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Zero localization and asymptotic behavior of orthogonal polynomials of Jacobi-Sobolev

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ABSTRACT. In this article we consider the Sobolev orthogonal polynomials associated to the Jacobi's measure on $[-1, 1]$. It is proven that for the class of monic Jacobi-Sobolev orthogonal polynomials, the smallest closed interval that contains its real zeros is $[-\sqrt{1+2C}, \sqrt{1+2C}]$ with C a constant explicitly determined. The asymptotic distribution of those zeros is studied and also we analyze the asymptotic comparative behavior between the sequence of monic Jacobi-Sobolev orthogonal polynomials and the sequence of monic Jacobi orthogonal polynomials under certain restrictions.

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1. Introduction

Let $\mathbb{P}$ be the linear space of all the polynomials with real coefficients. Given $\lambda > 0$ let us define a inner product on $\mathbb{P}$ as

$$\langle p, q\rangle_S = \int_{-1}^{1} p(x)q(x)d\mu(x) + \lambda \int_{-1}^{1} p'(x)q'(x)d\mu(x), \quad p, q \in \mathbb{P}. \tag{1}$$

By the Gram-Schmidt's method there is an unique sequence of monic orthogonal polynomials associated with that product such that there is a representative for each degree. We will denote the corresponding monic orthogonal polynomial of degree n by $Q_n^{(\alpha,\beta)}$. The sequence $\{Q_n^{(\alpha,\beta)}\}_n$ is called the monic Jacobi-Sobolev orthogonal polynomials relative to the inner product (1).

Sobolev orthogonal polynomials have been a subject of increasing interest, but only recently an important advance in the study of its asymptotic properties for a sufficiently general class has taken place. In this connection, we refer to [5] and [6], in which the asymptotic properties of Sobolev polynomials are studied in the continuous case. Particularly, in [5] there is an extensive study of the properties of Gegenbauer-Sobolev polynomials and in [6] the asymptotic behavior of Gegenbauer-Sobolev polynomials and the asymptotic behavior of the zeros and norms of these polynomials is studied. Another important reference is [8], where it is proved that multiplication operator is bounded under certain assumptions and a characterization for the boundedness of the multiplication operator in terms of admissible measures is obtained.

2. **Preliminary results**

As usual, throughout the paper, $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote respectively the natural, real and complex numbers. We denote by $supp(\mu)$ the support of the measure μ. $\overline{\mathbb{P}}_n$ denotes the set of polynomials with real coefficients and degree less than or equal to n.

Definition 2.1. The operator multiplication by x, M_x, is defined on the space $\overline{\mathbb{P}}$ as

$$M_x(p) = xp, \quad \text{for all } p \in \overline{\mathbb{P}}.$$

Definition 2.2. We will say that a family $\{P_n\}_{n\geq 0}$ is a sequence of standard orthogonal polynomials if the multiplication operator M_x is symmetric with respect to the inner product $\langle \cdot , \cdot \rangle$ to which that sequence is associated, i.e., M_x is the self-adjoint operator,

$$\langle M_x(p) \, , \, q\rangle = \langle p \, , M_x(q)\rangle,$$

for any $p, q \in \overline{\mathbb{P}}$.

Definition 2.3. The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$, are defined as the orthogonal polynomials with respect to the Jacobi inner product

$$\langle p, q\rangle_\omega = \int_{-1}^{1} p(x)q(x)\omega^{(\alpha,\beta)}(x)dx, \tag{2}$$

where $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha, \beta > -1.$

The monic Jacobi polynomials, $\{\tilde{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$, are those polynomials whose leading coefficient is 1 and they are orthogonal with respect to the Jacobi inner product $\langle \cdot , \cdot \rangle_\omega$.

Let us denote by

$$m_k = \int_{-1}^{1} x^k \omega^{(\alpha,\beta)}(x)dx,$$

the moments of the measure $d\mu(x) = \omega^{(\alpha,\beta)}dx$.

Definition 2.4. For each $a > 0$ *and* $n \in \mathbb{N}$, the Pochhammer symbol $(a)_n$ is defined as

$$(a)_n = a(a+1)\cdots(a+(n-1))$$

and by convention $(a)_0 = 1$.

Let us mention first some properties of the Jacobi orthogonal polynomials that will be needed in what follows.

Theorem 2.1. *The Jacobi orthogonal polynomials have the following explicit expression:*

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\beta+\alpha+1)_k(\alpha+k+1)_{n-k}\left(\frac{x-1}{2}\right)^k,$$

$$P_n^{(\alpha,\beta)}(1) = \frac{2^n n!\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\binom{n+\alpha}{n}, \tag{3}$$

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n\frac{2^n n!\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)}\binom{n+\beta}{n}.$$

Furthermore,

$$\begin{aligned} k_n^{(\alpha,\beta)} &:= \|\tilde{P}_n^{(\alpha,\beta)}\|_\omega^2 \\ &= 2^{\alpha+\beta+1+2n} n! \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+2)\Gamma(2n+\alpha+\beta+1)}. \end{aligned} \tag{4}$$

The sequence of monic Jacobi orthogonal polynomials is associated to a standard inner product and then

$$\frac{\langle x\tilde{P}_n^{(\alpha,\beta)}, \tilde{P}_n^{(\alpha,\beta)}\rangle_\omega}{\|\tilde{P}_n^{(\alpha,\beta)}\|_\omega^2} = \frac{2(\beta-\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}. \tag{5}$$

The quotient of the norms of two Jacobi consecutive orthogonal polynomials satisfies

$$\frac{k_n^{(\alpha,\beta)}}{k_{n-1}^{(\alpha,\beta)}} = \frac{\|\tilde{P}_n^{(\alpha,\beta)}\|_\omega^2}{\|\tilde{P}_{n-1}^{(\alpha,\beta)}\|_\omega^2} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)2(2n+\alpha+\beta-1)}. \tag{6}$$

Finally,

$$\lim_{n\to\infty}\frac{k_n^{(\alpha,\beta)}}{k_{n-1}^{(\alpha,\beta)}} = \frac{1}{4}. \tag{7}$$

Proof. For the proof of these results, we refer the reader to [9], Chap 4. ☑

Theorem 2.2. *The sequence* $\{\tilde{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ *of monic Jacobi orthogonal polynomials satisfies the following three term recurrence relations for* $n \geq 1$*:*

$$\tilde{P}_{n+1}^{(\alpha,\beta)}(x) = \left(x - \lambda_n^{(\alpha,\beta)}\right)\tilde{P}_n^{(\alpha,\beta)}(x) - \gamma_n^{(\alpha,\beta)}\tilde{P}_{n-1}^{(\alpha,\beta)}(x), \tag{8}$$

where,

$$\lambda_n^{(\alpha,\beta)} = \frac{\beta 2 - \alpha 2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$\gamma_n^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)2(2n+\alpha+\beta-1)}.$$

Proof. The relation (8) is an immediate consequence of the fact that the sequence of monic Jacobi orthogonal polynomials is a sequence of standard orthogonal polynomials with respect to the Jacobi inner product (2). ☑

The Jacobi orthogonal polynomials corresponding to different parameters (α, β) are related by the differentiation process.

Proposition 2.1. *For* $1 \leq \eta \leq n$

$$\mathcal{D}^\eta\left(P_n^{(\alpha,\beta)}(x)\right) = \frac{(n+\beta+\alpha+1)_\eta}{2^\eta}P_{n-\eta}^{(\alpha+\eta,\beta+\eta)}(x), \tag{9}$$

where $\mathcal{D}^\eta(\cdot)$ *denotes the* η*-th derivative with respect to the variable x.*

For $1 \leq \eta \leq n$*, the monic Jacobi orthogonal polynomials* $\tilde{P}_n^{(\alpha,\beta)}$ *satisfies*

$$\mathcal{D}^\eta\left(\tilde{P}_n^{(\alpha,\beta)}(x)\right) = \frac{n!}{(n-\eta)!}\tilde{P}_{n-\eta}^{(\alpha+\eta,\beta+\eta)}(x). \tag{10}$$

Proof. (9) is immediate by induction on η if we use the explicit representation of $P_n^{(\alpha,\beta)}(x)$.

(10) follows from (9) and the relation

$$\frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}\tilde{P}_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x). \qquad ☑$$

The following recurrence formulas allow us to relate different families of monic Jacobi orthogonal polynomials.

Proposition 2.2. *For* $n \geq 2$ *and* $1 \leq \eta \leq n$,

$$\begin{aligned}\tilde{P}_n^{(\alpha,\beta)}(x) &= \tilde{P}_n^{(\alpha+\eta,\beta+\eta)}(x)\\ &+ \sum_{k=1}^{\eta}\delta_k^{(1)}(n,\alpha,\beta)\tilde{P}_{n-1}^{(\alpha+k,\beta+k)}(x) - \delta_k^{(2)}(n,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha+k,\beta+k)}(x),\end{aligned} \tag{11}$$

where the coefficients $\delta_k^{(i)}(n,\alpha,\beta)$, $i=1,2$ *are given by*

$$\delta_k^{(1)}(n,\alpha,\beta)=\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta+2k-2)(2n+\alpha+\beta+2k)},$$

$$\delta_k^{(2)}(n,\alpha,\beta)=$$
$$\frac{4n(n-1)(n+\alpha+k-1)(n+\beta+k-1)}{(2n+\alpha+\beta+2k-1)(2n+\alpha+\beta+2k-2)^2(2n+\alpha+\beta+2k-3)}.$$

Additionally,

$$\lim_{n\to\infty}\delta_k^{(i)}(n,\alpha,\beta)=\begin{cases}0, & if\ i=1\\ \frac{1}{4}, & if\ i=2\end{cases}\tag{12}$$

for $k=1,\,2,\ldots,\,\eta$.

Proof. By induction on the order of differentiation. For $\eta=1$, we know, by Theorem 2.2, that

$$\tilde{P}_{n+1}^{(\alpha,\beta)}(x)=\left(x-\lambda_n^{(\alpha,\beta)}\right)\tilde{P}_n^{(\alpha,\beta)}(x)-\gamma_n^{(\alpha,\beta)}\tilde{P}_{n-1}^{(\alpha,\beta)}(x).$$

Differentiating both sides of the equation and applying (10) we have

$$\tilde{P}_n^{(\alpha,\beta)}(x)=\tilde{P}_n^{(\alpha+1,\beta+1)}(x)+\delta_1^{(1)}(n,\alpha,\beta)\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x)$$
$$-\delta_1^{(2)}(n,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha+1,\beta+1)}(x),$$

with

$$\delta_1^{(1)}(n,\alpha,\beta)=\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$\delta_1^{(2)}(n,\alpha,\beta)=\frac{4n(n-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)2(2n+\alpha+\beta-1)}.$$

Let us suppose now that the proposition is true for $0\le k\le\eta$. By the recurrence relation,

$$\tilde{P}_{n+1+\eta}^{(\alpha,\beta)}(x)=(x-\lambda_{n+\eta})\,\tilde{P}_{n+\eta}^{(\alpha,\beta)}(x)-\gamma_{n+\eta}\tilde{P}_{n+\eta-1}^{(\alpha,\beta)}(x).\tag{13}$$

Differentiating $(\eta+1)$ times both sides of (13) and applying the induction hypothesis we obtain the result. ☑

In what follows, we will need the asymptotic behavior of the monic Jacobi orthogonal polynomials so, let us state it next (for the proofs see [3] or [9]).

Theorem 2.3. *The monic Jacobi orthogonal polynomials have the following asymptotic behavior:*

$$\tilde{P}_n^{(\alpha,\beta)}(x)=2^{-\alpha-\beta}\frac{(\Phi(x))^{n+1/2}}{(x^2-1)^{1/4}}\frac{(\sqrt{x-1}+\sqrt{x+1})^{\alpha+\beta}}{(x-1)^{\alpha/2}(x+1)^{\beta/2}}(1+o(1)),\tag{14}$$

$$\frac{\tilde{P}_{n+1}^{(\alpha,\beta)}(x)}{\tilde{P}_{n}^{(\alpha,\beta)}(x)} = \Phi(x)(1+o(1)), \tag{15}$$

$$\frac{\mathcal{D}^1\left(\tilde{P}_{n+1}^{(\alpha,\beta)}(x)\right)}{n\tilde{P}_{n}^{(\alpha,\beta)}(x)} = \frac{1}{\sqrt{x^2-1}}(1+o(1)), \tag{16}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$, *where*

$$\Phi(x) = \frac{x+\sqrt{x^2-1}}{2}.$$

3. Related definitions and results for monic Jacobi-Sobolev orthogonal polynomials

Let us define a inner product on $\overline{\mathbb{P}}$ by

$$\langle p, q\rangle_S = \int_{-1}^{1} p(x)q(x)d\mu(x) + \lambda \int_{-1}^{1} p'(x)q'(x)d\mu(x), \tag{17}$$

where $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx \quad \alpha, \beta > -1$, and $\lambda > 0$. We will call it Jacobi-Sobolev inner product.

We will denote the sequence of monic orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle_S$ by $\{Q_n^{(\alpha,\beta)}\}_{n\geq 0}$ and call it the monic Jacobi-Sobolev orthogonal polynomials sequence. Let us denote by

$$K_n^{(\alpha,\beta)} := \|Q_n^{(\alpha,\beta)}\|_S^2, \tag{18}$$

the square of the Sobolev norm of the polynomial $Q_n^{(\alpha,\beta)}$.

It is easy to verify (see definition 3.3) that

$$Q_0^{(\alpha,\beta)}(x) = \tilde{P}_0^{(\alpha,\beta)}(x) = 1,$$
$$Q_1^{(\alpha,\beta)}(x) = \tilde{P}_1^{(\alpha,\beta)}(x) = x + \frac{(\alpha-\beta)}{(\alpha+\beta+2)}.$$

For $n > 1$, $\tilde{P}_n^{(\alpha,\beta)}(x) \neq Q_0^{(\alpha,\beta)}(x)$ due to the presence of the second summand in (17).

Now writing

$$Q_n^{(\alpha,\beta)}(x) = x^n + \sum_{i=0}^{n-1} b_{i,n}x^i, \quad b_{i,n} \in \mathbb{R},$$

the orthogonality condition implies the following system of equations:

$$\langle x^n, x^j\rangle_S + b_{n-1,n}\langle x^{n-1}, x^j\rangle_S + \cdots + \\ + b_{1,n}\langle x, x^j\rangle_S + b_{0,n}\langle 1, x^j\rangle_S = 0, \quad 0 \leq j \leq n-1. \tag{19}$$

If we put $c_{i,j} = \langle x^i , x^j \rangle_S$, then (19) can be written in terms of the moments m_j associated to the Jacobi's inner product

$$\begin{aligned} c_{i,0} &= \langle x^i , 1 \rangle_S = m_i, \ i \geq 0, \\ c_{0,j} &= \langle 1 , x^j \rangle_S = m_j, \ j \geq 0, \\ c_{i,j} &= m_{i+j} + ij\lambda m_{i+j-2}, \ i+j \geq 2. \end{aligned} \tag{20}$$

Applying Cramér's rule, we obtain the following expression for the monic orthogonal polynomial:

$$Q_n^{(\alpha,\beta)}(x) = \frac{\begin{vmatrix} c_{0,0} & c_{1,0} & \cdots & c_{n,0} \\ c_{0,1} & c_{1,1} & \cdots & c_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1} & c_{1,n-1} & \cdots & c_{n,n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}}{\begin{vmatrix} c_{0,0} & c_{1,0} & \cdots & c_{n-1,0} \\ c_{0,1} & c_{1,1} & \cdots & c_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1} & c_{1,n-1} & \cdots & c_{n-1,n-1} \end{vmatrix}}$$

Using (20), this last equality can be expressed in terms of the moments associated to the Jacobi inner product as follows:

$$Q_n^{(\alpha,\beta)}(x) = \frac{\begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ \frac{m_1}{\lambda} & \frac{m_2}{\lambda} + m_0 & \cdots & \frac{m_{n+1}}{\lambda} + nm_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_{n-1}}{\lambda} & \frac{m_n}{\lambda} + (n-1)m_{n-2} & \cdots & \frac{m_{2n-1}}{\lambda} + n(n-1)m_{2n-3} \\ 1 & x & \cdots & x^n \end{vmatrix}}{\begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ \frac{m_1}{\lambda} & \frac{m_2}{\lambda} + m_0 & \cdots & \frac{m_n}{\lambda} + (n-1)m_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_{n-1}}{\lambda} & \frac{m_n}{\lambda} + (n-1)m_{n-2} & \cdots & \frac{m_{2n-2}}{\lambda} + (n-1)^2 m_{2n-4} \end{vmatrix}} \tag{21}$$

Observe that each coefficient of $Q_n^{(\alpha,\beta)}$ is a rational function in λ, whose numerator and denominator have degree $n-1$. This property allows us to treat the polynomial $Q_n^{(\alpha,\beta)}$ as a function of two variables; that is,

$$Q_n^{(\alpha,\beta)}(x) = Q_n^{(\alpha,\beta)}(x, \lambda),$$

whenever $n \geq 0$. Now, taking $\lambda \to \infty$, we obtain

Definition 3.1. The limit polynomial with respect to λ associated to the sequence of monic orthogonal polynomials $\{Q_n^{(\alpha,\beta)}\}_{n\geq 0}$ is given by

$$R_0^{(\alpha,\beta)}(x) = Q_0^{(\alpha,\beta)}(x) = \tilde{P}_0^{(\alpha,\beta)}(x) = 1,$$
$$R_1^{(\alpha,\beta)}(x) = Q_1^{(\alpha,\beta)}(x) = \tilde{P}_1^{(\alpha,\beta)}(x) = x + \frac{(\alpha-\beta)}{\alpha+\beta+2}$$

and for $n \geq 2$:

$$R_n^{(\alpha,\beta)}(x) = \lim_{\lambda\to\infty} Q_n^{(\alpha,\beta)}(x,\lambda) = \frac{\begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ 0 & m_0 & \cdots & nm_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (n-1)m_{n-2} & \cdots & n(n-1)m_{2n-3} \\ 1 & x & \cdots & x^n \end{vmatrix}}{\begin{vmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ 0 & m_0 & \cdots & (n-1)m_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (n-1)m_{n-2} & \cdots & (n-1)^2 m_{2n-4} \end{vmatrix}}$$

Observe that $R_n^{(\alpha,\beta)}$ is a monic polynomial of degree n and independent of λ. The polynomials $\{R_n^{(\alpha,\beta)}\}$ satisfies the following properties:

Theorem 3.1. *For $n \geq 2$, we have:*

$$\int_{-1}^{1} R_n^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx = 0. \tag{22}$$

For $0 \leq k \leq n-2$,

$$\int_{-1}^{1} \mathcal{D}^1\left(R_n^{(\alpha,\beta)}(x)\right) x^k \omega^{(\alpha,\beta)}(x)dx = 0. \tag{23}$$

For $1 \leq \eta \leq n$,

$$\begin{aligned} \mathcal{D}^\eta\left(R_n^{(\alpha,\beta)}(x)\right) &= n(n-1)(n-2)\cdots(n-\eta+1)\tilde{P}_{n-\eta}^{(\alpha+\eta-1,\beta+\eta-1)}(x) \\ &= (n-\eta+1)_\eta \tilde{P}_{n-\eta}^{(\alpha+\eta-1,\beta+\eta-1)}(x). \end{aligned} \tag{24}$$

Proof. Since $\langle Q_n^{(\alpha,\beta)}, 1\rangle_S = 0,\ n \geq 2$, then

$$\int_{-1}^{1} Q_n^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx = 0.$$

Now, let us call $f_\lambda(x) = Q_n^{(\alpha,\beta)}(x,\lambda)$; then $f_\lambda \xrightarrow{a.e} R_n^{(\alpha,\beta)}$ as $\lambda \to \infty$. Using the Lebesgue's dominated convergence theorem, we get

$$\int_{-1}^{1} \left| f_\lambda(x) - R_n^{(\alpha,\beta)}(x) \right| \omega^{(\alpha,\beta)}(x)dx \longrightarrow 0, \text{ as } \lambda \to \infty,$$

and therefore

$$\lim_{\lambda\to\infty} \int_{-1}^{1} Q_n^{(\alpha,\beta)}(x,\lambda)\omega^{(\alpha,\beta)}(x)dx = \int_{-1}^{1} R_n^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx,$$

so we get (22).

Since for $0 \le k \le n-2$,

$$\langle Q_n^{(\alpha,\beta)}, x^{k+1} \rangle_S = 0,$$

then,

$$\int_{-1}^{1} \mathcal{D}^1 \left(Q_n^{(\alpha,\beta)}(x) \right) x^k \omega^{(\alpha,\beta)}(x)dx = -\frac{1}{\lambda(k+1)} \int_{-1}^{1} Q_n^{(\alpha,\beta)}(x) x^{k+1} \omega^{(\alpha,\beta)}(x)dx,$$

and taking $\lambda \to \infty$, we get

$$\lim_{\lambda\to\infty} \int_{-1}^{1} \mathcal{D}^1(Q_n^{(\alpha,\beta)}(x,\lambda)) x^k \omega^{(\alpha,\beta)}(x)dx$$

$$= -\lim_{\lambda\to\infty} \frac{1}{\lambda(k+1)} \int_{-1}^{1} Q_n^{(\alpha,\beta)}(x) x^{k+1} \omega^{(\alpha,\beta)}(x)dx = 0.$$

Therefore,

$$\lim_{\lambda\to\infty} \int_{-1}^{1} \mathcal{D}^1 \left(Q_n^{(\alpha,\beta)}(x,\lambda) \right) x^k \omega^{(\alpha,\beta)}(x)dx = 0. \tag{25}$$

Now, if we put $g_\lambda(x) = \mathcal{D}^1 \left(Q_n^{(\alpha,\beta)}(x,\lambda) \right) x^k$ then $g_\lambda \xrightarrow{a.e} \mathcal{D}^1 \left(R_n^{(\alpha,\beta)}(x) \right)$ as $\lambda \to \infty$. Again, Lebesgue's dominated convergence theorem, give us

$$\int_{-1}^{1} \left| g_\lambda(x) - \mathcal{D}^1 \left(R_n^{(\alpha,\beta)}(x) \right) x^k \right| \omega^{(\alpha,\beta)}(x)dx \longrightarrow 0, \ \lambda \to \infty.$$

Therefore

$$\lim_{\lambda\to\infty} \int_{-1}^{1} \mathcal{D}^1 \left(Q_n^{(\alpha,\beta)}(x,\lambda) \right) x^k \omega^{(\alpha,\beta)}(x)dx = \int_{-1}^{1} \mathcal{D}^1 \left(R_n^{(\alpha,\beta)}(x) \right) x^k \omega^{(\alpha,\beta)}(x)dx;$$

and using (25) , we obtain (23).

To verify (24) we use induction on η. For $\eta = 1$, as a consequence of (23) we have

$$\mathcal{D}^1 \left(R_n^{(\alpha,\beta)}(x) \right) = c\tilde{P}_{n-1}^{(\alpha,\beta)}(x),$$

with c a constant. Then comparing the leading coefficients of both polynomials, is clear that $n = c$. Therefore

$$\mathcal{D}^1 \left(R_n^{(\alpha,\beta)}(x) \right) = n\tilde{P}_{n-1}^{(\alpha,\beta)}(x).$$

Let us suppose that the proposition is true for $1 \leq s \leq \eta$, that is,

$$\mathcal{D}^s\left(R_n^{(\alpha,\beta)}(x)\right) = (n-s+1)_s\, \tilde{P}_{n-s}^{(\alpha+s-1,\beta+s-1)}(x), \quad s = 1, \cdots, \eta.$$

Then for $s = \eta + 1$

$$\begin{aligned}\mathcal{D}^{\eta+1}\left(R_n^{(\alpha,\beta)}(x)\right) &= (n-\eta+1)_\eta\, \mathcal{D}^1\left(\tilde{P}_{n-\eta}^{(\alpha+\eta-1,\beta+\eta-1)}(x)\right) \\ &= (n-\eta)_{\eta+1}\, \tilde{P}_{n-(\eta+1)}^{(\alpha+\eta,\beta+\eta)}(x).\end{aligned}$$

☑

Corollary 3.1. *For all $\alpha, \beta > -1$ and $n \geq 3$, the polynomial $R_n^{(\alpha+1,\beta+1)}$ is the monic Jacobi polynomial of degree n, $\tilde{P}_n^{(\alpha,\beta)}$, associated to the weight $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$.*

Proof. Using (24) with $\eta = 1$, we have

$$\mathcal{D}^1\left(R_n^{(\alpha+1,\beta+1)}(x)\right) = n\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x) = \mathcal{D}^1\left(\tilde{P}_n^{(\alpha,\beta)}(x)\right);$$

thus,

$$\mathcal{D}^1\left(R_n^{(\alpha+1,\beta+1)}(x) - \tilde{P}_n^{(\alpha,\beta)}(x)\right) = 0.$$

Then $R_n^{(\alpha+1,\beta+1)}(x) - \tilde{P}_n^{(\alpha,\beta)}(x) = C$, for some constant C and therefore

$$\int_{-1}^{1}\left(R_n^{(\alpha+1,\beta+1)}(x) - \tilde{P}_n^{(\alpha,\beta)}(x)\right)\omega^{(\alpha+1,\beta+1)}(x)dx = C\int_{-1}^{1}\omega^{(\alpha+1,\beta+1)}(x)dx.$$

But by (23) (Theorem 3.1),

$$\int_{-1}^{1} R_n^{(\alpha+1,\beta+1)}(x)\omega^{(\alpha+1,\beta+1)}(x)dx = 0;$$

therefore

$$-\int_{-1}^{1}\tilde{P}_n^{(\alpha,\beta)}(x)\omega^{(\alpha+1,\beta+1)}(x)dx = C\int_{-1}^{1}\omega^{(\alpha+1,\beta+1)}(x)dx.$$

Using Proposition 2.2,

$$\begin{aligned}C\int_{-1}^{1}\omega^{(\alpha+1,\beta+1)}(x)dx = -\int_{-1}^{1}&(\tilde{P}_n^{(\alpha+1,\beta+1)}(x) + \delta_1^{(1)}(n,\alpha,\beta)\,\tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x) \\ &- \delta_1^{(2)}(n,\alpha,\beta)\,\tilde{P}_{n-2}^{(\alpha+1,\beta+1)}(x))\omega^{(\alpha+1,\beta+1)}(x)dx = 0,\end{aligned}$$

hence $C = 0$ and thus

$$R_n^{(\alpha+1,\beta+1)}(x) = \tilde{P}_n^{(\alpha,\beta)}(x),$$

whenever $n \geq 3$. ☑

Theorem 3.2. *For* $\alpha, \beta > 0$, $n \geq 3$ *and* $x \notin [-1, 1]$,

$$\lim_{n\to\infty} \frac{R_n^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} = 1 - \frac{1}{4(\Phi(x))2}, \tag{26}$$

where

$$\Phi(x) = \frac{x + \sqrt{x^2 - 1}}{2}.$$

Proof. For each $n \geq 3$, by Corollary 3.1 and Proposition 2.2, we can write

$$\frac{R_n^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} = 1 + \delta_1^{(1)}(n,\alpha,\beta)\frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} - \delta_1^{(2)}(n,\alpha,\beta)\frac{\tilde{P}_{n-2}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)}; \tag{27}$$

now using this and Theorem 2.3, we get the result. ☑

Theorem 3.3. *For* $\alpha, \beta > 0$, $n \geq 3$ *and* λ *sufficiently large, there are two sequences of real numbers* $\{d_n(\lambda)\}$ *and* $\{d'_n(\lambda)\}$ *such that*

$$R_n^{(\alpha,\beta)}(x) = Q_n^{(\alpha,\beta)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(\alpha,\beta)}(x) - d'_{n-2}(\lambda)Q_{n-2}^{(\alpha,\beta)}(x). \tag{28}$$

Proof. Since we can express to the polynomial $R_n^{(\alpha,\beta)}$ in terms of the monic Jacobi-Sobolev polynomials, i.e.,

$$R_n^{(\alpha,\beta)}(x) = Q_n^{(\alpha,\beta)}(x) + \sum_{i=0}^{n-1} a_i^{(n)}(\lambda)Q_i^{(\alpha,\beta)}(x), \tag{29}$$

by orthogonality, we have

$$a_i^{(n)}(\lambda) = \frac{\langle R_n^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_S}{\|Q_i^{(\alpha,\beta)}\|_S^2} \qquad i = 0, \ldots, n-1.$$

Now, for each i, $0 \leq i \leq n-1$,

$$\begin{aligned}\langle R_n^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_S &= \langle R_n^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_\omega + \lambda n\langle \tilde{P}_{n-1}^{(\alpha,\beta)}, \mathcal{D}^1(Q_i^{(\alpha,\beta)})\rangle_\omega \\ &= \langle R_n^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_\omega.\end{aligned} \tag{30}$$

On the other hand, using Corollary 3.1 and Proposition 2.2 one can get that for $n \geq 3$

$$\begin{aligned}&\langle R_n^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_S = \\ &\qquad \langle \tilde{P}_n^{(\alpha,\beta)} + \delta_1^{(1)}(n,\alpha,\beta)\tilde{P}_{n-1}^{(\alpha,\beta)} - \delta_1^{(2)}(n,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_\omega.\end{aligned} \tag{31}$$

By orthogonality of $\tilde{P}_n^{(\alpha,\beta)}$,

$$\langle R_n^{(\alpha,\beta)}, Q_i^{(\alpha,\beta)}\rangle_S = 0 \quad i = 0, \ldots, n-3;$$

but

$$\langle R_n^{(\alpha,\beta)}, Q_{n-2}^{(\alpha,\beta)}\rangle_S = -\delta_1^{(2)}(n,\alpha,\beta)\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2$$

and

$$\langle R_n^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_S = \delta_1^{(1)}(n,\alpha,\beta)\|\tilde{P}_{n-1}^{(\alpha,\beta)}\|_\omega^2 - \delta_1^{(2)}(n,\alpha,\beta)\langle \tilde{P}_{n-2}^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_\omega,$$

that is,

$$a_i^{(n)}(\lambda) = \begin{cases} 0, & if \;\; i = 0,\dots,n-3 \\ \dfrac{\delta_1^{(2)}(n,\alpha,\beta)\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-2}^{(\alpha,\beta)}\|_S^2}, & if \;\; i = n-2 \\ \dfrac{\delta_1^{(1)}(n,\alpha,\beta)\|\tilde{P}_{n-1}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-1}^{(\alpha,\beta)-}\|_S^2} - \dfrac{\delta_1^{(2)}(n,\alpha,\beta)\langle \tilde{P}_{n-2}^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_\omega}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2}, & \text{if } i = n-1. \end{cases}$$

Putting $d_{n-1}(\lambda) = a_{n-1}^{(n)}(\lambda)$ and $d'_{n-2}(\lambda) = -a_{n-2}^{(n)}(\lambda)$, we obtain (28). ☑

The limit polynomial also allows us to study the behavior of the norm of the Jacobi-Sobolev polynomials, as we will see in the following theorem:

Theorem 3.4. *For all $\alpha,\beta > 0$, $n \geq 3$, $k_n^{(\alpha,\beta)}$ defined as in (4) and $K_n^{(\alpha,\beta)}$ defined as in (18), we have*

$$k_n^{(\alpha,\beta)} + \lambda n^2 k_{n-1}^{(\alpha,\beta)} \leq K_n^{(\alpha,\beta)} \leq k_n^{(\alpha,\beta)} + \left[\left(\delta_1^{(1)}(n,\alpha,\beta)\right)^2 + \lambda n^2\right] k_{n-1}^{(\alpha,\beta)} + \left(\delta_1^{(2)}(n,\alpha,\beta)\right)^2 k_{n-2}^{(\alpha,\beta)}; \tag{32}$$

furthermore,

$$\lim_{n\to\infty} \frac{K_n^{(\alpha,\beta)}}{n^2 k_{n-1}^{(\alpha,\beta)}} = \lambda. \tag{33}$$

Proof. By the well known extremal property of the Jacobi norm,

$$k_n^{(\alpha,\beta)} = \inf\{\langle p,p\rangle_\omega : \deg(p) = n,\; p \text{ monic}\};$$

therefore,

$$K_n^{(\alpha,\beta)} \geq k_n^{(\alpha,\beta)} + \lambda n^2 k_{n-1}^{(\alpha,\beta)}. \tag{34}$$

On the other hand,

$$\|R_n^{(\alpha,\beta)}\|_S^2 = \|\tilde{P}_n^{(\alpha,\beta)}\|_\omega^2 + \left[\left(\delta_1^{(1)}(n,\alpha,\beta)\right)^2 + \lambda n^2\right]\|\tilde{P}_{n-1}^{(\alpha,\beta)}\|_\omega^2 + \left(\delta_1^{(2)}(n,\alpha,\beta)\right)^2\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2.$$

Using the extremal property of $K_n^{(\alpha,\beta)}$ we get

$$K_n^{(\alpha,\beta)} \leq \|R_n^{(\alpha,\beta)}\|_S^2 = k_n^{(\alpha,\beta)} + \left[\left(\delta_1^{(1)}(n,\alpha,\beta)\right)^2 + \lambda n^2\right] k_{n-1}^{(\alpha,\beta)} + \left(\delta_1^{(2)}(n,\alpha,\beta)\right)^2 k_{n-2}^{(\alpha,\beta)}; \tag{35}$$

therefore, from (34) and (35) we obtain (32). Dividing (32) by $n^2 k_{n-1}^{(\alpha,\beta)}$, we obtain

$$\frac{k_n^{(\alpha,\beta)}}{n^2 k_{n-1}^{(\alpha,\beta)}} + \lambda \le \frac{K_n^{(\alpha,\beta)}}{n^2 k_{n-1}^{(\alpha,\beta)}} \le \frac{k_n^{(\alpha,\beta)}}{n^2 k_{n-1}^{(\alpha,\beta)}} + \lambda + \left(\frac{\delta_1^{(1)}(n,\alpha,\beta)}{n}\right)2 + \left(\frac{\delta_1^{(2)}(n,\alpha,\beta)}{n}\right)2\frac{k_{n-2}^{(\alpha,\beta)}}{k_{n-1}^{(\alpha,\beta)}}.$$

Taking limit as $n \to \infty$ in the previous chain of inequalities and using (6) and (7), we obtain (33). ☑

In what follows we will relate the Jacobi-Sobolev polynomials with the Jacobi polynomials using the previous results. The purpose of this study is to establish a comparison criterion by the limit between the sequences $\{Q_n^{(\alpha,\beta)}\}_n$ and $\{\tilde{P}_n^{(\alpha,\beta)}\}_n$; for $\alpha, \beta > 0$ and λ sufficiently large when we are outside of the support of the measure of orthogonality, that is to say, we want to determine the limit

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)},$$

for $x \notin [-1, 1]$.

In the first place, the following technical results is needed.

Theorem 3.5. *For $\alpha, \beta > 0$, $n \ge 3$ and λ sufficiently large, there are two sequences of real numbers $\{d_n(\lambda)\}$ and $\{d'_n(\lambda)\}$ such that*

$$Q_n^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\alpha,\beta)}(x) + \delta_1^{(1)}(n,\alpha,\beta)\tilde{P}_{n-1}^{(\alpha,\beta)}(x) - \delta_1^{(2)}(n,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha,\beta)}(x) - d_{n-1}(\lambda)Q_{n-1}^{(\alpha,\beta)}(x) + d'_{n-2}(\lambda)Q_{n-2}^{(\alpha,\beta)}(x). \tag{36}$$

Furthermore,

$$\lim_{n\to\infty} d_n(\lambda) = 0, \tag{37}$$

$$\lim_{n\to\infty} d'_n(\lambda) = 0. \tag{38}$$

Proof. By Corollary (3.1), Proposition 2.2 and Theorem 3.3, it is deduced that if $n \ge 3$, $\alpha, \beta > 0$ and λ is sufficiently large

$$\tilde{P}_n^{(\alpha,\beta)}(x) + \delta_1^{(1)}(n,\alpha,\beta)\tilde{P}_{n-1}^{(\alpha,\beta)}(x) - \delta_1^{(2)}(n,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha,\beta)}(x) = Q_n^{(\alpha,\beta)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(\alpha,\beta)}(x) - d'_{n-2}(\lambda)Q_{n-2}^{(\alpha,\beta)}(x),$$

and from this (36) is obtained. Now, since

$$d'_{n-2}(\lambda) = \delta_1^{(2)}(n,\alpha,\beta)\frac{\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-2}^{(\alpha,\beta)}\|_S^2} = \delta_1^{(2)}(n,\alpha,\beta)\frac{k_{n-2}^{(\alpha,\beta)}}{K_{n-2}^{(\alpha,\beta)}}$$
$$= \delta_1^{(2)}(n,\alpha,\beta)\left(\frac{k_{n-2}^{(\alpha,\beta)}}{k_{n-3}^{(\alpha,\beta)}}\right)\left(\frac{k_{n-3}^{(\alpha,\beta)}}{K_{n-2}^{(\alpha,\beta)}}\right),$$

by (14) we have $d'_{n-2}(\lambda) > 0$ for $\alpha,\beta > 0$ and $n \geq 3$. Using Theorem 3.4 we have

$$k_{n-2}^{(\alpha,\beta)} + \lambda(n-2)^2 k_{n-3}^{(\alpha,\beta)} \leq K_{n-2}^{(\alpha,\beta)};$$

therefore

$$d'_{n-2}(\lambda) \leq \delta_1^{(2)}(n,\alpha,\beta)\left(\frac{k_{n-2}^{(\alpha,\beta)}}{k_{n-3}^{(\alpha,\beta)}}\right)\frac{1}{\lambda(n-2)^2}$$

and, since

$$\lim_{n\to\infty}\frac{k_{n-2}^{(\alpha,\beta)}}{k_{n-3}^{(\alpha,\beta)}} = \frac{1}{4} = \lim_{n\to\infty}\delta_1^{(2)}(n,\alpha,\beta) \text{ and } \lim_{n\to\infty}\frac{1}{\lambda(n-2)^2} = 0,$$

we get

$$\lim_{n\to\infty} d'_{n-2}(\lambda) = 0.$$

In a similar way, we will prove that $\lim_{n\to\infty} d_{n-1}(\lambda) = 0$. As

$$d_{n-1}(\lambda) = \frac{\delta_1^{(1)}(n,\alpha,\beta)\|\tilde{P}_{n-1}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2} - \frac{\delta_1^{(2)}(n,\alpha,\beta)\langle\tilde{P}_{n-2}^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_\omega}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2},$$

the sequence of positive terms $\left\{\frac{\|\tilde{P}_{n-1}^{(\alpha,\beta)d}\|_\omega^2}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2}\right\}_n$ is bounded and

$$\lim_{n\to\infty}\delta_1^{(1)}(n,\alpha,\beta) = 0;$$

then, our problem reduces to demonstrate that

$$\lim_{n\to\infty}\frac{\delta_1^{(2)}(n,\alpha,\beta)\langle\tilde{P}_{n-2}^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_\omega}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2} = 0.$$

By (36), we have

$$Q_{n-1}^{(\alpha,\beta)}(x) = \tilde{P}_{n-1}^{(\alpha,\beta)}(x) + \delta_1^{(1)}(n-1,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha,\beta)}(x) - \delta_1^{(2)}(n-1,\alpha,\beta)\tilde{P}_{n-3}^{(\alpha,\beta)}(x)$$
$$- d_{n-2}(\lambda)Q_{n-2}^{(\alpha,\beta)}(x) + d'_{n-3}(\lambda)Q_{n-3}^{(\alpha,\beta)}(x).$$

Then, by the orthogonality of $\tilde{P}_{n-2}^{(\alpha,\beta)}$, we get

$$\delta_1^{(2)}(n,\alpha,\beta)\frac{\langle \tilde{P}_{n-2}^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_\omega}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2} = \delta_1^{(2)}(n,\alpha,\beta)\left[\delta_1^{(1)}(n-1,\alpha,\beta) - d_{n-2}(\lambda)\right]\frac{k_{n-2}^{(\alpha,\beta)}}{K_{n-1}^{(\alpha,\beta)}}. \tag{39}$$

But then, applying the Cauchy-Schwarz inequality we have

$$|d_{n-2}(\lambda)| \le \left|\delta_1^{(1)}(n-1,\alpha,\beta)\right| \frac{\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2} + \left|\delta_1^{(2)}(n-1,\alpha,\beta)\right| \frac{\left|\langle \tilde{P}_{n-3}^{(\alpha,\beta)}, Q_{n-2}^{(\alpha,\beta)}\rangle\right|}{\|Q_{n-2}^{(\alpha,\beta)}\|_S^2},$$

$$|d_{n-2}(\lambda)| \le \left|\delta_1^{(1)}(n-1,\alpha,\beta)\right| \frac{\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2} + \left|\delta_1^{(2)}(n-1,\alpha,\beta)\right| \frac{\|\tilde{P}_{n-3}^{(\alpha,\beta)}\|_\omega}{\|Q_{n-2}^{(\alpha,\beta)}\|_S}. \tag{40}$$

On the other hand, the sequence of positive terms $\left\{\frac{\|\tilde{P}_{n-2}^{(\alpha,\beta)}\|_\omega^2}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2}\right\}_{n\ge 3}$ is uniformly bounded since

$$\frac{k_{n-2}^{(\alpha,\beta)}}{K_{n-1}^{(\alpha,\beta)}} \le \frac{1}{\lambda(n-1)^2} \tag{41}$$

and

$$\lim_{n\to\infty} \delta_1^{(2)}(n,\alpha,\beta)d_{n-2}(\lambda) = 0 = \lim_{n\to\infty} \delta_1^{(2)}(n,\alpha,\beta)\delta_1^{(1)}(n-1,\alpha,\beta).$$

From (40) and (41) it follows that the sequence $\{d_n\}_n$ is also uniformly bounded. Therefore

$$\lim_{n\to\infty} \frac{\delta_1^{(2)}(n,\alpha,\beta)\langle \tilde{P}_{n-2}^{(\alpha,\beta)}, Q_{n-1}^{(\alpha,\beta)}\rangle_\omega}{\|Q_{n-1}^{(\alpha,\beta)}\|_S^2} = 0;$$

thus

$$\lim_{n\to\infty} d_{n-1}(\lambda) = 0.$$ ☑

By the previous result we can obtain the relative asymptotic behavior of $\left\{Q_n^{(\alpha,\beta)}\right\}_n$ with respect to $\left\{\tilde{P}_n^{(\alpha,\beta)}\right\}_n$.

Theorem 3.6. *For α, $\beta > 0$, we have*

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} = 1 - \frac{1}{4(\Phi(x))^2}, \tag{42}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$, where $\Phi(x) = \frac{x+\sqrt{x2-1}}{2}$.

Proof. From the relation (36) we have

$$Q_n^{(\alpha,\beta)}(x) + d_{n-1}(\lambda)Q_{n-1}^{(\alpha,\beta)}(x) - d'_{n-2}(\lambda)Q_{n-2}^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\alpha,\beta)}(x) + \delta_1^{(1)}(n,\alpha,\beta)\tilde{P}_{n-1}^{(\alpha,\beta)}(x) - \delta_1^{(2)}(n,\alpha,\beta)\tilde{P}_{n-2}^{(\alpha,\beta)}(x).$$

Dividing by $\tilde{P}_n^{(\alpha,\beta)}$ both members of the previous equality and putting

$$Y_n(x) := \frac{Q_n^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)}, \quad \alpha_n(x) := d_n(\lambda)\frac{\tilde{P}_n^{(\alpha,\beta)}(x)}{\tilde{P}_{n+1}^{(\alpha,\beta)}(x)}, \quad \alpha'_n(x) := d'_n(\lambda)\frac{\tilde{P}_n^{(\alpha,\beta)}(x)}{\tilde{P}_{n+2}^{(\alpha,\beta)}(x)},$$

$$\beta_n(x) := 1 + \delta_1^{(1)}(n,\alpha,\beta)\frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} - \delta_1^{(2)}(n,\alpha,\beta)\frac{\tilde{P}_{n-2}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)},$$

we get

$$Y_n(x) + \alpha_{n-1}(x)Y_{n-1}(x) - \alpha'_{n-2}(x)Y_{n-2}(x) = \beta_n(x), \tag{43}$$

$n \geq 3$. The sequence $\{Y_n\}_n$ is a sequence of analytic functions on $\mathbb{C}\setminus[-1, 1]$, with $Y_0 \equiv Y_1 \equiv Y_2 \equiv 1$. Therefore,

$$\begin{aligned} |Y_n(x)| &= |\beta_n(x) - \alpha_{n-1}(x)Y_{n-1}(x) + \alpha'_{n-2}(x)Y_{n-2}(x)|, \\ &\leq |\beta_n(x)| + |\alpha_{n-1}(x)|\,|Y_{n-1}(x)| + |\alpha'_{n-2}(x)|\,|Y_{n-2}(x)| . \end{aligned} \tag{44}$$

Using (17), (37) and (38), we deduce that there exist $n_0, n_1 \in \mathbb{N}$, such that

$$|\alpha_n(x)| < \frac{1}{4}, \quad n \geq n_0, \tag{45}$$

$$|\alpha'_n(x)| < \frac{1}{4}, \quad n \geq n_1. \tag{46}$$

On the other hand,

$$\begin{aligned} |\beta_n(x)| &= \left| 1 + \delta_1^{(1)}(n,\alpha,\beta)\frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} - \delta_1^{(2)}(n,\alpha,\beta)\frac{\tilde{P}_{n-2}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} \right| \\ &\leq 1 + \left|\delta_1^{(1)}(n,\alpha,\beta)\right| \left|\frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)}\right| \\ &\quad + \left|\delta_1^{(2)}(n,\alpha,\beta)\right| \left|\frac{\tilde{P}_{n-2}^{(\alpha,\beta)}(x)}{\tilde{P}_{n-1}^{(\alpha,\beta)}(x)}\right| \left|\frac{\tilde{P}_{n-1}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)}\right|, \end{aligned} \tag{47}$$

Using (40), Proposition 2.2 and the inequality $|\Phi(x)| > \frac{1}{2}$, for $x \notin [-1, 1]$, we deduce that there exists $M > 0$ y $n_2 \in \mathbb{N}$, such that

$$|\beta_n(x)| < M, \quad n \geq n_2. \tag{48}$$

Taking $n_3 = \max\{n_0, n_1, n_2\}$ and using (45), (46) and (48) in (44), one gets

$$|Y_n(x)| < \frac{1}{2}(|Y_{n-1}(x)| + |Y_{n-2}(x)|) + M, \quad n \geq n_3. \tag{49}$$

Let us prove now that the sequence $\{Y_n\}_n$ is uniformly bounded. For this consider the auxiliary sequence

$$Z_n(x) = \begin{cases} |Y_n(x)|, & n \leq n_3, \\ \frac{1}{2}\left(Z_{n-1}(x) + Z_{n-2}(x)\right) + M, & n > n_3. \end{cases}$$

For $m > n_3$ fixed,

$$Z_{m+r}(x) = \frac{a_{r+1}}{2^r} Z_m(x) + \frac{a_r}{2^r} Z_{m-1}(x) + 2M\left(1 - \frac{1}{2^r}\right), \quad r = 1, 2, \ldots \tag{50}$$

where the increasing sequence of positive terms $\{a_r\}_r$, satisfies the following recurrence relation:

$$2a_{r-1} + a_r = a_{r+1} \quad \text{for all } r \geq 2,$$

with $a_1 = a_2 = 1$. Taking limit in (50), as $r \to \infty$, we obtain that the sequence $\{Z_n\}_n$ is uniformly bounded for n sufficiently large. Now, $0 < |Y_n(x)| \leq Z_n(x)$, for all $n \in \mathbb{N}$, therefore $\{Y_n\}_n$ is uniformly bounded. Finally, taking limit as $n \to \infty$ in (43), using the Proposition 3.2, (37) and (38), we obtain

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} = 1 - \lim_{n\to\infty} \delta_1^{(2)}(n,\alpha,\beta)\frac{\tilde{P}_{n-2}^{(\alpha,\beta)}(x)}{\tilde{P}_n^{(\alpha,\beta)}(x)} = 1 - \frac{1}{4\left(\Phi(x)\right)^2}. \qquad ☑$$

Corollary 3.2. *Under the conditions of the previous theorem, the asymptotic behavior of the monic Jacobi-Sobolev orthogonal polynomials is*

$$Q_n^{(\alpha,\beta)}(x) = \frac{2^{-\alpha-\beta-2}\left[4\left(\Phi(x)\right)^{n+\frac{1}{2}} - \left(\Phi(x)\right)^{n-\frac{3}{2}}\right]\left(\sqrt{x-1}+\sqrt{x+1}\right)^{\alpha+\beta}}{\left(x^2-1\right)\left(x-1\right)^{\frac{\alpha}{2}}\left(x+1\right)^{\frac{\beta}{2}}}\left(1+o(1)\right), \tag{51}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$, *where* $\Phi(x) = \frac{x+\sqrt{x^2-1}}{2}$.

Proof. Theorems 2.3 and 3.6 give the result immediately. ☑

Let us recall the results on the distribution of the zeros of the family of orthogonal polynomials $\{P_n\}$ with respect to a measure ν, (see [3] and [9]).

Theorem 3.7. *i) The zeros of the orthogonal polynomial* P_n *are real, simple and contained in the convex hull of support of the measure* ν.
ii) The zeros $x_{n,1} < \cdots < x_{n,n}$ *of* P_n *separate those of* P_{n+1}, *more precisely,*

$$x_{n+1,k} < x_{n,k} < x_{n+1,k+1} \quad (k = 1, \ldots, n).$$

iii) The zeros of the orthogonal polynomials form a dense set in the support of the measure ν; that is, for any subinterval I in $\operatorname{supp}(\nu)$ *such that $\int_I d\nu(x) > 0$ and n sufficiently large, all polynomials P_n has at least a zero in I. Moreover,*

$$\overline{\lim_{n\to\infty}}\{x : P_n(x) = 0\} = \operatorname{supp}(\nu).$$

The study of the zeros of the Jacobi-Sobolev polynomials has been carried out in the following particular cases:

(1) $\alpha = \beta = 0$. In this case P. Althammer [1] demonstrates that Legendre-Sobolev polynomials of degree n possess exactly n real zeros in the interval $[-1, 1]$.
(2) $\alpha = \beta = \alpha' - \frac{1}{2}$, $\alpha' > -\frac{1}{2}$. E. A. Cohen [2] shows that the zeros of the Legendre polynomials of degree $n-1$ separate the zeros of the Legendre-Sobolev polynomial of degree n, when $\lambda \geq \frac{2}{n}$.
(3) In his study on coherent couples of measures and zeros of orthogonal polynomials type Sobolev, H. G. Meijer studies the zeros of the Gegenbauer-Sobolev polynomials when λ is sufficiently large, (see [7]). Following him, we will also use the fact that for $\alpha, \beta > 0$, λ sufficiently large and $n \geq 3$, the zeros of the polynomial $Q_n^{(\alpha,\beta)}$ behave as the zeros of the limit polynomial $R_n^{(\alpha,\beta)}$. The following theorems are devoted to establishing results on the distribution of zeros of Jacobi-Sobolev orthogonal polynomials.

Theorem 3.8. *For $\alpha, \beta > 0$, λ sufficiently large and $n \geq 3$, the n-th monic Jacobi-Sobolev orthogonal polynomial $Q_n^{(\alpha,\beta)}$, has n different real zeros that are intertwined with the zeros of the polynomial $\tilde{P}_n^{(\alpha-1,\beta-1)}$ and at least $n-2$ of them are contained in the interval $[-1, 1]$.*

Proof. If α, $\beta > 0$ and $n \geq 3$, by definition

$$R_n^{(\alpha,\beta)}(x) = \lim_{\lambda\to\infty} Q_n^{(\alpha,\beta)}(x,\lambda).$$

By Corollary 3.1 we have

$$R_n^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\alpha-1,\beta-1)}(x);$$

then Theorem 3.7 shows that $R_n^{(\alpha,\beta)}$ has only real, simple zeros and they are contained in the interval $[-1, 1]$. Therefore, for λ sufficiently large, the zeros of $Q_n^{(\alpha,\beta)}$ are real and intertwined with those of $\tilde{P}_n^{(\alpha,\beta)}$.

We denote by $z_{n,j}$, $(1 \leq j \leq n)$ the zeros of $R_n^{(\alpha,\beta)}$. As

$$\mathcal{D}^1\left(R_n^{(\alpha,\beta)}(x)\right) = n\tilde{P}_{n-1}^{(\alpha,\beta)}(x),$$

the critical points of $R_n^{(\alpha,\beta)}$ are the zeros of $\tilde{P}_{n-1}^{(\alpha,\beta)}$. Given the zeros of $\tilde{P}_{n-1}^{(\alpha-1,\beta-1)}$ such that

$$x_{n-1,1} < x_{n-1,2} < \cdots < x_{n-1,n-1},$$

then, the polynomial $R_n^{(\alpha,\beta)}$ is monotone in each open interval of the form

$$(x_{n-1,1}, x_{n-1,2}), (x_{n-1,2}, x_{n-1,3}), \ldots, (x_{n-1,n-1}, \infty),$$

so that each one of the zeros of $R_n^{(\alpha,\beta)}$ is contained in each one of these intervals. This fact makes us conclude that at least $n-2$ zeros of $Q_n^{(\alpha,\beta)}$ are in the interval $[-1, 1]$ and there is a pair of zeros (not necessarily symmetrical) that can be outside of the interval $[-1, 1]$, depending on the sign of $R_n^{(\alpha,\beta)}(1)$ and $R_n^{(\alpha,\beta)}(-1)$. Furthermore, since for each $x \in (-1, 1)$

$$\forall\, \varepsilon > 0\ \exists\, r > 0 : \left| Q_n^{(\alpha,\beta)}(x,\lambda) - R_n^{(\alpha,\beta)}(x) \right| < \varepsilon, \text{ whenever } \lambda \geq r,$$

these $n-2$ zeros of the n-th Jacobi-Sobolev polynomial are distributed with respect to the zeros of the polynomial $\tilde{P}_{n-1}^{(\alpha-1,\beta-1)}$ in the following way

$$x_{n-1,k} < y_{n,k} < x_{n-1,k+1}, \quad k = 1, \ldots, n-2. \qquad ☑$$

In Section 2 the multiplication operator by x, $M_x : \overline{\mathbb{P}} \to \overline{\mathbb{P}}$, was introduce. Now we will use the fact that this operator is bounded on $\overline{\mathbb{P}}$ to find a compact set in the real line that contains the zeros of the Jacobi-Sobolev orthogonal polynomials.

Theorem 3.9. *There is a positive constant C, such that if x_0 is a real zero of $Q_n^{(\alpha,\beta)}$ and $n \geq 1$ then $|x_0| \leq \sqrt{1+2C}$. Therefore, all the zeros of the polynomial $Q_n^{(\alpha,\beta)}$ are in the interval*

$$[-\sqrt{1+2C}, \sqrt{1+2C}\,].$$

Proof. Let x_0 a zero of $Q_n^{(\alpha,\beta)}$. Hence, there exists $p \in \overline{\mathbb{P}}_{n-1}$ such that

$$Q_n^{(\alpha,\beta)}(x) = (x - x_0)p(x).$$

Considering the Sobolev norm of the polynomial xp we have

$$\|xp\|_S^2 = \|Q_n^{(\alpha,\beta)}\|_S^2 + \|x_0 p\|_S^2;$$

thus

$$|x_0|\,\|p\|_S \leq \|xp\|_S$$

and therefore

$$|x_0| \leq \frac{\|xp\|_S}{\|p\|_S} = \frac{\|M_x(p)\|_S}{\|p\|_S}.$$

So, it is enough to show that $M_x : \overline{\mathbb{P}} \to \overline{\mathbb{P}}$ is bounded. Now by definition of Sobolev inner product we have

$$\begin{aligned}\|xp\|_S^2 &= \|xp\|_\omega^2 + \lambda\|(xp)'\|_\omega^2 \\ &\leq \|xp\|_\omega^2 + 2\lambda\|xp'\|_\omega^2 + 2\lambda\|p\|_\omega^2 \\ &\leq \|xp\|_S^2 + 2\lambda\|xp'\|_\omega^2 + 2\lambda\|p\|_\omega^2.\end{aligned}$$

Taking $C_1 = \max\{|x| : -1 \leq x \leq 1\}$ and $C_2 = \max\{1, \lambda\}$ we get

$$\|xp\|_S^2 \leq C_1\|p\|_S^2 + 2\lambda\|p'\|_\omega^2 + 2\lambda\|p\|_\omega^2 \leq (1 + 2C_2)\|p\|_S^2. \quad \boxtimes$$

4. Concluding remarks and observations

In a natural way arises the consideration of the inner product of Jacobi-Sobolev that involves derivatives of order higher than one,

$$\langle p, q\rangle_m = \sum_{k=0}^{m} \lambda_k \int_{-1}^{1} p^{(k)}(x)q^{(k)}(x)\omega^{(\alpha,\beta)}(x)dx = \sum_{k=0}^{m} \lambda_k \langle p^{(k)}(x), q^{(k)}(x)\rangle_\omega$$

where $m \in \mathbb{Z}_+$ is fixed and the $(m+1)$-uple of real numbers $(\lambda_0, \ldots \lambda_m)$ satisfies

$$\lambda_0 = 1, \quad \lambda_k > 0, \quad k = 1, \ldots, m.$$

Let us denote the monic orthogonal polynomials corresponding to this Sobolev product by

$$Q_{n,\nu}^{(\alpha,\beta)}, \quad \nu = 1, \ldots, m.$$

Notice that for each $\nu = 1, \ldots, m$, the coefficients of $Q_{n,\nu}^{(\alpha,\beta)}$ are rational functions in λ in which the degrees of the numerator and denominator coincide. Thus,

$$Q_{n,\nu}^{(\alpha,\beta)}(x) = Q_{n,\nu}^{(\alpha,\beta)}(x, \lambda_\nu), \quad \nu = 1, \ldots, m.$$

Therefore, we can also define for each $\nu = 1, \ldots, m$, and $n \geq 3$ the limit polynomial associated,

$$R_{n,\nu}^{(\alpha,\beta)}(x) := \lim_{\lambda_\nu \to \infty} Q_{n,\nu}^{(\alpha,\beta)}(x, \lambda_\nu),$$

From our results, we can formulate the following open questions:

1. Can we determine the asymptotic behavior of the polynomials $Q_{n,\nu}^{(\alpha,\beta)}$?
2. How are the zeros of $Q_{n,\nu}^{(\alpha,\beta)}$ distributed in the complex plane?

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