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Some polynomial values in binary recurrences

To the memory of Professor Péter Kiss

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ABSTRACT. We study the diophantine equations $G_n = \binom{x}{k}$ and $G_n = \sum_{i=1}^{x} i^k$ in integers $n \geq 0$ and $x \geq k$ ($x \geq 1$) with fixed positive integer k, where G_n denotes the n^{th} term of a binary recurrence of certain types. We give a simple practical algorithm which can solve the first equation if $k = 3$ and the second equation if $k = 2$. As a demonstration, this algorithm is applied to provide the solutions of the second equation in case of the Fibonacci, Lucas and Pell sequences. Further, using different methods, the problems $G_n = \binom{x}{4}$ and $G_n = \sum_{i=1}^{x} i^3$ will be solved for the three famous recurrences.

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1. Introduction

In 1962 OGILVY [16], one year later MOSER and CARLITZ [14], and ROLLETT [19] posed the following problem. In the Fibonacci series the first, second and twelfth terms are squares. Are there any others? The answer was given by COHN [2, 3] and WYLER [23] independently, who applying elementary methods, proved that the only square Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$ and $F_{12} = 144$. A similar result for the Lucas numbers was obtained by ALFRED [1] and by COHN [4]: if $L_n = x^2$ then $n = 1$ or $n = 3$. PETHŐ [18] and later independently COHN [5] found all the perfect powers in the Pell sequence. They

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showed that the equation $P_n = x^q$ has only for $q = 2$ a non-trivial solution, namely $(n, x) = (7, 13)$. Here, and in the sequel, denote by F_n, L_n and P_n the n^{th} term of the Fibonacci, the Lucas and the Pell sequences, respectively. These recursions are defined by the recurrence relations $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, $F_0 = 0$, $F_1 = 1$; $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$, $L_0 = 2$, $L_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ $(n \geq 2)$, $P_0 = 0$, $P_1 = 1$; respectively.

Another special interest was to determine the triangular numbers $T_x = \frac{x(x+1)}{2}$ in certain recurrences. HOGGATT conjectured that there are only five triangular Fibonacci numbers. This problem was originally posed by TALLMAN [22] in the Fibonacci Quarterly. In 1989 MING [8] proved HOGATT's conjecture by showing that the only Fibonacci numbers which are triangular are $F_0 = 0$, $F_1 = F_2 = 1$, $F_4 = 3$, $F_8 = 21$ and $F_{10} = 55$. Moreover the triangular Lucas numbers are $L_1 = 1$, $L_2 = 3$ and $L_{18} = 5778$ (MING, [9]), and the only triangular Pell number is $P_1 = 1$ (MCDANIEL, [7]).

These results were followed by others and lot of similar questions have been arised concerning occurrence of figurate numbers in recurrences. In this paper we pose two problems coming from the results connected to triangular numbers as well as polynomial values, and certain cases will be solved.

Let G_n be a linear recurrence of arbitrary order.

Problem 1. *Since the number T_{x-1} is equal to the binomial coefficient $\binom{x}{2}$, it is natural to ask whether the terms $\binom{x}{k}$ occur in recurrence G or not. More exactly, solve the diophantine equation*

$$G_n = \binom{x}{k} \tag{1}$$

in integers $n \geq 0$ and $x \geq k$, where the integer $k \geq 2$ is fixed.

Problem 2. *On the other hand the number T_x coincides the sum $\sum_{i=1}^{x} i$. We are also interested in the occurrence of the numbers of type $\sum_{i=1}^{x} i^k$ among the terms G_n. More exactly, solve the diophantine equation*

$$G_n = \sum_{i=1}^{x} i^k \tag{2}$$

in integers $n \geq 0$ and $x \geq 1$, where the integer $k \geq 1$ is fixed.

As aforesaid, the case $k = 2$ of Problem 1 and the case $k = 1$ of Problem 2 coincide and has already been solved for the Fibonacci, the Lucas and the Pell sequences.

Let $U := \{U_n\}_{n=0}^{\infty}$ be a binary recurrence defined by the integer initial terms U_0, U_1 and by the recurrence relation

$$U_n = AU_{n-1} + BU_{n-2} \quad (n \geq 2), \tag{3}$$

with arbitrary integer $A \neq 0$ and $B = \pm 1$. Let α and β denote the zeros of the characteristic polynomial $x^2 - Ax - B$ of U. Further, let $D = A^2 + 4B$

the discriminant of U, $a_u = U_1 - \beta U_0$, $b_u = U_1 - \alpha U_0$, and $C = a_u b_u = U_1^2 - AU_1U_0 - BU_0^2$. The binary recurrence U is called non-degenerated if $C \neq 0$ and $\frac{\alpha}{\beta}$ is not a root of unity. Denote by $V := \{V_n\}_{n=0}^{\infty}$ the associate sequence of U, i.e. $V_0 = 2U_1 - AU_0$, $V_1 = AU_1 + 2BU_0$, and V satisfies the same recurrence relation as U does.

In [21], with $U_0 = 0$ and $U_1 = 1$, SZALAY treated the equations

$$U_n = \binom{x}{3}, \qquad V_n = \binom{x}{3} \tag{4}$$

and showed that equations (4) have only finitely many solutions (n, x). Further a procedure for solving (4) has also been developed, and it was applied to show that $F_n = \binom{x}{3}$ implies $(n, x) = (1, 3)$ or $(2, 3)$, $L_n = \binom{x}{3}$ implies $(n, x) = (1, 3)$ or $(3, 4)$, finally $P_n = \binom{x}{3}$ gives $(n, x) = (1, 3)$ as the only solution.

Now let $p(x) \in \mathbb{Q}[x]$ be a polynomial of the form

$$p(x) = \frac{1}{d}\left(ax^3 + 3abx^2 + cx + \left(bc - 2ab^3\right)\right) \tag{5}$$

where $a \neq 0$, b, c, $d \neq 0$ are rational integers. This polynomial —by substituting $x_1 = x + b$— can be put in the form $\frac{1}{d}(ax_1^3 + ex_1)$, where e is a suitable integer. This shows how the reader can decide whether or not one can have a polynomial of the form (5).

In the present paper we extend the algorithm of [21] to the polynomial values $p(x)$, i.e., we describe a general practical algorithm for computing explicitely all integral solutions of the equation

$$U_n = p(x) \tag{6}$$

if it has finitely many solutions. Theorem 1 provides a sufficient condition of finiteness. Theorem 2 contains the results of application of the method to three famous binary recurrences. Theorem 3 is an immediate consequence of knowing all the squares among the terms of F_n, L_n and P_n. Finally, applying and completing the theorem of MING in [10] we solve Problem 1 for the Fibonacci sequence and its associate if $k = 4$.

Suppose now, that U is a non-degenerated binary recurrence. Then $C \neq 0$, and further $A \neq 0$ and $B = \pm 1$ guarantee that $D > 0$. It is known that $U_n = \frac{a_u\alpha^n - b_u\beta^n}{\alpha - \beta}$, $V_n = a_u\alpha^n + b_u\beta^n$, moreover $V_n^2 - DU_n^2 = 4C(-B)^n$. Since $B = \pm 1$, we have

$$V_n^2 - DU_n^2 = \pm 4C. \tag{7}$$

Theorem 1. *Let $e = c - 3ab^2$. If $\pm 27Cad^2 \neq De^3$ then the equation*

$$U_n = p(x) \tag{8}$$

has only a finite number of solutions in integers n and x.

Remark 1. Althought the proof is a consequence of Theorem 3 of NEMES-PETHŐ [15] or Theorem 1 of PETHŐ [17], however the direct application of these results is not shorter than the alternate proof (based on a theorem of MORDELL [12, 13]) we describe in Section 3.

Remark 2. By choosing, for example, the Fibonacci sequence ($C=1, D=5$), and $a=5$, $b=0$, $c=3$, $d=1$ ($e=3$) the condition $27Cad^2 = De^3$ is satisfied.

Theorem 2. *All the integer solutions of the equation*

(a) $F_n = \sum_{i=1}^{x} i^2$ *are* $(n,x) = (1,1)$, $(2,1)$, $(5,2)$ *and* $(10,5)$;

(b) $L_n = \sum_{i=1}^{x} i^2$ *is* $(n,x) = (2,1)$;

(c) $P_n = \sum_{i=1}^{x} i^2$ *are* $(n,x) = (1,1)$, $(3,2)$.

Theorem 3. *The diophantine equations*

$$F_n = \sum_{i=1}^{x} i^3, \quad L_n = \sum_{i=1}^{x} i^3, \quad P_n = \sum_{i=1}^{x} i^3 \tag{9}$$

has only the trivial solution $x=1$ with some suitable integers n.

Theorem 4. *The solutions of the equation*

(a) $F_n = \binom{x}{4}$ *are* $(n,x) = (1,4), (2,4), (5,5)$;

(b) $L_n = \binom{x}{4}$ *is* $(n,x) = (1,4)$.

2. The algorithm

The purpose of this section is to describe an algorithm for solving completely equation (6) in integers n and x, if it has finitely many solutions. First we transform the equation into two elliptic equations, and then we use the computer algebraic system SIMATH [20] which is able to find all the integer points on the corresponding elliptic curves. The algorithms of SIMATH are based on some deep results of GEBEL, PETHŐ and ZIMMER [6].

Suppose that $p(x)$ is a polynomial defined above. Then

$$p(x) = \frac{1}{d}(ax^3 + 3abx^2 + cx + (bc - 2ab^3)) = \frac{1}{d}(a(x+b)^3 + e(x+b)) \tag{10}$$

with $e = c - 3ab^2$. Let $x_1 = x + b$.

Assume that (8) has finitely many solutions, and let $y = V_n$. By (7) we have

$$y^2 - D\left(\frac{ax_1^3 + ex_1}{d}\right)^2 = \pm 4C. \tag{11}$$

Multiplying it by d^2, reordering and introducing the notation $y_1 = dy$ and $x_2 = x_1^2$, it follows that

$$y_1^2 = D\left(a^2x_2^3 + 2aex_2^2 + e^2x_2\right) \pm 4Cd^2. \tag{12}$$

Now multiplying (12) by $27a^3$, let $y_2 = 3ay_1$ and $x_3 = 3ax_2 + 2e$, and using these new variables, we get

$$3ay_2^2 = Da^2x_3^3 - 3Da^2e^2x_3 - \left(2Da^2e^3 \mp 108Ca^3d^2\right). \tag{13}$$

Finally, multiplying equation (13) by 3^3D^2a, together with $k = 9Day_2$ and $l = 3Dax_3$ it follows that

$$k^2 = l^3 - (27D^2a^2e^2)\,l - (54D^3a^3e^3 \mp 2916CD^2a^4d^2)\,. \tag{14}$$

We have two elliptic equations in short Weierstrass normal form. Now apply the program package SIMATH to solve the equations (14). It is easy to see that a solution (k, l) of (14) implies that only

$$V_n = y = \frac{k}{27Da^2d} \tag{15}$$

and

$$x = \pm\sqrt{\frac{l - 6Dae}{9Da^2}} - b. \tag{16}$$

are possible. These values must be non-negative integers, and the suffix n —whose parity needs to be correct— can simply be computed from the sequence V.

3. Proof of the theorems

Proof of Theorem 1. By a theorem of MORDELL [12, 13] it is sufficient to show that except the cases $\pm 27Cad^2 \neq De^3$, the polynomial

$$u_1(l) = l^3 - (27D^2a^2e^2)\,l - (54D^3a^3e^3 \mp 2916CD^2a^4d^2) \tag{17}$$

have three distinct roots.

Suppose now, that the polynomial $u_1(l)$ has a multiple root $\tilde{l}$. Then $\tilde{l}$ satisfies the equation $u_1'(l) = 3l^2 - 27D^2a^2e^2 = 0$, i.e. $\tilde{l} = \pm 3Dae$. Since $u_1(-3Dae) = \pm 2916CD^2a^4d^2$ it follows that $CDad = 0$ which is impossible. Moreover, $u_1(3Dae) = -108D^3a^3e^3 \pm 2916CD^2a^4d^2 = 0$ implies that $De^3 = \pm 27Cad^2$, and in any other case $u_1(l)$ has three distinct zeros. ☑

Proof of Theorem 2. (Applications of the Algorithm.) Consider the polynomial $p(x) = \frac{1}{24}(x^3 + 3x^2 + 2x)$, which possesses $a = 1$, $b = 1$, $c = 2$ and $d = 24$. Clearly, $e = -1$ and $(C, D) = (1,5), (-5,5), (1,8)$ in case of Fibonacci, Lucas and Pell sequence, respectively. Since $\pm 27Cad^2 \neq De^3$, further $p(2x) = \frac{1}{6}x(x+1)(2x+1) = \sum_{i=1}^{x} i^2$, we can determine all Fibonacci, Lucas and Pell numbers which are the sum of the first x consecutive squares. By (14) one can compute the coefficients of the appropriate elliptic curves.

The calculations and results are summarized in Table 1. Every binary recurrence leads to two elliptic equations because of the even and odd suffixes.

The last step is to calculate x and y from the solutions (l, k). By the algorithm it follows that $x = \pm\sqrt{\frac{l-6Dae}{9Da^2}} - b$, $y = \frac{k}{27Da^2d}$. Except for some values x and y, they are not integers with the given conditions. The exceptions (origined from the $\boxed{\text{framed points}}$) provide all the solutions of equation (14). ☑

Recurrence	*Elliptic equation*	*All the integer solutions* (l,k)
F_n (n odd)	$k^2 = l^3 - 675l - 41983650$	$\boxed{(375,3240)}$, $(3475,204740)$, $(5631,422496)$, $(555,11340)$, $\boxed{(1095,35640)}$, $(1446,54594)$, $(5919,455328)$, $(9278754,28264056408)$
F_n (n even)	$k^2 = l^3 - 675l + 41997150$	$(771,22356)$, $(15,6480)$, $(82959,23894352)$, $(-30,6480)$, $(-345,1080)$, $(27915,4663980)$, $(555,14580)$, $(951,30024)$, $(2850,152280)$, $(271,7856)$, $(3435,201420)$, $(-201,5832)$, $(195,7020)$, $\boxed{(5415,398520)}$, $(-281,4472)$, $\boxed{(375,9720)}$, $(1635,66420)$, $(-309,3564)$, $(1255,44920)$, $(114,6588)$, $(17295,2274480)$, $(8079,726192)$, $(406036495,8181776377520)$
L_n (n odd)	$k^2 = l^3 - 675l - 209945250$	$(10504,57376)$
L_n (n even)	$k^2 = l^3 - 675l + 209958750$	$(-525,8100)$, $(3651,221076)$, $(879,29808)$, $(231,14904)$, $\boxed{(375,16200)}$, $(-581,3772)$, $(46050,9882000)$
P_n (n odd)	$k^2 = l^3 - 1728l - 107467776$	$\boxed{(600,10368)}$, $(10504,57376)$, $(564,8424)$, $\boxed{(1752,72576)}$, $(271320,141326208)$, $(2832,150336)$
P_n (n even)	$k^2 = l^3 - 1728l + 107523072$	$(-48,10368)$, $(82968,23898240)$, $(24,10368)$

TABLE 1

Proof of Theorem 3. Since $\sum_{i=1}^{x} i^3 = \left(\frac{x(x+1)}{2}\right)^2$ it follows that the solutions of the equations

$$U_n = \sum_{i=1}^{x} i^3, \qquad V_n = \sum_{i=1}^{x} i^3 \tag{18}$$

can be given from the square terms U_n and V_n, which are well known if $U_n = F_n$ or $V_n = L_n$ or $U_n = P_n$ (see e.g. [2, 3] or [23], [4] and [18], respectively). We apply these results to conclude that the equations

$$F_n = \sum_{i=1}^{x} i^3, \qquad L_n = \sum_{i=1}^{x} i^3, \qquad P_n = \sum_{i=1}^{x} i^3 \tag{19}$$

has only the trivial solution $x = 1$. ☑

Proof of Theorem 4. Let $x_1 = x^2 - 3x + 1$. Then $\binom{x}{4} = \frac{x_1^2-1}{24}$ and to solve the equations $F_n = \binom{x}{4}$ or $L_n = \binom{x}{4}$ is equivalent to find all Fibonacci and Lucas numbers for which $24F_n + 1$ and $24L_n + 1$ is a square, respectively.

MING [11] showed that $24L_n + 1$ is a square if and only if $n = 0, 1$ or 4, i.e. $L_n = 2, 1$ or 7. Consequently

$$x^2 - 3x + 1 = 7 \text{ or } 5 \text{ or } 13, \tag{20}$$

and $x = 4$ only the integer satisfies (20) and the conditions. This proves the second part of Theorem 4.

In case of Fibonacci sequence we also refer to MING. In [10] the author first proved that if $24F_n + 1$ is a square and $n \equiv 0, \pm 1, 2, \pm 5 \pmod{3168}$ then $n = 0, 1, 2, 5$. Next he showed directly that if $n \equiv 3 \pmod 6$ then F_n is not a pentagonal number, i.e. an integer of the form $\frac{1}{2}m(3m-1)$, or equivalently $24F_n + 1 \neq (6m-1)^2$. Finally, for the remaining cases MING concluded that $24F_n + 1$ is not a square.

Now let $x_2 = x - 2$ and suppose that n is an integer satisfying $n \equiv 3 \pmod 6$ and $F_n = \binom{x_2+2}{4}$, i.e.

$$24F_n + 1 = (x_2^2 + x_2 - 1)^2. \tag{21}$$

Observe that $x_2^2 + x_2$ is always even. Therefore if $3 \mid (x_2^2 + x_2)$ then $x_2^2 + x_2 - 1$ is an integer of the form $6m - 1$, and we see from the result of MING that this is impossible. Thus we may assume that $3 \nmid x_2^2 + x_2$ which gives $x_2 \equiv 1 \pmod 3$. Put $x_2 = 3x_3 + 1$ and $x_4 = \frac{x_3(x_3+1)}{2}$. Thus

$$\left(x_2^2 + x_2 - 1\right)^2 = \left(18x_4 + 1\right)^2 \equiv 1 \pmod{18}. \tag{22}$$

On the other hand it is easy to see that if $n = 6m + 3$ ($m \in \mathbb{N}$) then

$$24F_{6m+3} + 1 \equiv 13 \quad \text{or} \quad 24F_{6m+3} + 1 \equiv 7 \pmod{18}, \tag{23}$$

which leads to contradiction, i.e. the equation $F_n = \binom{x}{4}$ has no solution if $n \equiv 3 \pmod 6$. For the remaining case the result of MING implies that $24F_n + 1$ is a square if and only if $n = 0, 1, 2, 5$, which proves the first part of Theorem 4. ☑

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References

[1] B. U. ALFRED , *On square Lucas numbers*, Fib. Quarterly, **2** (1964), 11–12.

[2] J. H. E. COHN, *Square Fibonacci numbers, etc.*, Fib. Quarterly, **2** (1964), 109–113.

[3] J. H. E. COHN, *On square Fibonacci numbers*, J. London Math. Soc., **39** (1964), 537–540.

[4] J. H. E. COHN, *Lucas and Fibonacci numbers and some Diophantine equations*, Proc. Glasgow Math. Assoc., **7** (1965), 24–28.

[5] J. H. E. COHN, , *Perfect Pell powers*, Glasgow Math. J., **38** (1996), 19–20.
[6] L. GEBEL, A. PETHŐ AND H. G. ZIMMER, *Computing integral points on elliptic curves*, Acta Arithm. **68** (1994), 171–192.
[7] W. L. MCDANIEL, *Triangular numbers in the Pell sequence*, Fib. Quarterly, **34** (1996), 105–107.
[8] L. MING, *On triangular Fibonacci numbers*, Fib. Quarterly, **27** (1989), 98–108.
[9] L. MING, *On triangular Lucas numbers*, Applications of Fibonacci Numbers, Vol 4., 231–240, Kluwer Acad. Publ., Dordrecht, 1991.
[10] L. MING, *Pentagonal numbers in the Fibonacci sequence*, Applications of Fibonacci numbers, Vol. 6, 349–354, Kluwer Acad. Publ., Dordrecht, 1996.
[11] L. MING, *Pentagonal numbers in the Lucas sequence*, Portugal. Math., **53** (1996), 325–329.
[12] L. J. MORDELL, *Note on the integer solutions of the equation* $Ey^2 = Ax^3 + Bx^2 + Cx + D$, Messenger Math., **51** (1922), 169–171.
[13] L. J. MORDELL, *On the integer solutions of the equation* $ey^2 = ax^3 + bx^2 + cx + d$, Proc. London Math. Soc. (2), **21** (1923), 415–419.
[14] L. MOSER AND L. CARLITZ, *Problem H-2*, Fib. Quarterly, **1** (1963), 46.
[15] I. NEMES AND A. PETHŐ, *Polynomial values in linear recurrences*, II., J. Number Theory, **24** (1986), 47–53.
[16] C. S. OGILVY, *Tomorrow's math, unsolved problems for the amateur*, Oxford Univ. Press, 1962, p. 100.
[17] A. PETHŐ, *On the solution of the equation* $G_n = P(x)$, Fibonacci Numbers and Their Applications, D. Reidel Publ. Comp., 1986, 193–201.
[18] A. PETHŐ, *The Pell sequence contains only trivial perfect powers*, Colloq. Math. Soc. János Bolyai 60, Sets, Graphs and Numbers Budapest (Hungary), 1991, 561–568.
[19] A. P. ROLLETT, *Problem 5080*, Amer. Math. Monthly, **70** (1963), 216.
[20] *SIMATH Manual*, Saarbrücken, 1996.
[21] L. SZALAY, *On the resolution of the equations* $U_n = \binom{x}{3}$ *and* $V_n = \binom{x}{3}$, Fib. Quarterly, **40** (2002), 9–12.
[22] M. H. TALLMAN, Fib. Quarterly, **1** (1963), 47.
[23] O. WYLER, *In the Fibonacci series* $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ *the first, second and twelfth terms are squares*, Amer. Math. Monthly, **71** (1964), 221–222.

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