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Convergence analysis of a one-step intermediate Newton iterative scheme

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ABSTRACT. We study a one-step intermediate method for the iterative computation of a zero of the sum of two nonlinear operators. The proposed method contains Newton scheme and the Modified Newton scheme as special cases and therefore provides a unified setting for the study of both methods.

Key words and phrases. Nonlinear equations, Newton's method, intermediate Newton method, majorant error bounds.

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Let X and Y be Banach spaces, let $u_0 \in X$ and let f, g be continuous operators mapping a closed ball $B[u_0, T]$ into Y, assumed Fréchet differentiable on the open ball $B(u_0, T)$. We are interested in the equation

$$f(u) + g(u) = 0. \tag{1}$$

The method of Newton, defined by the iterations

$$u_{m+1} = u_m - [f'(u_m) + g'(u_m)]^{-1}[f(u_m) + g(u_m)], \quad m = 0, 1, \ldots \tag{2}$$

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and the modified Newton scheme, defined by the iterations

$$u_{m+1} = u_m - [f'(u_0) + g'(u_0)]^{-1}[f(u_m) + g(u_m)], \quad m = 0, 1, \ldots \tag{3}$$

are the best known methods for the iterative solution of this equation.

It is well-known that, in general, Newton's method converges faster than the modified Newton method. However, the Newton's method updates derivatives at each iteration step, while the modified method performs only one derivative evaluation. This led several authors to propose intermediate Newton methods which converge faster than the modified method and are easier to implement than the classical scheme. The best known scheme of this type is the standard two-step intermediate scheme (cf. [1,2,5,6]) which when applied to problem (1) leads to the iterations

$$v_m = u_m - [f'(u_m) + g'(u_m)]^{-1}[f(u_m) + g(u_m)], \tag{4a}$$

$$u_{m+1} = v_m - [f'(u_m) + g'(u_m)]^{-1}[f(v_m) + g(v_m)], \quad m = 0, 1, \ldots \tag{4b}$$

in which each derivative is updated after every two iterations.

In this paper, we study the simpler one-step intermediate Newton scheme

$$u_{m+1} = u_m - [f'(u_m) + g'(u_0)]^{-1}[f(u_m) + g(u_m)], \quad m = 0, 1, \ldots. \tag{5}$$

When $g \doteq 0$, this scheme becomes the classical Newton scheme for the equation $f(u) = 0$, and when $f = 0$, it becomes the Modified Newton scheme for the equation $g(u) = 0$. Therefore the intermediate scheme (5) provides a unified setting for the study of both Newton's method and the modified Newton's method.

Convergence results for Newton schemes (2)–(3) have been given by many authors, especially Kantorovich and Akilov [6] (cf. also [4,5]), Zabrejko and Nguen [6], and convergence results for the intermediate scheme (4) have been given by Argyros [2] and Appel, de Pacale, Evkuta and Zabrejko [1]. In the sequel we give convergence results for the intermediate Newton scheme (5) under Zabrejko–Nguen conditions. In the following Proposition we define the majorant sequence that we will use and give its main properties.

Proposition 1. *Let $a > 0$, let $\alpha(t)$ and $\beta(t)$ be non-negative non-decreasing functions defined on an interval $[0, T]$ such that $\alpha(t) + \beta(t) > 0$ for all $t \in [0, T]$. Let*

$$\kappa(t) = \int_0^t \alpha(s)\, ds,$$

$$\sigma(t) = a + \int_0^t \kappa(s)\, ds - t,$$

$$\pi(t) = \int_0^t \beta(s)\, ds,$$

$$\tau(t) = \int_0^t \pi(s)\, ds$$

and suppose that the function $\mu(t) = \sigma(t) + \tau(t)$ has a unique zero t_* in $[0, T]$. Let $t_0 = 0$, and for $m = 0, 1, \ldots$, let

$$t_{m+1} = t_m - [\sigma(t_m) + \tau(t_m)]/\sigma'(t_m). \tag{6}$$

Then the t_m are well defined and, for $m = 1, 2, \ldots$, we have

$$t_{m-1} < t_m < t_*, \tag{7}$$

$$\lim_{m\to\infty} t_m = t_*. \tag{8}$$

Proof. The hypotheses imply that $\sigma(t)$ and $\tau(t)$ are convex and $\mu(t) = \sigma(t) + \tau(t)$ is strictly convex, and hence that, whenever $0 \le t < s \le T$, we have

$$\begin{aligned}
\tau(t) + \sigma(t) &< \tau(s) + \sigma(s) + \sigma'(t)(t-s) + \tau'(t)(t-s), \\
&= \tau(s) + \sigma(s) + \sigma'(t)(t-s) + \pi(t)(t-s) \\
&\le \tau(s) + \sigma(s) + \sigma'(t)(t-s).
\end{aligned} \tag{9}$$

Also, since $\mu(0) = a > 0$ and t_* is the only zero of μ in $[0, T]$, we see that $\mu(t) \ge 0$ for all $t \in [0, t_*]$, with equality if and only if $t = t_*$. Furthermore, if $\sigma'(\bar{t}) = 0$ for some $\bar{t} \in [0, t_*]$, then convexity implies that $\bar{t}$ is a minimal point of $\sigma(t)$, and hence that $0 \le \mu(\bar{t}) = \sigma(\bar{t}) + \tau(\bar{t}) \le \sigma(t_*) + \tau(t_*) = 0$ which implies that $\bar{t} = t_*$. Since $\sigma'(0) = -1 < 0$, it follows that $\sigma'(t) \le 0$ for all $t \in [0, t_*]$, with equality if and only if $t = t_*$.

In (9), if we set $t = 0$ and $s = t_*$, we see that $t_0 = 0 < t_1 = a = \sigma(0) + \tau(0) < \sigma(t_*) + \tau(t_*) + t_* = t_*$ which shows that (7) holds when $m = 1$.

Suppose now, by induction, $m \ge 1$ and that (7) holds. Then on using (9) and the fact that $\mu(t_m) > 0$ and $\sigma'(t_m) < 0$, we obtain the relations:

$$\begin{aligned}
t_* - t_{m+1} &= t_* - t_m + [\sigma(t_m) + \tau(t_m)]/\sigma'(t_m) \\
&= [\sigma'(t_m)(t_* - t_m) + \sigma(t_m) + \tau(t_m)]/\sigma'(t_m) \\
&> [\sigma(t_*) + \tau(t_*)]/\sigma'(t_m) = 0 \\
t_{m+1} - t_m &= -\mu(t_m)/\sigma'(t_m) > 0
\end{aligned}$$

which show that (7) holds when m is replaced with $m + 1$ and hence, by induction, that it holds for all positive integral values of m.

It follows that t_m is monotone increasing sequence that is bounded above by t_*. Hence it converges, as m tends to infinity, to a real number $\bar{t}$, with the property that $0 \le \bar{t} \le t_*$. If $\sigma'(\bar{t}) \ne 0$, then on letting m tend to infinity in (6), we see that $\mu(t) = 0$, and hence that $\bar{t} = t_*$. If $0 = \sigma'(\bar{t})$ then it follows from the comments in the first paragraph of the proof that $\bar{t} = t_*$. In either case, $\bar{t}$ is a root of $\mu(t)$. ☑

The following result will be used repeatedly in the sequel. The proof can be found in [6, Proposition 1].

Lemma 1. *Let v be a function defined on the closed ball $B[u_0, T]$ in the Banach space X, with values in the Banach space Y. Suppose that there exists a non-decreasing function $\theta(t)$ defined on the closed interval $[0, T]$ such that, for all $0 \leq t \leq T$, we have*

$$\|v(x) - v(y)\| \leq \theta(t)\|x - y\| \quad \forall x, y \in B(u_0, t).$$

Then, whenever $0 \leq t \leq s \leq T$, $x \in B[u_0, t]$ and $y \in B[x, s - t]$ we have $\|v(x) - v(y)\| \leq \int_t^s \theta(s)\, ds$.

We next prove the convergence of the generalized Newton-type scheme (5) under Zabrejko-Nguen-type hypotheses of the kind used in [6].

Theorem 1. *Let $a \geq 0$, $u_0 \in X$ and let f and g be functions defined in $B[u_0, T]$, with values in Y, Fréchet differentiable on $B(u_0, T)$. Suppose further that $J_0 = f'(u_0) + g'(u_0)$ is invertible, that $\|J_0^{-1}[f(u_0) + g(u_0)]\| \leq a$, and that whenever $0 \leq t \leq s \leq T$, $x \in B(u_0, t)$ and $y \in B(x, s - t)$ we have*

$$\|J_0^{-1}[f'(x) - f'(y)]\| \leq \alpha(t)\|x - y\|, \tag{10}$$

$$\|J_0^{-1}[g'(x) - g'(y)]\| \leq \beta(t)\|x - y\| \tag{11}$$

where $\alpha(t)$ and $\beta(t)$ satisfy the hypotheses of Proposition 1. Then the intermediate Newton iterates in (5) are all well defined and converge to a solution u of equation (1) in $B[u_0, T]$, with error estimates

$$\|u_m - u_{m-1}\| \leq t_m - t_{m-1}, \tag{12}$$

$$\|u_m - u_0\| \leq t_m, \tag{13}$$

$$\|u - u_m\| \leq t_* - t_m \tag{14}$$

where the sequence t_m is defined as in Proposition 1.

Proof. Let $\sigma(t)$, $\tau(t)$ and $\mu(t)$ be defined as in Proposition 1.

If $a = 0$, then $u = u_0$ solves equation (1) and, since $u_m = u_0$ and $t_m = t_0$ for all m, the estimates (12)–(14) hold trivially. In the rest of the proof we assume $a > 0$.

Since $\|u_1 - u_0\| = a \leq t_1 - t_0$, we see that (12)–(13) hold when $m = 1$.

Suppose now, by induction, that $m \geq 1$ and that the u_m are well defined and satisfy (12)–(13). Then, on letting $J_m \equiv f'(u_m) + g'(u_0) = J_0(I + A)$ or, equivalently, $A = J_0^{-1}[f'(u_m) - f'(u_0)]$, and applying Lemma 1, we see that $\|A\| \leq \kappa(t_m) < 1$. Therefore $(I + A)^{-1}$ exists, with

$$\|(I + A)^{-1}\| \leq 1/[1 - \kappa(t_m)] = -1/\sigma'(t_m),$$

and it follows that J_m is invertible, and that $J_m^{-1}J_0 = (I+A)^{-1}$. Hence $\|J_m^{-1}J_0\| \le -1/\sigma'(t_m)$, and it follows from Lemma 1 and the induction hypotheses that

$$\|u_{m+1}-u_m\| = \|J_m^{-1}[f(u_m)+g(u_m)]\| \le \|J_m^{-1}J_0\|\|J_0^{-1}[f(u_m)+g(u_m)]\|.$$

But

$$\begin{aligned} f(u_m)+g(u_m) &= [f(u_m)-f(u_{m-1})-f'(u_{m-1})(u_m-u_{m-1})] \\ &\quad + [g(u_m)-g(u_{m-1})-g'(u_0)(u_m-u_{m-1})]. \end{aligned}$$

Hence

$$\begin{aligned} \|J_0^{-1}[f(u_m)+g(u_m)]\| &\le \|J_0^{-1}[f(u_m)-f(u_{m-1})-f'(u_{m-1})(u_m-u_{m-1})]\| \\ &\quad + \|J_0^{-1}[g(u_m)-g(u_{m-1})-g'(u_0)(u_m-u_{m-1})]\| \\ &\le \Big\| \int_0^1 J_0^{-1}[f'(u_m+s(u_m-u_{m-1})) \\ &\qquad - f'(u_{m-1})](u_m-u_{m-1})\,ds\Big\| \\ &\quad + \Big\| \int_0^1 J_0^{-1}[g'(u_m+s(u_m-u_{m-1})) \\ &\qquad - g'(u_0)](u_m-u_{m-1})\,ds\Big\| \\ &\le \int_0^1 \Big[\int_{t_{m-1}}^{t_m+s(t_m-t_{m-1})} \alpha(w)\,dw\Big](t_m-t_{m-1})\,ds \\ &\quad + \int_0^1 \Big[\int_0^{t_m+s(t_m-t_{m-1})} \beta(w)\,dw\Big](t_m-t_{m-1})\,ds \\ &= \int_0^1 [\kappa(t_m+s(t_m-t_{m-1}))-\kappa(t_{m-1})](t_m-t_{m-1})\,ds \\ &\quad + \int_0^1 \pi(t_m+s(t_m-t_{m-1}))(t_m-t_{m-1})\,ds \\ &= \int_{t_{m-1}}^{t_m} \kappa(s)\,ds - \kappa(t_{m-1})(t_m-t_{m-1}) + \int_{t_{m-1}}^{t_m} \pi(s)\,ds \\ &= (t_m-t_{m-1})(1-\kappa(t_{m-1})) + \sigma(t_m) \\ &\qquad - \sigma(t_{m-1}) + \tau(t_m) - \tau(t_{m-1}) \\ &= \sigma(t_m)+\tau(t_m)-(t_m-t_{m-1})\sigma'(t_{m-1}) \\ &\qquad - \tau(t_{m-1}) - \sigma(t_{m-1}) \\ &= \sigma(t_m)+\tau(t_m), \\ \|u_{m+1}-u_m\| &\le \|J_m^{-1}J_0\|[\sigma(t_m)+\tau(t_m)] \\ &\le -[\sigma(t_m)+\tau(t_m)]/\sigma'(t_m) = t_{m+1}-t_m, \\ \|u_{m+1}-u_0\| &\le \|u_{m+1}-u_m\| + \|u_m-u_0\| \\ &\le t_{m+1}-t_m+t_m = t_{m+1}. \end{aligned}$$

It follows that (12) and (13) also hold when m is replaced with $m+1$ and hence, by induction, that they hold for all positive integral values of m.

This implies that

$$\|u_{m+q} - u_m\| \leq \sum_{k=m+1}^{m+q} \|u_k - u_{k-1}\| \leq \sum_{k=m+1}^{m+q} (t_k - t_{k-1}) = t_{m+q} - t_m.$$

Since t_m is a Cauchy sequence, it follows that u_m is also a Cauchy sequence converging to some $u \in B[u_0, T]$. On letting q tend to infinity we see that (14) holds. It follows from (5) that $[f'(u_m)+g'(u_0)](u_{m+1}-u_m)+f(u_m)+g(u_m) = 0$ and on letting m tend to infinity we see that u solves equation (1). ☑

Remark 1. The issue of uniqueness be settled from Theorem 4 of [6] which implies that if the conditions of Theorem 1 hold and $\mu(T) \leq 0$, then the solution of equation (1) is unique in $B(u_0, T)$.

Remark 2. The error estimate —analogous to (14)— satisfied by the Newton scheme (2) under the hypotheses of Theorem 1 is

$$\|u - u_m\| \leq t_* - s_m, \tag{15}$$

with $s_0 = 0$ and $s_{m+1} = s_m - [\sigma(s_m) + \tau(s_m)]/[\sigma'(s_m) + \tau'(s_m)]$. The bound for the modified Newton scheme (3) under the same conditions is

$$\|u - u_m\| \leq t_* - k_m, \tag{16}$$

with $k_0 = 0$ and $k_{m+1} = k_m + \sigma(k_m) + \tau(k_m)$. It is no difficult to see that for $m = 0, 1, \ldots$, we have

$$k_m \leq t_m \leq s_m. \tag{17}$$

The inequality holds trivially when $m = 0$. Suppose, by induction that (17) holds for a certain m. Then, on using (9) and the fact that $t + \sigma(t) + \tau(t)$ is a increasing in $[0, t_*)$ (since $[t+\sigma(t)+\tau(t)]' = 1+\sigma'(t)+\tau'(t) = \kappa(t)+\pi(t) \geq 0$) and $\sigma'(t_m) \leq \sigma'(s_m)$, we see that

$$\begin{aligned}
k_{m+1} &= k_m + \sigma(k_m) + \tau(k_m) \\
&\leq t_m + \sigma(t_m) + \tau(t_m) \\
&\leq t_m + \sigma(t_m) + \tau(t_m) + [1 + \sigma'(t_m)](t_{m+1} - t_m) \\
&= \sigma(t_m) + \tau(t_m) + \sigma'(t_m)(t_{m+1} - t_m) + t_{m+1} = t_{m+1} \\
&= t_m - [\sigma(t_m) + \tau(t_m)]/\sigma'(t_m) \\
&= s_m - [\sigma(t_m) + \tau(t_m) + \sigma'(t_m)(s_m - t_m)]/\sigma'(t_m) \\
&\leq s_m - [\sigma(s_m) + \tau(s_m)]/\sigma'(t_m) \\
&\leq s_m - [\sigma(s_m) + \tau(s_m)]/\sigma'(s_m) \\
&\leq s_m - [\sigma(s_m) + \tau(s_m)]/[\sigma'(s_m) + \tau'(s_m)] \\
&= s_{m+1}.
\end{aligned}$$

This shows, by induction, that (17) holds for all m.

The inequality (17) suggests that (5) is an intermediate scheme between the Newton scheme (2) and the modified Newton scheme (3). However, under the hypotheses of Theorem 1, $J_0^{-1}[f'(x)+g'(x)]$ will have a Lipschitz constant $\gamma(t)$ that is smaller than $\alpha(t)+\beta(t)$, which implies that the error estimates (15) and (16) are too coarse. However, all numerical examples that one cares to do will confirm that, under the hypotheses of Theorem 1, the intermediate scheme (5) is indeed an intermediate scheme between the Newton scheme and the modified Newton scheme.

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