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On ultra-products of some families of composition operators between certain finite dimensional ℓ^p spaces

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Abstract. Let $0 < \sigma < 1$ and $1 < p, r < \infty$ be such that $1/r + (1 - \sigma)/p' = 1$. We show that for every continuous linear map T between Banach spaces E, F such that its restriction to every finite dimensional subspace N of E factorizes through a chain of type

$$\ell^\infty(\Omega_N, \mu_N) \to \ell^r(\Omega_N, \mu_N) \to \ell^1(\Omega_N, \mu_N) + \ell^p(\Omega_N, \mu_N)$$

where (Ω_N, μ_N) is a discrete measure space with a finite number of atoms, there is a σ–finite measure space (Ω, μ) such that $T \in \mathcal{L}(E, F'')$ factorizes through the chain of "continuous spaces

$$L^\infty(\Omega, \mu) \to L^r(\Omega, \mu) \to L^1(\Omega, \mu) + L^p(\Omega, \mu).$$

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1. Introduction

It is well known (see [2], [3], [10]) that the ultra-product map of a family of maps between finite dimensional spaces ℓ^∞ and ℓ^p is a fundamental tool in the characterization of p-integral operators in the class of infinite dimensional Banach spaces. Matter, in his study of absolutely continuous operators of Niculescu (see [9]) has introduced in [8] the ideal $\mathcal{P}_{p,\sigma}$, $(0 < \sigma < 1, 1 \leq p \leq$

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∞) of (p,σ)-absolutely continuous operators, which is strictly larger than the classical ideal of $p-$absolultely summing operators. Since $\mathcal{P}_{p,\sigma}$ is a maximal ideal in the sense of Pietsch (see [10]), it is associated with some tensor norm according to the theory developed by Defant and Floret in [2].

We have find in [6] the tensor norm $g_{p,\sigma}$ such that for any Banach spaces E, F the equality $(E \otimes_{g_{p,\sigma}} F)' = \mathcal{P}_{p',\sigma}(F, E')$ holds. The natural problem is the characterization of $g_{p,\sigma}$-integral operators. To do this, we are in a situation similar to the classical one of $p-$integral operators, but with a more complicated finite dimensional subjacent factorizations which we explain now:

It is clear that every continuous linear map $T \in \mathcal{L}(E,F)$, E and F Banach spaces, defines canonically a linear map T_M from E into the dual M' of every subspace $M \subset F'$ by

$$\forall x \in E \quad y \in M \quad \langle T_M(x), y\rangle = \langle T(x), y\rangle.$$

On the other hand, it is easy to check that, given a measure space (Ω,μ), $0<\sigma<1$ and $1<p<\infty$, there is always an inclusion map

$$L^\infty(\mu)\cap L^{p'}(\mu) \subset L^{\frac{p'}{1-\sigma}}(\mu)$$

and hence an inclusion map $J^\mu_r : L^r(\mu) \subset L^1(\mu) + L^p(\mu)$ such that $\|J^\mu_r\| \leq 1$. Then, suppose moreover that T is such that for every finite dimensional subspace $M \subset F'$, the restriction of T_M to every finite dimensional subspace $N \subset E$ factorizes in the way

$$N \longrightarrow \ell^\infty(\Omega_N,\mu_N) \xrightarrow{D_N} \ell^r(\Omega_N,\mu_N) \xrightarrow{J^{\mu_N}_r} \ell^1(\Omega_N,\mu_N) + \ell^p(\Omega_N,\mu_N) \longrightarrow M'$$

where every (Ω_N,μ_N) is a discrete measure space with a finite number of atoms and every D_N is a positive diagonal operator.

Our purpose on this paper is to show that, in this case, there is a $\sigma-$finite measure space (Ω,μ) such that $J_F T$ factorizes in the way

$$E \longrightarrow L^\infty(\Omega,\mu) \xrightarrow{B_w} L^r(\Omega,\mu) \xrightarrow{J^\mu_r} L^1(\Omega,\mu) + L^p(\Omega,\mu) \longrightarrow F''$$

where B_w is a diagonal operator. This is a very technical result and applications of it will be given in the forthcoming paper [7].

Our notation is standard. If E is a Banach space, B_E will be the closed unit ball of E, E' the topological dual Banach space and J_E the natural embedding of E into its bi-dual E''. Sometimes, to call attention on the norm of the involved Banach space E, we shall write $\|.\|_E$. Given $p \in [1,\infty]$, a measure space $(\Omega,\mathcal{M},\mu)$ and a measurable non null everywhere real function g, we denote by $L^p(\Omega,\mathcal{M},g,\mu)$ (or simply $L^p(g,\mu)$ if there is no risk of confusion) the Banach space of classes of functions f such that fg belongs to the Lebesgue space $L^p(\Omega,\mathcal{M},\mu)$, provided with the norm $\|f\| = \|fg\|_{L^p(\mu)}$.

From now on, *in all the paper, σ and p will be real numbers such that* $0<\sigma<1$ *and* $1<p<\infty$. Given such numbers, $r \in]1,\infty[$ will be always the real number such that $1/r + (1-\sigma)/p' = 1$. *All Banach spaces of this paper will be* *defined over the field of real numbers* since we shall use results of the theory of Banach lattices (see [1] for questions concerning this topic).

Given a compatible couple (A_0, A_1) of Banach spaces (i.e. two Banach spaces which are vector subspaces of a larger vector space E), the spaces $A_0 + A_1$ and $A_0 \cap A_1$ will be always endowed with its canonical norms

$$\|x\|_{A_0+A_1} = \inf\{\|a\|_{A_0} + \|b\|_{A_1} \mid x = a + b, a \in A_0,\ b \in A_1\}$$

and

$$\|x\|_{A_0 \cap A_1} = \max\{\|x\|_{A_0}, \|x\|_{A_1}\}$$

respectively.

2. On ultra-products of some factorizations

We refer the reader to [4] for definitions and basic results about ultra-products of Banach spaces. Let D be an index set an $\mathcal{D}$ a non-trivial ultrafilter on D. Given a family $\{A_d |\ d \in D\}$ of Banach spaces, $(A_d)_{\mathcal{D}}$ will denote its ultra-product by $\mathcal{D}$ and $(x_d)_{\mathcal{D}}$ will be the class of $(x_d) \in \Pi_{d\in D} A_d$ in $(A_d)_{\mathcal{D}}$. Analogously, if we have a family of maps $\{T_d \in \mathcal{L}(A_d, F_d),\ \ d \in D\}$ in such a way that $\sup_{d\in D} \|T_d\| < \infty$, we denote by $(T_d)_{\mathcal{D}} \in \mathcal{L}((A_d)_{\mathcal{D}}, (F_d)_{\mathcal{D}})$ the canonical ultra-product linear map. If every $A_d,\ d \in D$ is a Banach lattice, $(A_d)_{\mathcal{D}}$ has a canonical order which makes it a Banach lattice.

Suppose now that for every $d \in D$ we have the chain of Banach spaces and continuous mappings

$$A^d_\infty \xrightarrow{T_d} A^d_r \xrightarrow{J_r^{\mu_d}} A^d_1 + A^d_p$$

where $A^d_i = l^i(\Omega_d, \mu_d), i = 1, r, p, \infty$ for some atomic measure space (Ω_d, μ_d) and T_d is a positive operator such that $sup_{d\in\mathcal{D}}\|T_d\| < \infty$. We put $\mathcal{U}_i = (A^d_i)_{\mathcal{D}}$ for every $i = 1, r, p, \infty$. Now, the map $(T_d)_{\mathcal{D}} : \mathcal{U}_\infty \to \mathcal{U}_r$ is well defined.

The ultra-products $\mathcal{S} = (A^d_1 + A^d_p)_{\mathcal{D}}$ and $\mathcal{U}_i,\ i = 1, p, r$, are Banach lattices under its canonical order. For $i = 1, p$ and $d \in D$, let $J^d_i : A^d_i \to A^d_1 + A^d_p$ be the canonical map. The ultra-product maps $J_i = (J^d_i)_{\mathcal{D}} : \mathcal{U}_i \to \mathcal{S}, i = 1, p$ and $J_r := (J_r^{\mu_d})_{\mathcal{D}} : \mathcal{U}_r \to \mathcal{S}$ are well defined lattice homomorphisms. In this situation, the main technical problem is that every J_i *can not be an injective map*. To overpass this inconvenient we need to work with quotient spaces which makes more involved the argumentation.

Each kernel $Ker(J_i), i = 1, p, r$ is a closed ideal in $\mathcal{U}_i$. Let H_i be the quotient Banach lattice of $\mathcal{U}_i$ by $Ker(J_i)$, $K_i \in \mathcal{L}(\mathcal{U}_i, H_i)$ the canonical quotient map and $\overline{J}_i$ the canonical injective *positive* map from H_i into $\mathcal{S}$. We put $E_i = \overline{J}_i(H_i)$ provided with the topology of H_i. It is easy to check that E_i is an abstract L^i-space and $\overline{J}_i$ is an order isomorphism onto the sublattice E_i of $\mathcal{S}$.

For every $((x^d_i)) \in \Pi_{d\in D} A^d_r$ we define $((y^d_i))$ and $((z^d_i))$ in $\Pi_{d\in D} A^d_r$ such that for every $d \in D$ and every $i \in \Omega_d$, $y^d_i := x^d_i$ if $x^d_i \leq 1$, $y^d_i := 0$ if $x^d_i > 1$, $z^d_i := 0$

if $x_i^d < 1$ and $z_i^d := x_i^d$ if $x_i^d \geq 1$. Since $1 < r < p$ we have $((y_i^d)) \in \Pi_{d\in\mathcal{D}} A_p^d$ and $((z_i^d)) \in \Pi_{d\in\mathcal{D}} A_1^d$. Now we define

$$P_1(((x_i^d))_{\mathcal{D}}) = \overline{J}_p K_p(((y_i^d))_{\mathcal{D}}) \quad \text{and} \quad P_2(((x_i^d))_{\mathcal{D}}) = \overline{J}_1 K_1(((z_i^d))_{\mathcal{D}}).$$

Hence

$$\begin{aligned}\overline{J}_r K_r(((x_i^d)_{\mathcal{D}}) &= \overline{J}_r K_r(((y_i^d)_{\mathcal{D}}) + \overline{J}_r K_r(((z_i^d)_{\mathcal{D}}) \\ &= P_1(((x_i^d))_{\mathcal{D}}) + P_2(((x_i^d))_{\mathcal{D}}) \in E_1 + E_p\end{aligned}$$

and for every $((x_i^d))_{\mathcal{D}}$ in the open unit ball of $\mathcal{U}_r$ we have

$$\begin{aligned}\|\overline{J}_r K_r(((x_i^d)_{\mathcal{D}})\|_{E_1+E_p} &\leq \|\overline{J}_r K_r(((y_i^d)_{\mathcal{D}})\|_{E_p} + \|\overline{J}_r K_r(((z_i^d)_{\mathcal{D}})\|_{E_1} \\ &\leq \|\overline{J}_r K_r(((x_i^d)_{\mathcal{D}})\|_{\mathcal{U}_r}^{\frac{r}{p}} + \|\overline{J}_r K_r(((x_i^d)_{\mathcal{D}})\|_{\mathcal{U}_r}^{r} \leq 2.\end{aligned}$$

In consequence, since E_r and $E_1 + E_p$ are subsets of $\mathcal{S}$, there is a continuous inclusion $E_r \subset E_1 + E_p$. Clearly, E_r is a sublattice of $E_1 + E_p$ and $\overline{J}_r$ is an order isomorphism.

Lemma 1. *Suppose $E_r \neq \{0\}$. Then $E_1 \cap E_p \neq \{0\}$.*

Proof. Clearly, the diagram (arrows without characters are inclusion maps)

$$\begin{array}{ccc} \mathcal{U}_r & \xrightarrow{\quad J_r \quad} & \mathcal{S} \\ \Big\downarrow{\scriptstyle \overline{J}_r K_r} & & \Big\uparrow \\ E_r & \xrightarrow{\qquad\quad} & E_1 + E_p \end{array}$$

is commutative by definition of the involved mappings. Let $0 \neq w_0 := ((w_i^d))_{\mathcal{D}} \in E_r \subset E_1 + E_p$. Put $y := P_1(w_0) \in E_p$ and $z := P_2(w_0) \in E_1$. Then $w_0 = y + z$ and hence $y \neq 0$ or $z \neq 0$. Suppose $y \neq 0$. Then there is $\varepsilon > 0$ such that, for every representation $y = u + v$ with $u \in E_1, v \in E_p$, we have $\varepsilon < \|u\|_{E_1} + \|v\|_{E_p}$. Since $r < p$ we obtain

$$\sup_{d\in D} \sum_{i\in\Omega_d} |y_i^d|^{p^2} \leq \sup_{d\in D} \sum_{i\in\Omega_d} |y_i^d|^{p} \leq \sup_{d\in D} \sum_{i\in\Omega_d} |y_i^d|^{r} \leq \sup_{d\in D} \sum_{i\in\Omega_d} |w_i^d|^{r} < \infty.$$

Hence

$$\overline{J}_r K_r(((|y_i^d|^p))_{\mathcal{D}}) = \overline{J}_1 K_1(((|y_i^d|^p)_{\mathcal{D}}) = \overline{J}_p K_p(((|y_i^d|^p)_{\mathcal{D}}) \in E_1 \cap E_p.$$

Now, if $f := ((f_i^d)) \in ((|y_i^d|^p))_{\mathcal{D}}$ in the ultra-product $\mathcal{U}_1$, we have

$$\begin{aligned}\varepsilon^p < \|y\|_{E_p}^p \leq \lim_{\mathcal{D}} \sum_{i\in\Omega_d} |y_i^d|^p \mu_i^d &\leq \lim_{\mathcal{D}} \left(\sum_{i\in\Omega_d} \big||y_i^d|^p - f_i^d\big| \,\mu_i^d \right) + \lim_{\mathcal{D}} \sum_{i\in\Omega_d} |f_i^d| \mu_i^d \\ &= \|((|y_i^d|^p))_{\mathcal{D}} - f\|_{\mathcal{U}_1} + \|f\|_{\mathcal{U}_1} = \|f\|_{\mathcal{U}_1}.\end{aligned}$$

Then, taking the infimum over f we obtain $\varepsilon^p < \|y\|_{E_p}^p \leq \|K_1(((|y_i^d|^p))_{\mathcal{D}})\|_{H_1}$. Since $\overline{J}_1$ is injective, we have $0 \neq \overline{J}_r K_r(((|y_i^d|^p))_{\mathcal{D}})$. Analogously, if $z \neq 0$, there is $\varepsilon > 0$ such that, for every representation $z = u + v$ with $u \in E_1, v \in E_p$, we have $\varepsilon < \|u\|_{E_1} + \|v\|_{E_p}$. Now we have

$$\sup_{d\in D} \sum_{i\in\Omega_d} |z_i^d|^{\frac{1}{p}} \leq \sup_{d\in D} \sum_{i\in\Omega_d} |z_i^d| \leq \sup_{d\in D} \sum_{i\in\Omega_d} |z_i^d|^r \leq \sup_{d\in D} \sum_{i\in\Omega_d} |w_i^d|^r < \infty.$$

Hence

$$\overline{J}_r K_r(((|z_i^d|^{\frac{1}{p}}))_{\mathcal{D}}) = \overline{J}_1 K_1(((|z_i^d|^{\frac{1}{p}})_{\mathcal{D}}) = \overline{J}_p K_p(((|z_i^d|^{\frac{1}{p}})_{\mathcal{D}}) \in E_1 \cap E_p.$$

Now, if $f := ((f_i^d)) \in ((|z_i^d|^{\frac{1}{p}}))_{\mathcal{D}}$ in the ultra-product $\mathcal{U}_p$, we have

$$\begin{aligned} \varepsilon^{\frac{1}{p}} < \|z\|_{E_1} &\leq \lim_{\mathcal{D}} \left(\sum_{i\in\Omega_d} |z_i^d| \mu_i^d \right)^{\frac{1}{p}} \\ &\leq \lim_{\mathcal{D}} \left(\sum_{i\in\Omega_d} \left| |z_i^d|^{\frac{1}{p}} - f_i^d \right|^p \mu_i^d \right)^{\frac{1}{p}} + \lim_{\mathcal{D}} \left(\sum_{i\in\Omega_d} |f_i^d|^p \mu_i^d \right)^{\frac{1}{p}} \\ &= \|((|z_i^d|^{\frac{1}{p}}))_{\mathcal{D}} - f\|_{\mathcal{U}_p} + \|f\|_{\mathcal{U}_p} = \|f\|_{\mathcal{U}_p}. \end{aligned}$$

Then, taking the infimum over f we obtain $\varepsilon^{\frac{1}{p}} < \|y\|_{E_p}^{\frac{1}{p}} \leq \|K_p(((|y_i^d|^{\frac{1}{p}}))_{\mathcal{D}})\|_{H_p}$. Since $\overline{J}_p$ is injective, we have $0 \neq \overline{J}_r K_r(((|z_i^d|^{\frac{1}{p}}))_{\mathcal{D}})$. Consequently $E_1 \cap E_p \neq \{0\}$. ☑

Let E_0 be the closure of $E_1 \cap E_p$ in $E_1 + E_p$. Clearly E_0 is a closed sublattice of $E_1 + E_p$.

Lemma 2. *The norm of $E_1 + E_p$ is order continuous in E_0.*

Proof. Let $x \in E_1 \cap E_p$ be the supremum of an increasing non negative net $\{x_\alpha,\ \alpha \in A\}$ in $E_1 \cap E_p$. We have

$$\|x - x_\alpha\|_{E_1+E_p} \leq \|x - x_\alpha\|_{E_1}.$$

Since E_1 is an abstract L-space, it has order continuous norm. Then $\lim_\alpha x_\alpha = x$ in $E_1 + E_p$. By a result of Luxemburg, (see theorem 12.10 in [1]), E_0 has order continuous norm. ☑

Lemma 3. *Suppose $J_r(T_d)_{\mathcal{D}} \neq 0$. There is a set $\mathcal{E} \subset E_1 \cap E_p \cap E_r$ which is a maximal system of pairwise disjoint elements in E_0 and hence also in E_r.*

Proof. Since E_0 has order continuous norm (lemma 2), there is a topological vector space $\mathcal{F}$ of measurable real functions defined on some measure space (Ω, Σ, ν) and a continuous order isomorphism $\Psi : E_0 \to \mathcal{F}$ when $\mathcal{F}$ is provided with its canonical order. (See for instance section 1 of Pisier's paper [11] for the detailed construction of Ψ).

On the other hand, the element $u = ((u_i^d))_{\mathcal{D}} \in \mathcal{U}_\infty$ such that $u_i^d = 1$ for every $d \in D$ and $i \in \Omega$, is a strong unit in $\mathcal{U}_\infty$. Then $(T_d)_{\mathcal{D}}(\mathcal{U}_\infty)$ is contained in the band generated by $w := (T_d)_{\mathcal{D}}(u)$ in $\mathcal{U}_r$. Let $w_0 := J_r(w) \in E_1 + E_p$. Since $J_r(T_d)_{\mathcal{D}} \neq 0$, necessarily we have $0 \neq w_0$. By lemma 1, $E_1 \cap E_p \neq \{0\}$. By Zorn's lemma, there is a maximal system $\mathcal{E} = \{e_v | v \in \mathcal{V}\}$ of pairwise disjoint vectors in $E_1 \cap E_p$ such that $0 < \|e_v\|_{E_p} < 1$.

Let us see that every $z \in E_1 \cap E_p$ is the sum of a series $z = \sum_{n=1}^{\infty} x_n$ (in the topology of E_0) with every x_n in the band $e_{v_n}^{\perp\perp}$ generated in E_0 by some $e_{v_n} \in \mathcal{E}$. By Zorn's lemma, there is a set $\mathcal{E}'$ such that $\mathcal{E} \cup \mathcal{E}'$ is a maximal set of pairwise disjoint elements in the order continuous lattice E_0 (lemma 2). Then, by a well known result of Kakutani (see proposition 1.a.9 in [5]), there are sequences $(v_n)_{n=1}^{\infty} \subset \mathcal{E}$ and $(w_n)_{n=1}^{\infty} \subset \mathcal{E}'$ such that $z = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$ with every $x_n \in e_{v_n}^{\perp\perp}$ and $y_n \in e_{w_n}^{\perp\perp}$. Then $\Psi(z) = \sum_{n=1}^{\infty} \Psi(x_n) + \sum_{n=1}^{\infty} \Psi(y_n)$ and hence $|\sum_{n=1}^{\infty} \Psi(y_n)| \leq |\Psi(z)|$ and $|\sum_{n=1}^{\infty} y_n| \leq |z|$. In consequence $\sum_{n=1}^{\infty} y_n \in E_1 \cap E_p$ and $z - \sum_{n=1}^{\infty} x_n \in E_1 \cap E_p$. Moreover,

$$\left|z - \sum_{n=1}^{\infty} x_n\right| \wedge e_v = \left|\sum_{n=1}^{\infty} y_n\right| \wedge e_v \leq \sum_{n=1}^{\infty} |y_n| \wedge e_v = 0$$

for all $e_v \in \mathcal{E}$. Hence $z = \sum_{n=1}^{\infty} x_n$.

Now, suppose there is $z \in E_0$ such that $z \wedge e_v = 0$ for every $v \in \mathcal{V}$. Then $\Psi(z) \wedge \Psi(e_v) = 0$ for all $v \in \mathcal{V}$. On the other hand there is a sequence $(z_n)_{n=1}^{\infty} \subset E_1 \cap E_p$ such that $z = \lim_n z_n$ in E_0. Hence $\Psi(z) = \lim_n \Psi(z_n)$ in $\mathcal{F}$. By the result of last paragraph, for every z_n, $n \in \mathbb{N}$, there is a sequence $(v_{kn})_{k=1}^{\infty} \subset \mathcal{V}$ such that $z_n = \sum_{n=1}^{\infty} x_{v_{kn}}$ in E_0. In consequence, if we put $\Omega_v = \{t \in \Omega \mid \Psi(e_v)(t) \neq 0\}$, it follows that $\Psi(z_n) = \sum_{n=1}^{\infty} \Psi(x_{v_{kn}})$ is null on a measurable set $\Omega \setminus (\cup_{k\in\mathbb{N}} \Omega_{v_{kn}})$ for each $n \in \mathbb{N}$ and hence we have that $\Psi(z)$ is null on a measurable set $\Omega \setminus (\cup_{h\in\mathbb{N}} \Omega_{v_h})$ for some sequence $(v_h)_{h=1}^{\infty} \subset \mathcal{V}$. But $\Psi(z)$ is also null on Ω_v for all $v \in \mathcal{V}$ since $\Psi(z) \wedge \Psi(e_v) = 0$. Then $\Psi(z) = 0$, $z = 0$ and $\mathcal{E}$ is maximal in E_0.

Finally, let us see that $\mathcal{E} \subset E_r$. Given $e_v \in \mathcal{E} \subset E_1 \cap E_p$, we can write $e_v = ((e_i^d))_{\mathcal{D}}$ such that

$$\alpha := \sup \left\{ \sup_{d\in D} \sum_{i\in\Omega_d} |e_i^d| \mu_i, \ \sup_{d\in D} \sum_{i\in\Omega_d} |e_i^d|^p \mu_i \right\} < \infty.$$

But $r < p$ from definition of r. Then, putting

$$\Omega_d^1 = \{i \in \Omega_d \mid \ |e_i^d| \geq 1\} \text{ and } \Omega_d^2 = \{i \in \Omega_d \mid \ |e_i^d| < 1\}$$

we have

$$\sup_{d\in D} \sum_{i\in\Omega_d} |e_i^d|^r \mu_i \leq \sup_{d\in D} \left(\sum_{i\in\Omega_d^1} |e_i^d|^p \mu_i + \sum_{i\in\Omega_d^2} |e_i^d| \mu_i \right) \leq 2\alpha$$

and hence $e_v \in E_r$. As above, since $E_r \subset E_0$ we get that $\mathcal{E}$ must be maximal in E_r too. ☑

Once again by the quoted result of Kakutani ([5], Proposition 1.a.9) and by Lemma 2, there is a set $\mathcal{V}_0 = \{v_n, n \in \mathbb{N}\} \subset \mathcal{V}$ such that $w_0 = \sum_{n=1}^{\infty} x_{v_n}$ in E_0 with every x_{v_n} in the band generated by e_{v_n} in $E_1 + E_p$. For every $i = 1, p, r$ we define the *complemented* subspaces

$$G_i = \{z \in E_i \mid |z| \wedge e_v = 0 \ \ \forall v \notin \mathcal{V}_0\}.$$

Now we can state the main theorem of this paper:

Theorem 4. *Suppose* $1 < p < \infty$. *Then there are a* σ*-finite measure space* $(\Omega, \mathcal{M}, \nu)$*, a multiplication operator* C_h *and suitable operators for the following vertical arrows, such that the diagram*

$$\begin{array}{ccccc}
\mathcal{U}_\infty & \xrightarrow{(T_d)_\mathcal{D}} & \mathcal{U}_r & \xrightarrow{(I_r^{\mu_d})_\mathcal{D}} & \mathcal{S} \\
\downarrow & & \downarrow & & \uparrow \\
L^\infty(\nu) & \xrightarrow{C_h} & L^r(\nu) & \xrightarrow{J_r^\nu} & L^1(\nu) + L^p(\nu)
\end{array}$$

is commutative.

Proof. Fix $v_n \in \mathcal{V}_0$ and let $i = 1, r, p$. Let $B_i(e_{v_n})$ be the band generated by e_{v_n} in E_i. Let $\mathcal{A}_n$ be the boolean algebra $\mathcal{A}_n$ of the components of e_{v_n} in E_0

$$\mathcal{A}_n := \{x \in E_0 \mid x \wedge (e_{v_n} - x) = 0\}.$$

By lemma 2 and theorems 12.9 and 3.15 in [1], $\mathcal{A}_n$ is Dedekind complete and by the Stone representation theorem, it is isomorphic to the boolean algebra $\mathcal{O}_n$ of the clopen sets of a separated compact extremally disconnected topological space Ω_n. Since $e_{v_n} \in E_1 \cap E_r \cap E_p$, we have $\mathcal{A}_n \subset B_i(e_{v_n})$ and we can define the following set of functions on $\mathcal{O}_n$: if $x \in \mathcal{A}_n$ and S_x is its image in $\mathcal{O}_n$, we put

$$\omega_n^i(S_x) = \|\overline{J}_i^{-1}(x)\|_{H_i}^i.$$

As H_i is an abstract L^i-space, ω_n^i is a finitely additive measure on $\mathcal{O}_n$ and hence a measure, Ω_n being extremally disconnected. By the Carathéodory extension procedure we get an other measure $\omega_n^{i^*}$, $i = 1, r, p$, which is a measure (again denoted by ω_n^i) when restricted to the σ-algebra $\mathcal{M}_n^i$ of $\omega_n^{i^*}$-measurable sets of Ω_n. Considering the σ-algebra $\mathcal{M}_n = \mathcal{M}_n^1 \cap \mathcal{M}_n^r \cap \mathcal{M}_n^p$, every ω_n^i, $i = 1, r, p$ is a measure on $\mathcal{M}_n$ and $\omega_n^i(\Omega) = \|\overline{J}_i^{-1}(e_{v_n})\|_{H_i}^i < \infty$. It is easy to see that the map

$$\Psi_i\left(\sum_{h=1}^{k} \alpha_h x_h\right) = \sum_{h=1}^{k} \alpha_h \chi_{S_{x_h}}$$

$z \in \mathcal{A}_n$, is a well defined isometry from

$$\mathcal{F}_n := \left\{ \sum_{h=1}^{k} \alpha_h x_h \mid \alpha_h \in \mathbb{R}, x_h \in \mathcal{A}_n, x_h \wedge x_j = 0, h \neq j; h, j = 1, ..., k, \ k \in \mathbb{N} \right\}$$

(with the induced topology of G_i) into the linear span of measurable characteristic functions of $L^i(\Omega_n, \mathcal{M}_n, \omega_n^i)$. Let us see that Ψ_i can be extended to an isometric lattice homomorphism (again denoted by Ψ_i) from $B_i(e_{v_n})$ onto $L^i(\Omega_n, \mathcal{M}_n, \omega_n^i)$.

Let $S = \cup_{h=1}^{\infty} S_{x_h}$ with $x_h \in \mathcal{A}_n$ and $x_h \leq_{h+1} \leq e_{v_n}$ for every $h \in \mathbb{N}$. Then there exists $x = \sup_{h \in \mathbb{N}} x_h \in \mathcal{A}_n$. By the order continuity of the norm in E_i we have

$$\omega_n^i(S) = \lim_{h \to \infty} \omega_n^i(S_{x_h}) = \lim_{h \to \infty} \|\overline{J_i}^{-1}(x_h)\|_{H_i}^i = \|\overline{J_i}^{-1}(x)\|_{H_i} = \omega_n^i(S_x).$$

Then we obtain that $S \subset S_x$ and $\omega_n^i(S) = \omega_n^i(S_x)$.

Now, if S is a $\mathcal{M}_n$-measurable set in Ω_n we choose a sequence $(y_h^i)_{h=1}^{\infty} \subset \mathcal{A}_n$ such that $S \subset S_{y_{h+1}^i} \subset S_{y_h^i}$ for every $h \in \mathbb{N}$ and $\omega_n^i(S) = \lim_{h \to \infty} \omega_n^i(S_{y_h^i})$. Then exists $y^i = \inf_{h \in \mathbb{N}} y_h^i \in \mathcal{A}_n$ and it satisfies $S \subset S_y^i$ and $\omega_n^i(S) = \omega_n^i(S_y^i)$. Therefore $\chi_S = \chi_{S_y^i} = \Psi_i(y^i)$ holds, which shows that Ψ_i is a map onto the linear span of measurable characteristic functions of $L^i(\Omega_n, \mathcal{M}_n, \omega_n^i)$. Since this set and $\mathcal{F}_n$ are dense in $L^i(\Omega_n, \mathcal{M}_n, \omega_n^i)$ and $B_i(e_{v_n})$ respectively (by the Freudenthal's spectral theorem, see for instance theorem 6.8 in [1]), we get the announced isometry.

Now, we define $\Omega := \cup_{n=1}^{\infty} \Omega_n$ and the σ-algebra $\mathcal{M}$ and the measure ω^i in $\mathcal{M}$ such that

$$\mathcal{M} = \{M \subset \Omega \mid M \cap \Omega_n \in \mathcal{M}_n\} \quad \text{and} \quad \omega^i(M) = \sum_{n=1}^{\infty} \omega_n^i(M \cap \Omega_n) \ \ \forall\, M \in \mathcal{M}$$

Now it is easy to show that Ψ_i can be extended to an isometric and lattice isomorphism (again denoted by Ψ_i) from $G_i, i = 1, r, p$, onto $L^i(\Omega, \mathcal{M}, \omega^i)$. Hence there is a natural isometric order isomorphism Ψ from $G_1 + G_p$ onto $L^1(\Omega, \mathcal{M}, \omega^1) + L^p(\Omega, \mathcal{M}, \omega^p) \subset \mathcal{B}(\Omega, \mathcal{M})$, the space of measurable scalar functions on Ω.

Next we define in $\mathcal{M}$ the σ-finite measure $\omega = \omega^r + \omega^1 + \omega^p$. Every ω^i, $i = r, 1, p$ is absolutely continuous with respect to ω. By the Radon-Nikodym theorem, there is a measurable function g_i such that

$$\omega^i(A) = \int_A g_i d\omega \quad \forall A \in \mathcal{M}$$

and

$$\int_{\Omega} |f|^i d\omega^i = \int_{\Omega} |f|^i g_i d\omega \quad \forall f \in L^i(\Omega, \mathcal{M}, \omega^i)$$

and hence for every $i = r, 1, p$, the identity on $L^i(\Omega, \mathcal{M}, \omega^i)$ is an isometry onto the space $L^i(\Omega, \mathcal{M}, g_i^{1/i}, \omega)$. Consider the measure $\nu = (g_1^p/g_p)^{1/(p-1)}.\omega$ on $(\Omega, \mathcal{M})$, which is σ-finite again as it is easily checked. Let

$$W_1 : L^1(\Omega, \mathcal{M}, g_1, \omega) + L^p(\Omega, \mathcal{M}, g_p^{1/p}, \omega) \to L^1(\Omega, \mathcal{M}, \omega^1) + L^p(\Omega, \mathcal{M}, \omega^p)$$

be the identity map and let

$$W_2 : L^1(\Omega, \mathcal{M}, \nu) + L^p(\Omega, \mathcal{M}, \nu) \to L^1(\Omega, \mathcal{M}, g_1, \omega) + L^p(\Omega, \mathcal{M}, g_p^{1/p}, \omega)$$

be such that $W_2(f) = f(g_1^p/g_p)^{1/(p-1)}(1/g_1)$. Straightforward calculations show that W_1 and W_2 are isometric maps. Now consider the multiplication operators C_w and C_g where $w = g_r^{1/r}(g_1^p/g_p)^{-1/r(p-1)}$ and $g = w^{-1}(g_p/g_1)^{1/(p-1)} = g_r^{-1/r} g_1^{\sigma} g_p^{(1-\sigma)/p}$. C_w is an isometry from $L^r(\Omega, \mathcal{M}, \omega^r)$ onto $L^r(\Omega, \mathcal{M}, \nu)$. Let Q_r be a continuous projection from E_r onto G_r. We have the commutative diagram (arrows without character means natural inclusions)

$$\begin{array}{ccccccc}
\mathcal{U}_\infty & \xrightarrow{Q_r K_r \overline{J}_r (T_d)_{\mathcal{D}}} & G_r & \longrightarrow & G_1 + G_p & \longrightarrow & E_1 + E_p \longrightarrow S \\
 & & \downarrow \Psi_r & & \uparrow \Psi^{-1} & & \\
 & & L^r(\omega^r) & \longrightarrow & L^1(\omega^1) + L^p(\omega^p) & & \\
 & & \downarrow C_w & & \uparrow W_1 W_2 & & \\
 & & L^r(\nu) & \xrightarrow{C_g} & L^1(\nu) + L^p(\nu) & &
\end{array}$$

Given $\varepsilon > 0$, put $A := \{t \in \Omega \mid |g(t)| > (1+\varepsilon)\|C_g\|\}$ and suppose that $\nu(A) > 0$. The transposed map $C_g' : L^\infty(\nu) \cap L^{p'}(\nu) \longrightarrow L^{r'}(\nu)$ verifies

$$\begin{aligned}
(1+\varepsilon)\|C_g\| \|f\chi_A\|_{L^{r'}(\nu)} &\le \|gf\chi_A\|_{L^\infty(\nu)\cap L^{p'}(\nu)} \\
&\le \|C_g'\| \|f\chi_A\|_{L^\infty(\nu)\cap L^{p'}(\nu)}
\end{aligned}$$

for all $f \in L^\infty(\nu) \cap L^{p'}(\nu)$. Since we have the inclusion map $L^\infty(A, \nu) \cap L^{p'}(A, \nu) \subset L^{r'}(A, \nu)$ with norm less or equal than 1, there must be $(1 + \varepsilon)\|C_g\| \le \|C_g\|$ which is a contradiction. Then $|g(t)| \le \|C_g\|$ $\nu-$everywhere and $g \in L^\infty(\nu)$. As the map $C_\omega \Psi_r Q_r K_r \overline{J}_r (T_d)_{\mathcal{D}}$ is positive, it is enough to apply 7.3 and 18.9 of [2] (Maurey's factorization theorem) to get the result. ☑

Theorem 5. *Let $T \in \mathcal{L}(E, F)$ be such that for every finite dimensional subspace $M \subset F'$, the restriction of T_M to every finite dimensional subspace*

$N \subset E$ factorizes in the way

$$N \longrightarrow \ell^{\infty}(\Omega_N, \mu_N) \xrightarrow{D_N} \ell^r(\Omega_N, \mu_N) \xrightarrow{J_r^{\mu N}} \ell^1(\Omega_N, \mu_N) + \ell^p(\Omega_N, \mu_N) \longrightarrow M'$$

where every (Ω_N, μ_N) is a discrete measure space with a finite number of atoms and every D_N is a positive diagonal operator. Then there is a $\sigma-$finite measure space (Ω, μ) such that $J_F T$ factorizes in the way

$$E \longrightarrow L^{\infty}(\Omega, \mu) \xrightarrow{B_w} L^r(\Omega, \mu) \xrightarrow{J_r^{\mu}} L^1(\Omega, \mu) + L^p(\Omega, \mu) \longrightarrow F''$$

where B_w is a diagonal operator.

Proof. The proof goes along the same lines than in the classical case of p-integral operators but using our theorem 4. See [3] for the detailed proof. ☑

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References

[1] C. D. Aliprantis O. and Burkinshaw, *Positive Operators*, Academic Press. New York, London, 1985.

[2] A. Defant and K. Floret, *Tensor norms and operator ideals*, North Holland Math. Studies, Amsterdam, 1993.

[3] J. E. Gilbert and T. J. Leih, *Factorization, tensor products and bilinear forms in Banach space theory*, in *Notes in Banach spaces*, 182–305, University of Texas Press, Austin, London, 1980.

[4] S. Heinrich, *Ultraproducts in Banach spaces theory*, J. reine angew. Math. **313** (1980), 72–104.

[5] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II.* Springer Verlag, Berlin, Heidelberg, New York, 1979.

[6] J. A. López Molina and E. A. Sánchez Pérez, *Ideales de operadores absolutamente continuos.* Rev. Real Acad. Ciencias. Madrid. **87** (1993), 349–378.

[7] J. A. López Molina and E. A. Sánchez Pérez, *Operator ideals related to $(p, \sigma)-$absolutely continuous operators.* Preprint.

[8] U. Matter, *Absolutely continuous operators and super-reflexivity.* Math. Nach. **130** (1987), 193–216.

[9] P. C. Niculescu, *Absolute continuity in Banach space theory.* Rev. Roum. Math. Pures et Appl. **24** No 3 (1979), 413–422.

[10] A. Pietsch, *Operator Ideals.* North Holland Math. Library. North Holland. Amsterdam, New York. 1980.

[11] G. Pisier, *Some applications of the complex interpolation method to Banach lattices.* J. d'Analyse Mathematique, **35** (1979), 264–281.

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