(1,2)$-symplectic if and only if $(d\Omega)^{1,2} = 0$. If $d^*\Omega = 0$ then the metric is said to be cosymplectic.*

Figure 4 is included in the known Salamon's paper [Sa] and it contains a classification of the almost Hermitian structures. This figure provides the following implications

Kähler $\Longrightarrow$ (1,2)-symplectic $\Longrightarrow$ cosymplectic.

For a complete classification see [GH].

The following result due to Mo and Negreiros [MN2], is very useful to study (1,2)-symplectic metrics on $F(n)$:

Theorem 2.3. *If J is a $U(n)$-invariant almost complex structure on $F(n)$, $n \geq 4$, such that $\mathcal{T}(J)$ contains one of 4-tournaments in the Figure 5 then J does not admit any invariant $(1,2)$-symplectic metric.*

A smooth map $\phi\colon (M, g) \to (N, h)$ between two Riemannian manifolds is said to be harmonic if and only if it is a critical point of the energy functional

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g, \tag{2.24}$$

where $|d\phi|$ is the Hilbert–Schmidt norm of the linear map $d\phi$, i.e. ϕ is harmonic if and only if it satisfies the Euler–Lagrange equations

$$\delta E(\phi) = \left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = 0 \tag{2.25}$$

for all variation (ϕ_t) of ϕ and $t \in (-\varepsilon, \varepsilon)$ (see [EL]).

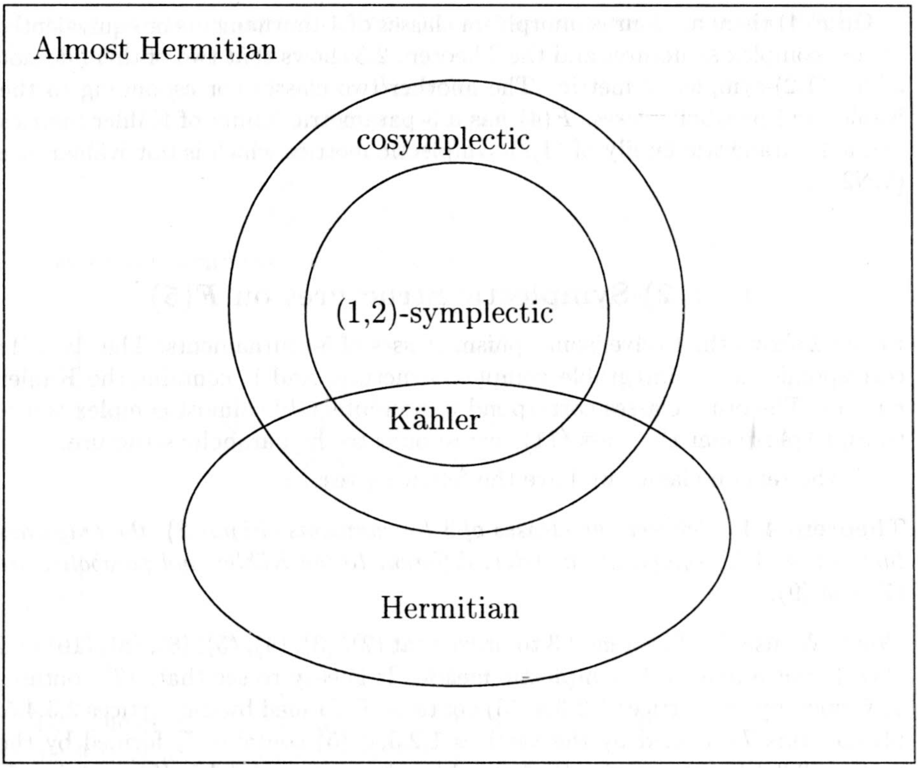

FIGURE 4. Almost Hermitian Structures

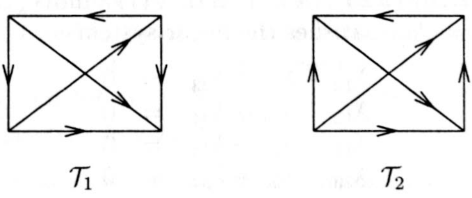

FIGURE 5. 4-tournaments of Theorem 2.3

3. (1, 2)-Symplectic Structures on $F(3)$ and $F(4)$

It is known that, on $F(3)$ there is a 2-parametric family of Kähler metrics and a 3-parametric family of (1,2)-symplectic metrics corresponding to the non-integrable almost complex structures class. Then each invariant almost complex structure on $F(3)$ admits a (1,2)-symplectic metric, see [ESa], [Bo].

On $F(4)$ there are four isomorphism classes of 4-tournaments or equivalently almost complex structures and the Theorem 2.3 shows that two of them do not admit (1,2)-symplectic metric. The another two classes corresponding to the Kähler and parabolic cases. $F(4)$ has a 3-parametric family of Kähler metrics and a 4-parametric family of (1,2)-symplectic metrics which is not Kähler, see [MN2].

4. $(1,2)$-Symplectic Structures on $F(5)$

Figure 2 shows the twelve isomorphism classes of 5-tournaments. The class (1) corresponds to the integrable complex structures and it contains the Kähler metrics. The other classes correspond to non-integrable almost complex structures, in particular the class (11) corresponds to the parabolic structure.

To the remain classes we have the following result:

Theorem 4.1. *Between the classes of* 5*-tournaments* (*Figure* 2), *the only ones that admit* $(1,2)$*-symplectic metrics, different to the Kähler and parabolic, are* (7) *and* (9).

Proof. We use the Theorem 2.3 to prove that (2), (3), (4), (5), (6), (8), (10) and (11) do not admit (1,2)-symplectic metric. It is easy to see that: (2) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4; (3) contains $\mathcal{T}_1$ formed by the vertices 2,3,4,5; (4) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4; (5) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5; (6) contains $\mathcal{T}_2$ formed by the vertices 1,3,4,5; (8) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5; (10) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4 and (11) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4. Then neither of them admit (1,2)-symplectic metric.

Using formulas (2.20)-(2.23), we obtain that (7) admits (1,2)-symplectic metric if and only if $\Lambda = (\lambda_{ij})$ satisfies the linear system

$$\begin{aligned}
\lambda_{12} - \lambda_{13} + \lambda_{23} &= 0 \\
\lambda_{12} - \lambda_{14} + \lambda_{24} &= 0 \\
\lambda_{13} - \lambda_{14} + \lambda_{34} &= 0 \\
\lambda_{23} - \lambda_{24} + \lambda_{34} &= 0 \\
\lambda_{23} - \lambda_{25} + \lambda_{35} &= 0 \\
\lambda_{24} - \lambda_{25} + \lambda_{45} &= 0 \\
\lambda_{34} - \lambda_{35} + \lambda_{45} &= 0
\end{aligned}$$

Then (7) admits (1,2)-symplectic metric if and only if $\Lambda = (\lambda_{ij})$ satisfies

$$\begin{aligned}
\lambda_{13} &= \lambda_{12} + \lambda_{23} \\
\lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\
\lambda_{24} &= \lambda_{23} + \lambda_{34} \\
\lambda_{25} &= \lambda_{23} + \lambda_{34} + \lambda_{45} \\
\lambda_{35} &= \lambda_{34} + \lambda_{45}
\end{aligned}$$

Similarly, we obtain that (9) admit (1,2)-symplectic metric if and only if $\Lambda = (\lambda_{ij})$ satisfies

$$\begin{aligned} \lambda_{13} &= \lambda_{12} + \lambda_{23} \\ \lambda_{14} &= \lambda_{12} + \lambda_{23} + \lambda_{34} \\ \lambda_{24} &= \lambda_{23} + \lambda_{34} \\ \lambda_{25} &= \lambda_{12} + \lambda_{15} \\ \lambda_{35} &= \lambda_{34} + \lambda_{45} \end{aligned}$$ ☑

Now we can write the respective matrices

$$\Lambda_{(7)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{15} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} \\ \lambda_{15} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 \end{pmatrix}$$

$$\Lambda_{(9)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{15} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{15} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} \\ \lambda_{15} & \lambda_{12}+\lambda_{15} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 \end{pmatrix}$$

The Theorem 4.1 says that $F(n)$ admits (1,2)-symplectic metrics, different to the Kähler and parabolic, if and only if $n \geq 5$.

5. (1, 2)-Symplectic Structures on $F(6)$

There are 56 isomorphism classes of 6-tournaments (see [M]), which are presented in Figures 6, 7 and 8. Again, the class (1) corresponds to the integrable complex structures. The other classes correspond to non-integrable almost complex structures, and the class (52) corresponds to the parabolic structure.

In this case we have the following result

Theorem 5.1. *Between the classes of* 6*-tournaments* (*Figure* 6, 7 *and* 8), *the only ones that admit* (1, 2)*-symplectic metrics, different to the Kähler and parabolic, are* (19), (31), (37) *and* (55).

Proof. We use the Theorem 2.3 to prove that each of the classes of 6-tournaments different to the (1), (19), (31), (37), (52) and (55) does not admit (1,2)-symplectic metrics:

- (2) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
- (3) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (4) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,5.
- (5) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.

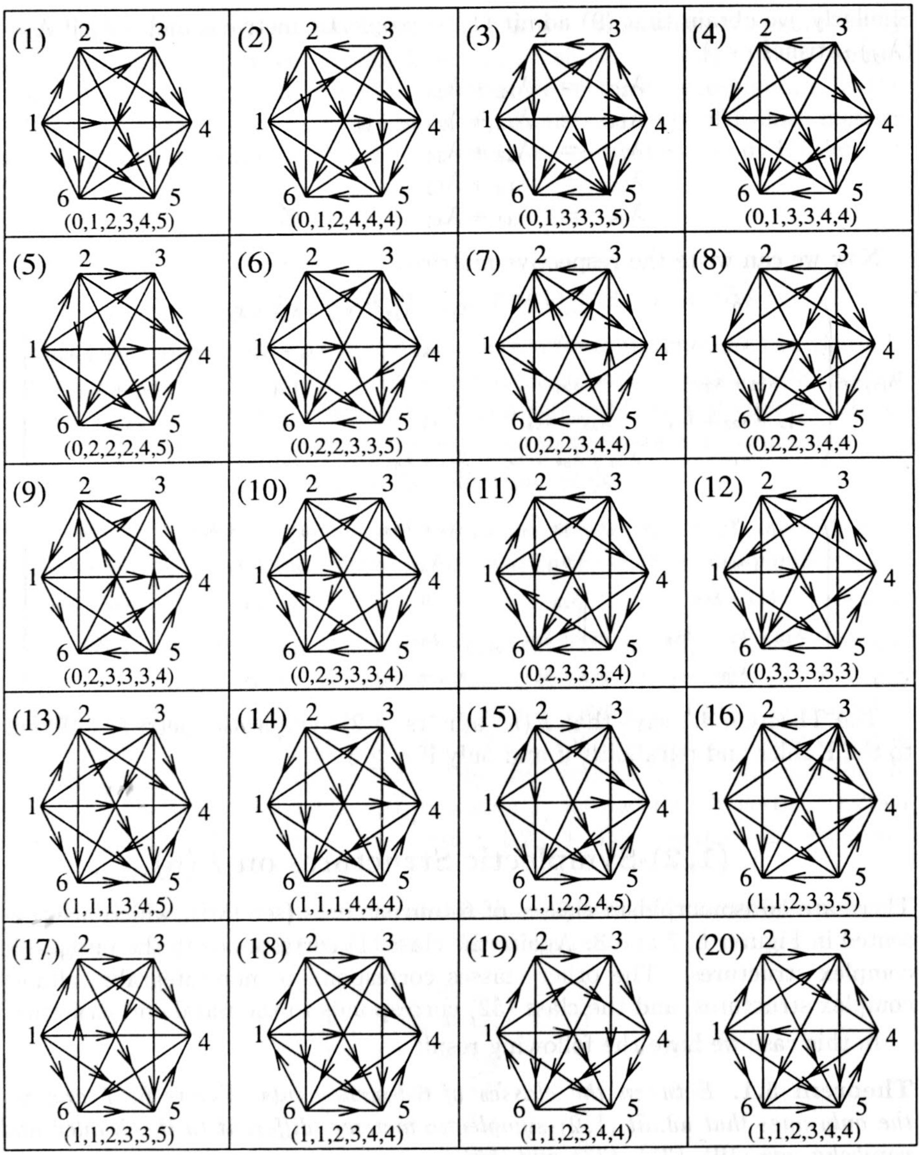

FIGURE 6. Isomorphism classes of 6-tournaments

- (6) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (7) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
- (8) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.

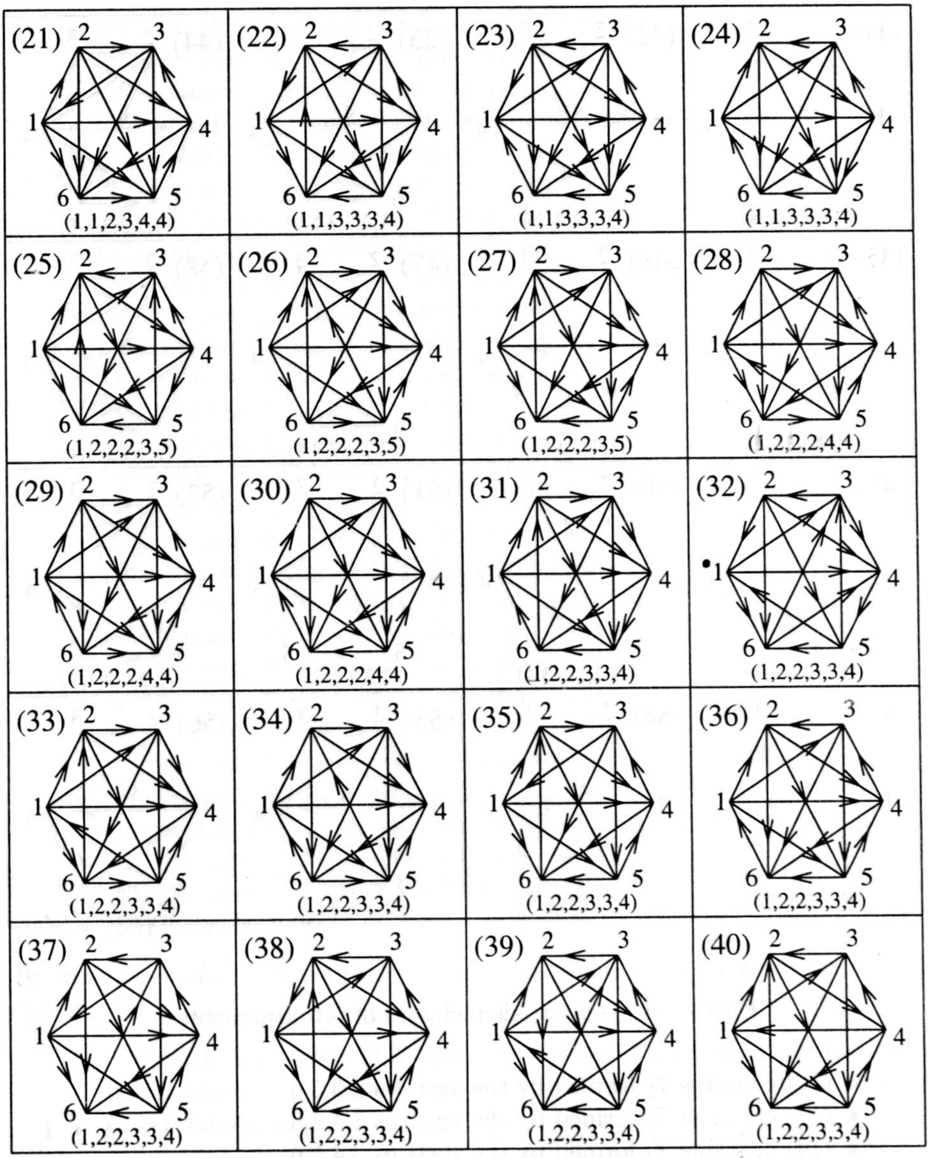

FIGURE 7. Isomorphism classes of 6-tournaments

- (9) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
- (10) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
- (11) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.

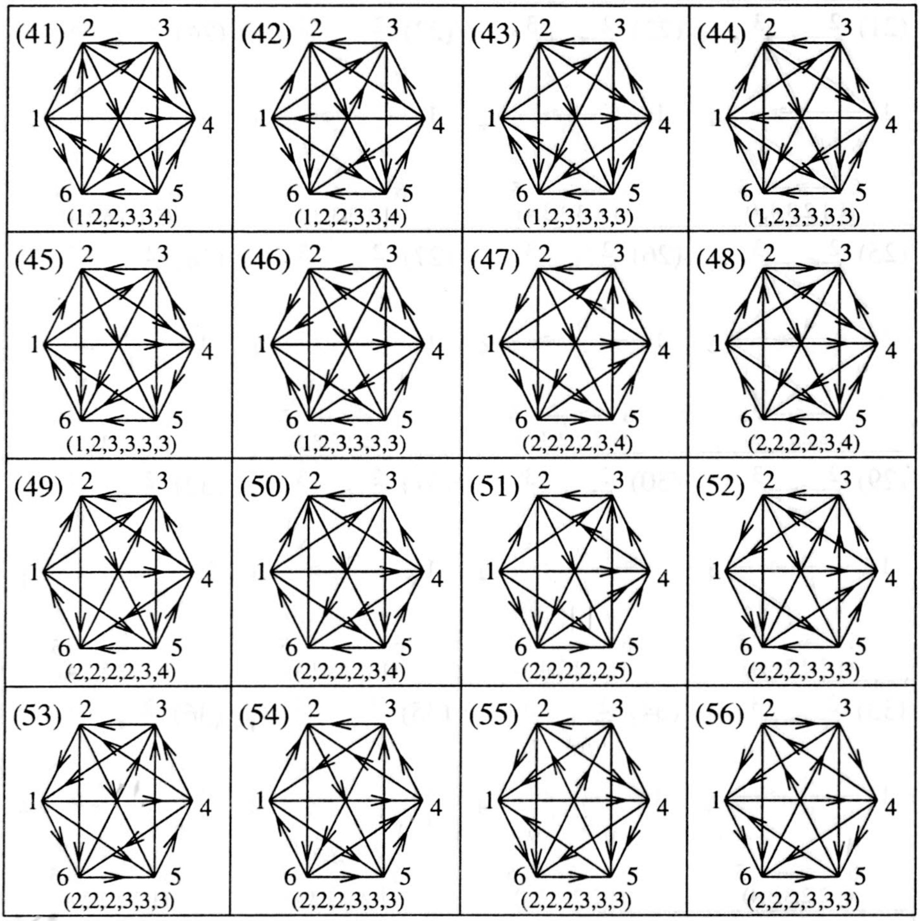

FIGURE 8. Isomorphism classes of 6-tournaments

- (12) contains $\mathcal{T}_1$ formed by the vertices 2,3,5,6.
- (13) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (14) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (15) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (16) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (17) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (18) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (20) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (21) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (22) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,5.
- (23) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,5.

- (24) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (25) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (26) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (27) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (28) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (29) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (30) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (32) contains $\mathcal{T}_1$ formed by the vertices 1,2,3,4.
- (33) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (34) contains $\mathcal{T}_2$ formed by the vertices 3,4,5,6.
- (35) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (36) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (38) contains $\mathcal{T}_1$ formed by the vertices 3,4,5,6.
- (39) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (40) contains $\mathcal{T}_1$ formed by the vertices 3,4,5,6.
- (41) contains $\mathcal{T}_1$ formed by the vertices 3,4,5,6.
- (42) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,6.
- (43) contains $\mathcal{T}_1$ formed by the vertices 3,4,5,6.
- (44) contains $\mathcal{T}_1$ formed by the vertices 3,4,5,6.
- (45) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (46) contains $\mathcal{T}_1$ formed by the vertices 2,3,5,6.
- (47) contains $\mathcal{T}_2$ formed by the vertices 1,3,4,6.
- (48) contains $\mathcal{T}_2$ formed by the vertices 2,3,4,5.
- (49) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (50) contains $\mathcal{T}_2$ formed by the vertices 1,2,3,4.
- (51) contains $\mathcal{T}_2$ formed by the vertices 1,3,5,6.
- (53) contains $\mathcal{T}_1$ formed by the vertices 1,2,4,6.
- (54) contains $\mathcal{T}_2$ formed by the vertices 1,2,4,5.
- (56) contains $\mathcal{T}_1$ formed by the vertices 1,2,4,6.

By making similar computations to we made in the proof of Theorem 4.1 we obtain:

- The class (19) admits (1,2)-symplectic metric if and only if the elements of corresponding matrix $\Lambda_{(19)} = (\lambda_{ij})$ satisfy the following system of linear equations

$$\begin{array}{rclrcl}
\lambda_{12}-\lambda_{13}+\lambda_{23} &=& 0 & \lambda_{12}-\lambda_{14}+\lambda_{24} &=& 0\\
\lambda_{12}-\lambda_{15}+\lambda_{25} &=& 0 & \lambda_{13}-\lambda_{14}+\lambda_{34} &=& 0\\
\lambda_{13}-\lambda_{15}+\lambda_{35} &=& 0 & \lambda_{14}-\lambda_{15}+\lambda_{45} &=& 0\\
\lambda_{23}-\lambda_{24}+\lambda_{34} &=& 0 & \lambda_{23}-\lambda_{25}+\lambda_{35} &=& 0\\
\lambda_{23}-\lambda_{26}+\lambda_{36} &=& 0 & \lambda_{24}-\lambda_{25}+\lambda_{45} &=& 0\\
\lambda_{24}-\lambda_{26}+\lambda_{46} &=& 0 & \lambda_{25}-\lambda_{26}+\lambda_{56} &=& 0\\
\lambda_{34}-\lambda_{35}+\lambda_{45} &=& 0 & \lambda_{34}-\lambda_{36}+\lambda_{46} &=& 0\\
\lambda_{35}-\lambda_{36}+\lambda_{56} &=& 0 & \lambda_{45}-\lambda_{46}+\lambda_{56} &=& 0\,.
\end{array}$$

Then the metric $ds^2_{\Lambda_{(19)}}$ is (1,2)-symplectic if and only if

$$\begin{array}{lllllll}
\lambda_{13} & = & \lambda_{12}+\lambda_{23} & & \lambda_{26} & = & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56}\\
\lambda_{14} & = & \lambda_{12}+\lambda_{23}+\lambda_{34} & & \lambda_{35} & = & \lambda_{34}+\lambda_{45}\\
\lambda_{15} & = & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & & \lambda_{36} & = & \lambda_{34}+\lambda_{45}+\lambda_{56}\\
\lambda_{24} & = & \lambda_{23}+\lambda_{34} & & \lambda_{46} & = & \lambda_{45}+\lambda_{56}\\
\lambda_{25} & = & \lambda_{23}+\lambda_{34}+\lambda_{45}. & & & &
\end{array}$$

- In similar way the class (31) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(31)} = (\lambda_{ij})$ satisfy the following relations

$$\begin{array}{lllllll}
\lambda_{13} & = & \lambda_{12}+\lambda_{23} & & \lambda_{26} & = & \lambda_{12}+\lambda_{16}\\
\lambda_{14} & = & \lambda_{12}+\lambda_{23}+\lambda_{34} & & \lambda_{35} & = & \lambda_{34}+\lambda_{45}\\
\lambda_{15} & = & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & & \lambda_{36} & = & \lambda_{34}+\lambda_{45}+\lambda_{56}\\
\lambda_{24} & = & \lambda_{23}+\lambda_{34} & & \lambda_{46} & = & \lambda_{45}+\lambda_{56}\\
\lambda_{25} & = & \lambda_{23}+\lambda_{34}+\lambda_{45}. & & & &
\end{array}$$

- Similarly, the class (37) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(37)} = (\lambda_{ij})$ satisfy the following relations

$$\begin{array}{lllllll}
\lambda_{14} & = & \lambda_{12}+\lambda_{25}+\lambda_{45} & & \lambda_{26} & = & \lambda_{25}+\lambda_{45}+\lambda_{46}\\
\lambda_{15} & = & \lambda_{12}+\lambda_{25} & & \lambda_{34} & = & \lambda_{36}+\lambda_{46}\\
\lambda_{16} & = & \lambda_{12}+\lambda_{25}+\lambda_{45}+\lambda_{46} & & \lambda_{35} & = & \lambda_{12}+\lambda_{13}+\lambda_{25}\\
\lambda_{23} & = & \lambda_{12}+\lambda_{13} & & \lambda_{56} & = & \lambda_{45}+\lambda_{46}\\
\lambda_{24} & = & \lambda_{25}+\lambda_{45}. & & & &
\end{array}$$

- Finally, the class (55) admits (1,2)-symplectic metric if and only if the elements of the corresponding matrix $\Lambda_{(55)} = (\lambda_{ij})$ satisfy the following relations

$$\begin{array}{lllllll}
\lambda_{13} & = & \lambda_{12}+\lambda_{25}+\lambda_{35} & & \lambda_{26} & = & \lambda_{12}+\lambda_{14}+\lambda_{46}\\
\lambda_{15} & = & \lambda_{12}+\lambda_{25} & & \lambda_{34} & = & \lambda_{36}+\lambda_{46}\\
\lambda_{16} & = & \lambda_{14}+\lambda_{46} & & \lambda_{45} & = & \lambda_{35}+\lambda_{36}+\lambda_{46}\\
\lambda_{23} & = & \lambda_{25}+\lambda_{35} & & \lambda_{56} & = & \lambda_{35}+\lambda_{36}\\
\lambda_{24} & = & \lambda_{12}+\lambda_{14} & & & &
\end{array}$$

☑

The matrices $\Lambda_{(19)}$, $\Lambda_{(31)}$, $\Lambda_{(37)}$ and $\Lambda_{(55)}$ correponding to the classes (19), (31), (37) and (55) are presented on the end of this paper.

6. (1, 2)-Symplectic Structures on $F(7)$

This case has a problem because it is not known any collection of tournament drawings for $n \geq 7$. The collection of tournaments drawings of $n = 2, 3, 4, 5, 6$, is contained in the Moon's book [M].

There are 456 isomorphism classes of 7-tornaments. In the Dias's M. Sc. Thesis [D] was obtained a representant matrix of each class of 7-tournament. The matrix $M(\mathcal{T}) = (a_{ij})$ of the tournament $\mathcal{T}$ is defined by

$$a_{ij} = \begin{cases} 0, & \text{if } j \xrightarrow{\mathcal{T}} i \\ 1, & \text{if } i \xrightarrow{\mathcal{T}} j. \end{cases}$$

Obviously, it has the matrix is equivalent to have the tournament drawing.

We used the matrices generated in [D] together with the Digraph computer program, created by Professor Davide Carlo Demaria, in order to know which 7-tournaments contain the tournaments in Figure 5. Table 1 shows the matrices of the 7-tournaments which admit (1,2)-symplectic metric. Using the matrices

$\begin{pmatrix} 0&1&1&1&1&1&1\\ 0&0&1&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix}$
$\begin{pmatrix} 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&0\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1\\ 1&1&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1\\ 1&1&1&0&0&0&0 \end{pmatrix}$
$\begin{pmatrix} 0&1&1&1&1&1&0\\ 0&0&1&1&1&1&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1\\ 1&1&1&1&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&0&0\\ 0&0&1&1&1&1&0\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 1&0&0&0&0&0&1\\ 1&1&0&0&0&0&0 \end{pmatrix}$
$\begin{pmatrix} 0&1&1&1&1&0&0\\ 0&0&1&1&1&1&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 1&0&0&0&0&0&1\\ 1&1&1&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 1&1&0&0&0&0&1\\ 1&1&0&0&0&0&0 \end{pmatrix}$
$\begin{pmatrix} 0&1&1&1&1&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 1&1&0&0&0&0&1\\ 1&1&1&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&0&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&1\\ 1&0&0&0&0&1&1\\ 1&1&0&0&0&0&1\\ 1&1&1&0&0&0&0 \end{pmatrix}$

TABLE 1. Matrices of the 7-tournaments which admit (1,2)-symplectic metric

in the Table 1 we construct the 7-tournament drawings which admit (1,2)-symplectic metric. Figures 9 and 10 show this 7-tournaments. Class (1) in the Figure 9 represents the integrable structures and the class (10) in Figure 10 corresponds to the parabolic structures. To the remain classes we have the following result.

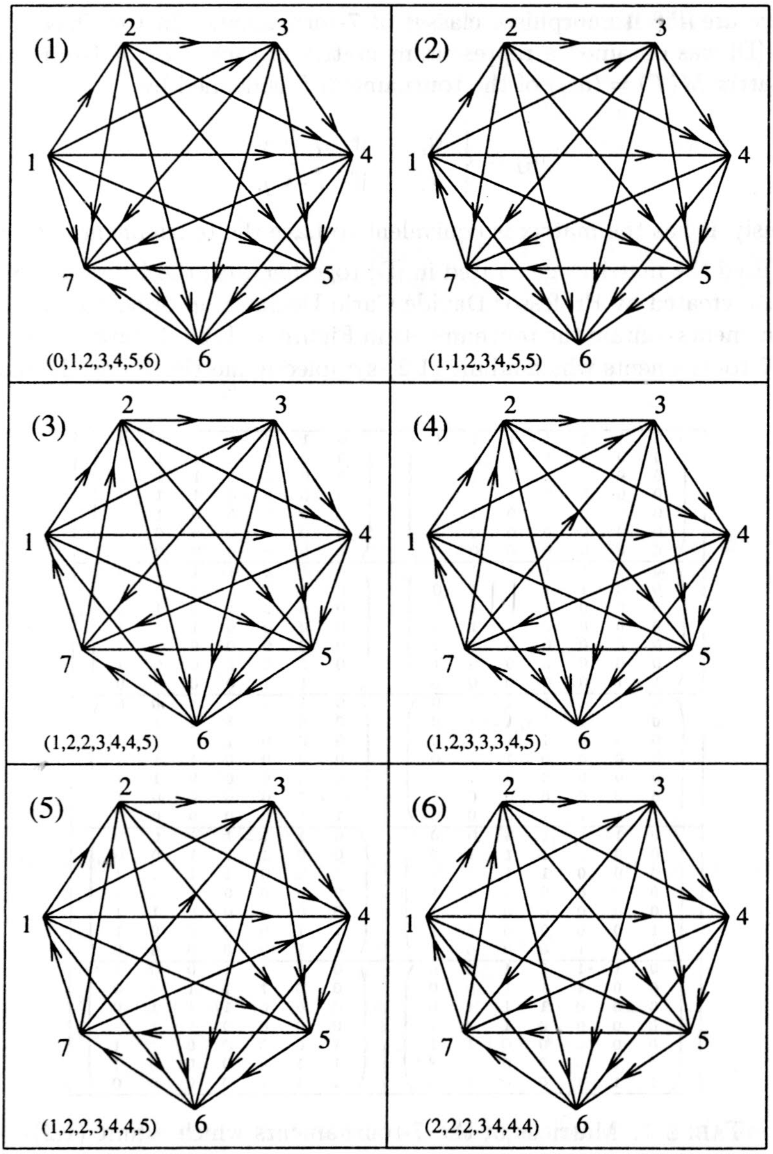

FIGURE 9. Isomorphism classes of 7-tournaments which admit (1,2)-symplectic metric

Theorem 6.1. *The classes of* 7*-tournaments* (2) *through* (9) *in the Figures* 9 *and* 10 *admit* $(1,2)$*-symplectic metrics, different to the Kähler and parabolic.*

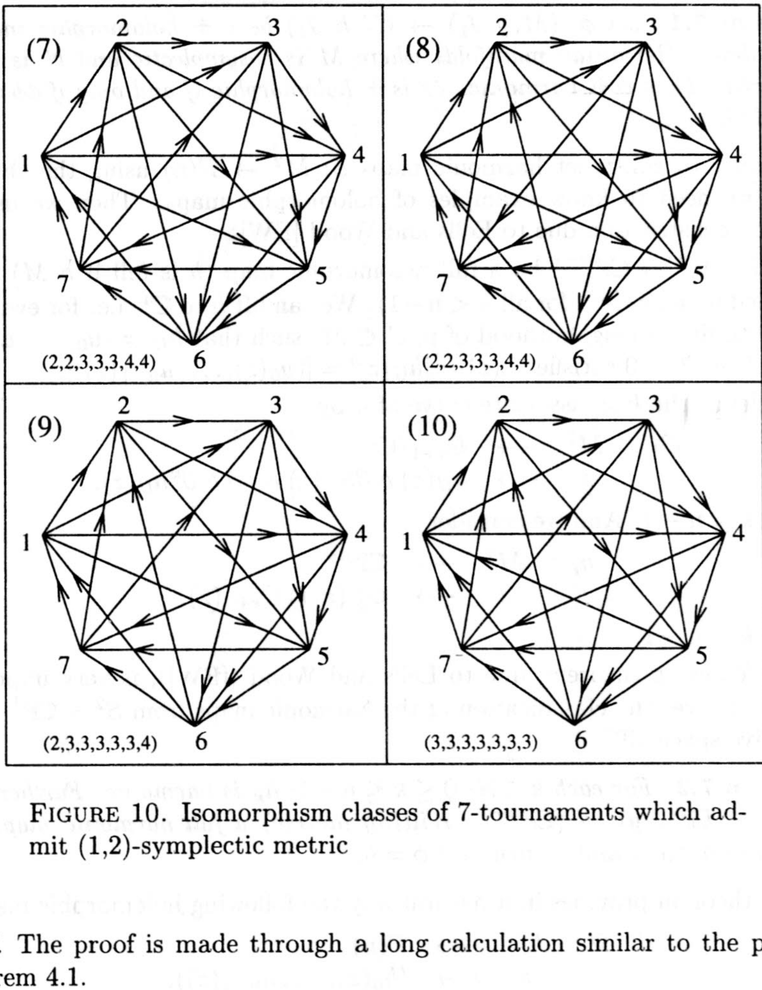

FIGURE 10. Isomorphism classes of 7-tournaments which admit (1,2)-symplectic metric

Proof. The proof is made through a long calculation similar to the proof of Theorem 4.1. ☑

The matrices $\Lambda_{(2)}$ through $\Lambda_{(9)}$ corresponding to the classes (2) through (9) are presented on the end of this paper.

Wolf and Gray [WG] proved that the normal metric on $F(n)$ is not (1,2)-symplectic for $n \geq 4$. Our results give a simple proof of this fact to $n = 5, 6, 7$.

7. Harmonic Maps

In this section we construct new examples of harmonic maps using the following result due to Lichnerowicz [L]:

Theorem 7.1. *Let ϕ: $(M, g, J_1) \to (N, h, J_2)$ be a $\pm$ holomorphic map between almost Hermitian manifolds where M is cosymplectic and N is $(1,2)$-symplectic. Then ϕ is harmonic. (ϕ is $\pm$ holomorphic if and only if $d\phi \circ J_1 = \pm J_2 \circ d\phi$).*

In order to construct harmonic maps ϕ: $M^2 \to F(n)$ using the theorem above, we need to know examples of holomorphic maps. Then we use the following construction due to Eells and Wood [EW].

Let $h : M^2 \to \mathbb{CP}^{n-1}$ be a full holomorphic map (h is full if $h(M)$ is not contained in none $\mathbb{CP}^k$, for all $k < n-1$). We can lift h to $\mathbb{C}^n$, i.e. for every $p \in M$ we can find a neighborhood of p, $U \subset M$, such that $h_U = (u_0, \ldots, u_{n-1}) : M^2 \supset U \to \mathbb{C}^n - 0$ satisfies $h(z) = [h_U(z)] = [(u_0(z), \ldots, u_{n-1}(z))]$.

We define the k-th associate curve of h by

$$\begin{array}{rccl} \mathcal{O}_k : & M^2 & \longrightarrow & \mathbb{G}_{k+1}(\mathbb{C}^n) \\ & z & \longmapsto & h_U(z) \wedge \partial h_U(z) \wedge \cdots \wedge \partial^k h_U(z), \end{array}$$

for $0 \leq k \leq n-1$. And we consider

$$\begin{array}{rccl} h_k : & M^2 & \longrightarrow & \mathbb{CP}^{n-1} \\ & z & \longmapsto & \mathcal{O}_k^{\perp}(z) \cap \mathcal{O}_{k+1}(z), \end{array}$$

for $0 \geq k \geq n-1$.

The following theorem, due to Eells and Wood ([EW]), is very important because it gives the classification of the harmonic maps from $S^2 \sim \mathbb{CP}^1$ into a projective space $\mathbb{CP}^{n-1}$.

Theorem 7.2. *For each $k \in \mathbb{N}$, $0 \leq k \leq n-1$, h_k is harmonic. Furthermore, given ϕ : $(\mathbb{CP}^1,\ g) \to (\mathbb{CP}^{n-1},$ Killing metric) a full harmonic map, then there are unique k and h such that $\phi = h_k$.*

This theorem provides in a natural way the following holomorphic maps

$$\begin{array}{rccl} \Psi : & M^2 & \longrightarrow & F(n) \\ & z & \longmapsto & (h_0(z), \ldots, h_{n-1}(z)), \end{array}$$

called by Eells–Wood's map (see [N2]).

We called $\mathfrak{M}_n$ the set of (1,2)-symplectic metrics on $F(n)$, for $n = 5$, 6 and 7 characterized in the sections above. Using Theorem 7.1 we obtain the following result

Theorem 7.3. *Let $\phi : M^2 \to \big(F(n), g\big)$, $g \in \mathfrak{M}$ a holomorphic map. Then ϕ is harmonic.*

In addition for maps from a flag manifold into a flag manifold we obtain the following result

Proposition 7.1. *Let ϕ : $(F(l), g) \to (F(k), h)$ a holomorphic map, with $g \in \mathfrak{M}_l$ and $h \in \mathfrak{M}_k$. Then ϕ is harmonic.*

$$\Lambda_{(19)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}\end{matrix} & \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}+\lambda_{45}\end{matrix} & \lambda_{16} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}\end{matrix} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}+\lambda_{56}\end{matrix} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \begin{matrix}\lambda_{34}+\lambda_{45}\\ +\lambda_{56}\end{matrix} \\ \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}\end{matrix} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} \\ \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}+\lambda_{45}\end{matrix} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}\end{matrix} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} \\ \lambda_{16} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}+\lambda_{56}\end{matrix} & \begin{matrix}\lambda_{34}+\lambda_{45}\\ +\lambda_{56}\end{matrix} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 \end{pmatrix}$$

$$\Lambda_{(31)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}\end{matrix} & \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}+\lambda_{45}\end{matrix} & \lambda_{16} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}\end{matrix} & \lambda_{12}+\lambda_{16} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \begin{matrix}\lambda_{34}+\lambda_{45}\\ +\lambda_{56}\end{matrix} \\ \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}\end{matrix} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} \\ \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}+\lambda_{45}\end{matrix} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}\end{matrix} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} \\ \lambda_{16} & \lambda_{12}+\lambda_{16} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 \end{pmatrix}$$

$$\Lambda_{(37)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{12}+\lambda_{25}+\lambda_{45} & \lambda_{12}+\lambda_{25} & \lambda_{12}+\lambda_{25}+\lambda_{45}+\lambda_{46} \\ \lambda_{12} & 0 & \lambda_{12}+\lambda_{13} & \lambda_{25}+\lambda_{45} & \lambda_{25} & \lambda_{25}+\lambda_{45}+\lambda_{46} \\ \lambda_{13} & \lambda_{12}+\lambda_{13} & 0 & \lambda_{36}+\lambda_{46} & \lambda_{12}+\lambda_{13}+\lambda_{25} & \lambda_{36} \\ \lambda_{12}+\lambda_{25}+\lambda_{45} & \lambda_{25}+\lambda_{45} & \lambda_{36}+\lambda_{46} & 0 & \lambda_{45} & \lambda_{46} \\ \lambda_{12}+\lambda_{25} & \lambda_{25} & \lambda_{12}+\lambda_{13}+\lambda_{25} & \lambda_{45} & 0 & \lambda_{45}+\lambda_{46} \\ \lambda_{12}+\lambda_{25}+\lambda_{45}+\lambda_{46} & \lambda_{25}+\lambda_{45}+\lambda_{46} & \lambda_{36} & \lambda_{46} & \lambda_{45}+\lambda_{46} & 0 \end{pmatrix}$$

$$\Lambda_{(55)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{25}+\lambda_{35} & \lambda_{14} & \lambda_{12}+\lambda_{25} & \lambda_{14}+\lambda_{46} \\ \lambda_{12} & 0 & \lambda_{25}+\lambda_{35} & \lambda_{12}+\lambda_{14} & \lambda_{25} & \lambda_{12}+\lambda_{14}+\lambda_{46} \\ \lambda_{12}+\lambda_{25}+\lambda_{35} & \lambda_{25}+\lambda_{35} & 0 & \lambda_{36}+\lambda_{46} & \lambda_{35} & \lambda_{36} \\ \lambda_{14} & \lambda_{12}+\lambda_{14} & \lambda_{36}+\lambda_{46} & 0 & \lambda_{35}+\lambda_{36}+\lambda_{46} & \lambda_{46} \\ \lambda_{12}+\lambda_{25} & \lambda_{25} & \lambda_{35} & \lambda_{35}+\lambda_{36}+\lambda_{46} & 0 & \lambda_{35}+\lambda_{36} \\ \lambda_{14}+\lambda_{46} & \lambda_{12}+\lambda_{14}+\lambda_{46} & \lambda_{36} & \lambda_{46} & \lambda_{35}+\lambda_{36} & 0 \end{pmatrix}$$

$$\Lambda_{(2)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

$$\Lambda_{(3)} = \begin{pmatrix}
0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{17} \\
\lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17} \\
\lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} \\
\lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56}+\lambda_{67} \\
\lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\
\lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\
\lambda_{17} & \lambda_{12}+\lambda_{17} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0
\end{pmatrix}$$

$$\Lambda_{(4)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}\end{matrix} & \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}+\lambda_{45}\end{matrix} & \begin{matrix}\lambda_{12}+\lambda_{23}+\lambda_{34}\\ +\lambda_{45}+\lambda_{56}\end{matrix} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}\end{matrix} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}+\lambda_{56}\end{matrix} & \lambda_{12}+\lambda 17 \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \begin{matrix}\lambda_{34}+\lambda_{45}\\ +\lambda_{56}\end{matrix} & \begin{matrix}\lambda_{12}+\lambda_{17}\\ +\lambda_{23}\end{matrix} \\ \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}\end{matrix} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \begin{matrix}\lambda_{45}+\lambda_{56}\\ +\lambda_{67}\end{matrix} \\ \begin{matrix}\lambda_{12}+\lambda_{23}\\ +\lambda_{34}+\lambda_{45}\end{matrix} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}\end{matrix} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \begin{matrix}\lambda_{12}+\lambda_{23}+\lambda_{34}\\ +\lambda_{45}+\lambda_{56}\end{matrix} & \begin{matrix}\lambda_{23}+\lambda_{34}\\ +\lambda_{45}+\lambda_{56}\end{matrix} & \begin{matrix}\lambda_{34}+\lambda_{45}\\ +\lambda_{56}\end{matrix} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{12}+\lambda_{17} & \begin{matrix}\lambda_{12}+\lambda_{17}\\ +\lambda_{23}\end{matrix} & \begin{matrix}\lambda_{45}+\lambda_{56}\\ +\lambda_{67}\end{matrix} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

$$\Lambda_{(5)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda 17 \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17}+\lambda_{23} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17}+\lambda_{23}+\lambda_{34} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{12}+\lambda_{17} & \lambda_{12}+\lambda_{17}+\lambda_{23} & \lambda_{12}+\lambda_{17}+\lambda_{23}+\lambda_{34} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

$$\Lambda_{(6)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{17}+\lambda_{67} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \lambda_{17}+\lambda_{67} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{12}+\lambda_{17} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

$$\Lambda_{(7)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{17}+\lambda_{67} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17}+\lambda_{23} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \lambda_{17}+\lambda_{67} & \lambda_{23}+\lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{12}+\lambda_{17} & \lambda_{12}+\lambda_{17}+\lambda_{23} & \lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

$$\Lambda_{(8)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{17}+\lambda_{67} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{12}+\lambda_{17}+\lambda_{67} & \lambda_{12}+\lambda_{17} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \lambda_{17}+\lambda_{67} & \lambda_{12}+\lambda_{17}+\lambda_{67} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{12}+\lambda_{17} & \lambda_{34}+\lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

$$\Lambda_{(9)} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{12}+\lambda_{23} & \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{17}+\lambda_{67} & \lambda_{17} \\ \lambda_{12} & 0 & \lambda_{23} & \lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{12}+\lambda_{17}+\lambda_{67} & \lambda_{12}+\lambda_{17} \\ \lambda_{12}+\lambda_{23} & \lambda_{23} & 0 & \lambda_{34} & \lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{12}+\lambda_{17}+\lambda_{23} \\ \lambda_{12}+\lambda_{23}+\lambda_{34} & \lambda_{23}+\lambda_{34} & \lambda_{34} & 0 & \lambda_{45} & \lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56}+\lambda_{67} \\ \lambda_{12}+\lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{23}+\lambda_{34}+\lambda_{45} & \lambda_{34}+\lambda_{45} & \lambda_{45} & 0 & \lambda_{56} & \lambda_{56}+\lambda_{67} \\ \lambda_{17}+\lambda_{67} & \lambda_{12}+\lambda_{17}+\lambda_{67} & \lambda_{34}+\lambda_{45}+\lambda_{56} & \lambda_{45}+\lambda_{56} & \lambda_{56} & 0 & \lambda_{67} \\ \lambda_{17} & \lambda_{12}+\lambda_{17} & \lambda_{12}+\lambda_{17}+\lambda_{23} & \lambda_{45}+\lambda_{56}+\lambda_{67} & \lambda_{56}+\lambda_{67} & \lambda_{67} & 0 \end{pmatrix}$$

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