A model of neck formation
on a rod under tension

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ABSTRACT. Stability of equilibrium of a circular cylinder under homogeneous axial stretching is investigated in the frame of 3–D nonlinear elasticity. The axisymmetric buckling modes describing developing of a "neck" on the stretched rod are studied. The isotropic incompressible material of the rod is described through the logarithmic strain tensor. The constitutive equations for the rod correspond to the power–law hardening of elastic–plastic materials. Solving the linearized stability equations of the stretched cylinder, we find the spectrum of critical values of longitudinal deformation and buckling eigenmodes of the rod. The bifurcation modes relating with the neck formation arise when the elongation of the rod insignificantly exceeds the maximum point on the diagram of stretching. It is noted that different buckling modes have close eigenvalues. This accumulation of the eigenvalues describes formation of the neck as a result of the superposition of many buckling modes. Similar results were established for a stretched rectangular beam under plane deformation [1].

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1. Axisymmetric deformation of a circular cylinder

Consider an elastic circular cylinder of radius r_0 and length l in the reference configuration. As the Lagrangian coordinates we use cylindrical coordinates

r, φ, z, so the body is described by the inequalities $0 \leq r \leq r_0$, $\quad 0 \leq \varphi \leq 2\pi$, $0 \leq z \leq l$. Assume the finite axisymmetric deformation of the cylinder is described by relations

$$R = R(r,z), \quad \Phi = \varphi, \quad Z = Z(r,z), \tag{1.1}$$

where R, Φ, Z are the cylindrical coordinates of body particle (r, φ, z) after deformation. With regard for the notation of the gradient operator in cylindrical coordinates

$$\mathrm{grad} \;= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z}, \tag{1.2}$$

and the expression for the position vector of a body particle in the deformed state

$$\mathbf{R} = R\mathbf{e}_r + Z\mathbf{e}_z, \tag{1.3}$$

we find the strain gradient $\mathbf{C} = \mathrm{grad}\; \mathbf{R}$ corresponding to transformation (1.1),

$$\mathbf{C} = \frac{\partial R}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{\partial Z}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_z + \frac{R}{r}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \frac{\partial R}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{\partial Z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z. \tag{1.4}$$

In (1.2)–(1.4) $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ are the unit vectors tangent to the coordinate lines of the cylindrical frame.

From (1.4) we define the Cauchy–Green's strain measure [2] $\mathbf{G} = \mathbf{C} \cdot \mathbf{C}^T$

$$\mathbf{G} = a\mathbf{e}_r \otimes \mathbf{e}_r + b\mathbf{e}_r \otimes \mathbf{e}_z + b\mathbf{e}_z \otimes \mathbf{e}_r + c\mathbf{e}_z \otimes \mathbf{e}_z + d\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi, \tag{1.5}$$

$$a = \left(\frac{\partial R}{\partial r}\right)^2 + \left(\frac{\partial Z}{\partial r}\right)^2, \quad b = \frac{\partial R}{\partial r}\frac{\partial R}{\partial z} + \frac{\partial Z}{\partial r}\frac{\partial Z}{\partial z},$$

$$c = \left(\frac{\partial R}{\partial z}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2, \quad d = \left(\frac{R}{r}\right)^2.$$

The equilibrium of an elastic body is described [2] by the equilibrium equations for Piola's stress tensor $\mathbf{D}$

$$\mathrm{div}\; \mathbf{D} \equiv \mathbf{e}_r \cdot \frac{\partial \mathbf{D}}{\partial r} + \frac{1}{r}\mathbf{e}_\varphi \cdot \frac{\partial \mathbf{D}}{\partial \varphi} + \mathbf{e}_z \cdot \frac{\partial \mathbf{D}}{\partial z} = 0 \tag{1.6}$$

and by constitutive equations

$$\mathbf{D} = d\Pi/d\mathbf{C} - \eta\, p\mathbf{C}^{-T}, \tag{1.7}$$

where div is the divergence operator in the reference configuration, Π is the specific strain–energy function, p is the pressure in the incompressible body which cannot be expressed through the strain measure, and $\mathbf{C}^{-T} \equiv (\mathbf{C}^T)^{-1} = (\mathbf{C}^{-1})^T$. Parameter η is equal to zero for a compressible material and to 1 for incompressible one. On the basis of (1.4), (1.5), (1.7) we conclude that for a homogeneous isotropic material with Π being a function of the invariants of tensor $\mathbf{G}$, tensor $\mathbf{D}$ has the form

$$\mathbf{D} = D_{rr}(r,z)\mathbf{e}_r \otimes \mathbf{e}_r + D_{rz}(r,z)\mathbf{e}_r \otimes \mathbf{e}_z + D_{\varphi\varphi}(r,z)\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + D_{zr}(r,z)\mathbf{e}_z \otimes \mathbf{e}_r + D_{zz}(r,z)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{1.8}$$

Due to (1.7), (1.8), for a compressible material, the vectorial equilibrium equation (1.6) reduces to two scalar equations with respect to two functions $R(r, z)$ and $Z(r, z)$.

Let the lateral surface of the cylinder be load–free and the ends be friction–less, so we suppose the tangent stresses to be zero and an uniform normal displacement to be given. This brings us the following boundary conditions

$$D_{rr}(r_0, z) = 0, \quad D_{rz}(r_0, z) = 0, \tag{1.9}$$

$$D_{zr}(r, 0) = D_{zr}(r, l) = 0, \quad Z(r, 0) = 0, \quad Z(r, l) = \lambda l, \tag{1.10}$$

where λ is a given positive constant.

Boundary conditions (1.10) mean that at stretching the vertical displacements of the cylinder ends are given, that is when cylinder loading is carried out with a hard testing machine.

Further we use the model of an incompressible isotropic material for which the deformation of any part of the body preserves its volume. In accordance with (1.4), at the axisymmetric deformation the incompressibility condition $\det \mathbf{C} = 1$ imposes the following restriction on $R(r, z)$ and $Z(r, z)$

$$\frac{R}{r} \left(\frac{\partial R}{\partial r} \frac{\partial Z}{\partial z} - \frac{\partial R}{\partial z} \frac{\partial Z}{\partial r} \right) = 1. \tag{1.11}$$

For representation of state equations of an isotropic incompressible material we apply the method implying application of the logarithmic strain tensor [2, 3]

$$\mathbf{H} = \frac{1}{2} \ln \mathbf{G}. \tag{1.12}$$

For an incompressible body tensor $\mathbf{H}$ is the deviator since at the constant volume deformation the trace of $\mathbf{H}$ is equal to zero. Denoting its second invariant by $-\Gamma^2/2$ we obtain

$$\Gamma = \sqrt{\operatorname{tr} \mathbf{H}^2}. \tag{1.13}$$

With use of (1.7), (1.12), for Piola's stress tensor in an isotropic incompressible body we get the constitutive equations

$$\mathbf{D} = (d\Pi/d\mathbf{H}) \cdot \mathbf{C}^{-T} - p\,\mathbf{C}^{-T}. \tag{1.14}$$

For a problem of axisymmetric deformation of incompressible body p, is an unknown function of coordinates r, z. Its appearance in the system of equilibrium is compensated by the additional equation, the incompressibility condition (1.11).

By the definition of logarithm of a positive definite tensor [2], with regard to (1.12) we have

$$\mathbf{H} = \sum_{i=1}^{3} \frac{1}{2} \ln G_i \, \mathbf{d}_i \otimes \mathbf{d}_i, \tag{1.15}$$

where G_i are the eigenvalues and $\mathbf{d}_i$ are the unit eigenvectors of tensor $\mathbf{G}$. With use of (1.5) we find

$$G_1 = d, \quad G_{2,3} = \frac{1}{2}\left(h \pm \sqrt{h^2 - 4/d}\right), \tag{1.16}$$

$$\mathbf{d}_1 \otimes \mathbf{d}_1 = \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi,$$

$$\mathbf{d}_2 \otimes \mathbf{d}_2 = (h - 2G_3)^{-1}(a\mathbf{e}_r \otimes \mathbf{e}_r + b\mathbf{e}_r \otimes \mathbf{e}_z + b\mathbf{e}_z \otimes \mathbf{e}_r$$
$$+ c\mathbf{e}_z \otimes \mathbf{e}_z - G_3\mathbf{e}_r \otimes \mathbf{e}_r - G_3\mathbf{e}_z \otimes \mathbf{e}_z,$$

$$\mathbf{d}_3 \otimes \mathbf{d}_3 = (h - 2G_2)^{-1}(a\mathbf{e}_r \otimes \mathbf{e}_r + b\mathbf{e}_r \otimes \mathbf{e}_z + b\mathbf{e}_z \otimes \mathbf{e}_r \tag{1.17}$$
$$+ c\mathbf{e}_z \otimes \mathbf{e}_z - G_2\mathbf{e}_r \otimes \mathbf{e}_r - G_2\mathbf{e}_z \otimes \mathbf{e}_z,$$

$$h = a + c = \left(\frac{\partial R}{\partial r}\right)^2 + \left(\frac{\partial Z}{\partial r}\right)^2 + \left(\frac{\partial R}{\partial z}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2.$$

Formulas (1.15)–(1.17) represent explicit expression of the logarithmic strain tensor through functions R and Z at axisymmetric deformation.

As a certain model of an isotropic incompressible body we consider the material with power–law hardening [4, 5] for which the elastic potential has the form

$$\Pi = A\Gamma^\beta, \quad A > 0, \quad \beta > 1, \tag{1.18}$$

where A, β are the constants. According to (1.14) Piola's stress tensor for the material has representation

$$\mathbf{D} = A\beta \left(\operatorname{tr} \mathbf{H}^2\right)^{\beta/2-1} \mathbf{H} \cdot \mathbf{C}^{-T} - p\mathbf{C}^{-T}. \tag{1.19}$$

For material (1.18), the relations between the principal true stresses σ_i, i.e. the eigenvalues of Cauchy's stress tensor, and the principal true strains H_i, i.e., the eigenvalues of the logarithmic strain tensor are given by the formulas

$$\sigma_i = A\beta\Gamma^{\beta-2}H_i - p \quad (i = 1, 2, 3). \tag{1.20}$$

The model (1.18) describes satisfactorily [4, 6] the behavior of a number of elastic–plastic constructional materials at active loading and can serve as the generalization of the deformation theory of plasticity for large deformation.

2. Homogeneous state of the uniaxial stretching

For an incompressible material with power–law hardening, the boundary value problem (1.6), (1.9), (1.10) has the following solution

$$R(r, z) = \lambda^{-1/2}r, \quad Z(r, z) = \lambda z,$$

$$p = p_0 = -\frac{1}{2}A\beta\left(\frac{3}{2}\ln^2\lambda\right)^{\beta/2-1}\ln\lambda. \tag{2.1}$$

To the solution (2.1) there corresponds a homogeneous stress–strain state of the cylinder at which all stresses, except the normal stress acting on the area

elements $z = const$, are equal to zero. According to (1.20), (2.1), the magnitude of the true stretching stress σ relates with the relative axial elongation $\delta = \lambda - 1$ by formula

$$\sigma(\lambda) = A\beta \left(\frac{3}{2}\right)^{\beta/2} \ln^{\beta-1} \lambda. \tag{2.2}$$

The stretching force applied on the ends of the cylinder is

$$P(\lambda) = \pi r_0^2 \lambda^{-1} \sigma = \pi r_0^2 A\beta \left(\frac{3}{2}\right)^{\beta/2} \lambda^{-1} \ln^{\beta-1} \lambda. \tag{2.3}$$

The dependence (2.3) describes the rod uniaxial stretching diagram; it is not monotonically increasing in contrast to equation (2.2). Function $P(\lambda)$ given by (2.3) has a point of a maximum $\lambda_* = \exp(\beta - 1)$. The descending part of the stretching diagram, where $\lambda > \lambda_*$, can be achieved at stretching of a rod with a hard testing machine.

3. The linearized boundary value problem of stability of homogeneous stretching state

From experiments with rod stretching it is well known that the process of homogeneous deformation becomes unstable when it reaches the maximum on the loading diagram. The cylindrical shape of the stretched sample is replaced by an axisymmetric equilibrium shape when a neck appears. Let us investigate the neck formation as the phenomenon of the loss of stability of a homogeneous state of the cylinder with use of the exact equations of the three–dimensional elastic bodies stability theory [2, 7]. Let us consider an axisymmetric equilibrium shape of the elastic cylinder that slightly differs from the one described in Section 2

$$\mathbf{R} = \mathbf{R}_0 + \varepsilon\mathbf{w}, \quad p(r, z) = p_0 + \varepsilon p^\bullet(r, z),$$
$$\mathbf{R}_0 = \lambda^{-1/2} r \mathbf{e}_r + \lambda z \mathbf{e}_z, \quad \mathbf{w} = u(r, z)\mathbf{e}_r + w(r, z)\mathbf{e}_z. \tag{3.1}$$

Here ε is the small parameter, $\mathbf{w}$ is the vector of additional displacement. Linearizing the equilibrium equations (1.6) with respect to the axisymmetric perturbations we obtain

$$\text{div } \mathbf{D}^\bullet = 0,$$
$$\mathbf{D}^\bullet = \frac{d}{d\varepsilon} \mathbf{D}(\mathbf{C}_0 + \varepsilon\text{grad } \mathbf{w})\Big|_{\varepsilon = 0}, \tag{3.2}$$
$$\mathbf{C}_0 = \text{grad } \mathbf{R}_0.$$

This is derived with regard to relation $\mathbf{C} = \mathbf{C}_0 + \varepsilon\text{grad } \mathbf{w}$.

By linearization of the constitutive equations (1.14) let us find perturbation $\mathbf{D}^\bullet$ of Piola's stress tensor

$$\mathbf{D}^\bullet = (d\Pi/d\mathbf{H})^\bullet \cdot \mathbf{C}_0^{-T} - (d\Pi/d\mathbf{H})_0 \cdot \mathbf{C}_0^{-T} \cdot (\text{grad } \mathbf{w})^T \cdot \mathbf{C}_0^{-T}$$
$$- p^\bullet \mathbf{C}_0^{-T} + p_0 \mathbf{C}_0^{-T} \cdot (\text{grad } \mathbf{w})^T \cdot \mathbf{C}_0^{-T}. \tag{3.3}$$

The substitution of (3.3) into (3.2) leads to the linearized equilibrium equations

$$A\beta \left(\frac{3}{2}\ln^2\lambda\right)^{\beta/2-1} \left\{ \lambda^{1/2}\frac{\partial^2 u}{\partial r^2} + \frac{3}{2}\ln\lambda \frac{\lambda^{-1/2}}{\lambda^2-\lambda^{-1}}\frac{\partial^2 u}{\partial z^2} + \right.$$
$$\left. + \left(\left[1-\frac{\beta}{2}\right]\lambda^{-1} + \frac{3}{2}\frac{\ln\lambda}{\lambda^4-\lambda}\right)\frac{\partial^2 w}{\partial r\partial z} + \frac{2}{r}\lambda^{1/2}\frac{\partial u}{\partial r} + \frac{\lambda^{-1}}{r}\frac{\partial w}{\partial z} \right\} - \frac{\partial p^\bullet}{\partial r} = 0,$$

$$A\beta \left(\frac{3}{2}\ln^2\lambda\right)^{\beta/2-1} \left\{ \frac{3}{2}\ln\lambda \frac{\lambda^{-1/2}\partial^2 u/\partial z\partial r + \lambda\,\partial^2 w/\partial r^2}{\lambda^2-\lambda^{-1}} + \right.$$
$$\left. + \lambda^{-1}\left(\beta-1-\frac{3}{2}\ln\lambda\right)\frac{\partial^2 w}{\partial z^2} + \frac{3}{2}\frac{\ln\lambda}{r}\frac{\lambda^{-1/2}\partial u/\partial z + \lambda\,\partial w/\partial r}{\lambda^2-\lambda^{-1}} \right\} - \frac{\partial p^\bullet}{\partial z} = 0. \tag{3.4}$$

The linearized incompressibility condition arises from (1.11), it is

$$\lambda^{-3/2}\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{u}{r} = 0. \tag{3.5}$$

According to (1.9), the boundary conditions on the lateral surface of the cylinder in the perturbed state are set by the vectorial equality $\mathbf{e}_r \cdot \mathbf{D}^\bullet(r_0, z) = 0$. This imposes the following restrictions on functions u, w, and $p^\bullet$ at $r = r_0$:

$$\left\{ \begin{array}{l} A\beta \left(\frac{3}{2}\ln^2\lambda\right)^{\beta/2-1} \left\{ \left(1-\frac{\beta}{2}\right)\lambda^{-1/2}\frac{\partial w}{\partial z} + \lambda\frac{\partial u}{\partial r} \right\} - \lambda^{1/2}p^\bullet = 0, \\ \\ \dfrac{\partial u}{\partial z} + \lambda^{3/2}\dfrac{\partial w}{\partial r} = 0. \end{array} \right. \tag{3.6}$$

Linearizing the boundary conditions (1.10) on the cylinder ends $z = 0$ and $z = l$ we get

$$w = 0,$$

$$\mathbf{e}_z \cdot \mathbf{D}^\bullet \cdot \mathbf{e}_r = A\beta \left(\frac{3}{2}\right)^{\beta/2} \ln^{\beta-1}\lambda \frac{\partial u/\partial z + \lambda^{-3/2}\,\partial w/\partial r}{\lambda^2-\lambda^{-1}} = 0. \tag{3.7}$$

Equations (3.4), (3.5) and the boundary conditions (3.6), (3.7) form the linear homogeneous boundary value problem which has the trivial solution $u = w = p^\bullet = 0$. According to the equilibrium stability bifurcation criterion [2, 7], the investigation of stability in small reduces to the determination of the spectrum of critical values of parameter λ, at which the above boundary value problem has nontrivial solutions and to the determination of eigenfunctions, the buckling modes.

Let us note that by virtue of "the concept of continuing loading", the bifurcation criterion is also suitable for investigation of the elastic–plastic body stability [8].

Let us search for a solution of the boundary value problem (3.4)–(3.7) of the form

$$u(r, z) = U(r) \cos \gamma z, \quad w(r, z) = W(r) \sin \gamma z,$$
$$p^\bullet(r, z) = AQ(r) \cos \gamma z, \quad \gamma = \frac{n\pi}{l}, \quad n = 1, 2, 3, \ldots \tag{3.8}$$

This solution form allows us to satisfy the boundary conditions (3.7) on the cylinder ends. Substitution of (3.8) into (3.4)–(3.6) leads to the ordinary differential equations system

$$\beta \left(\frac{3}{2} \ln^2 \lambda \right)^{\beta/2-1} \left\{ \lambda^{3/2} \left(U'' + \frac{2}{r} U' - \frac{3}{2} \gamma^2 \frac{\ln \lambda}{\lambda^3 - 1} U \right) \right.$$
$$\left. + \gamma \left(\frac{3}{2} \frac{\ln \lambda}{\lambda^3 - 1} + 1 - \frac{\beta}{2} \right) W' + \frac{\gamma}{r} W \right\} - \lambda Q' = 0,$$

$$\beta \left(\frac{3}{2} \ln^2 \lambda \right)^{\beta/2-1} \left\{ \frac{3}{2} \frac{\ln \lambda}{\lambda^2 - \lambda^{-1}} \left(-\lambda^{-1/2} \gamma \left[U' + \frac{1}{r} U \right] + \lambda W'' + \frac{\lambda}{r} W' \right) \right. \tag{3.9}$$
$$\left. + \lambda^{-1} \gamma^2 \left(\frac{3}{2} \ln \lambda - \beta + 1 \right) W \right\} + \gamma Q = 0,$$

$$U' + \frac{1}{r} U + \lambda^{-3/2} \gamma W = 0,$$

with the boundary conditions

$$\left[\beta \left(\frac{3}{2} \ln^2 \lambda \right)^{\beta/2-1} \left\{ \left(1 - \frac{\beta}{2} \right) \lambda^{-1/2} \gamma W + \lambda U' \right\} - \lambda^{1/2} Q \right] \Bigg|_{r = r_0} = 0, \tag{3.10}$$

$$\left(\gamma U - \lambda^{3/2} W' \right) \Big|_{r = r_0} = 0.$$

There are two boundary conditions needed at $r = 0$ besides the conditions (3.10) to solve the system (3.9). They arise because of the material continuity and the solution smoothness conditions on the cylinder axis. Their form is

$$U(0) = W'(0) = 0. \tag{3.11}$$

4. The properties of the critical elongations spectrum and the forms of the loss of stability

The linear homogeneous boundary value problem (3.9)–(3.11) was numerically solved by the finite–difference method described in [9]. It is determined that critical values of parameter λ exist only on the descending part of the stretching diagram, i. e. at $\lambda > \lambda_*$. This fact is consistent with the theorem of adjacent

equilibrium form absence on the ascending part of the rod stretching diagram proved [1] for an isotropic incompressible material of general type in case of plane deformation of a rectangular beam.

In Table 1 there are presented the values of critical elongations $\delta_n(\beta) = \lambda_n(\beta) - 1$ which depend on integer parameter of waveformation along the cylinder axis n and the hardening parameter β. It was accepted in these calculations that $l = 20$, $r_0 = 1$. In Table 2 there are presented the values $\delta_*(\beta)$ which correspond to the maximum point on the loading diagram. From comparison between Table 1 and Table 2 one can see that the first critical elongation δ_1 is located very close to the maximum point on the stretching diagram.

TABLE 1

n	$\beta = 1.01$	$\beta = 1.1$	$\beta = 1.5$	$\beta = 2.0$
1	0.01008	0.10543	0.64929	1.71870
2	0.01019	0.10621	0.65099	1.71995
3	0.01042	0.10756	0.65385	1.72204
4	0.01085	0.10959	0.65786	1.72495
10	0.02988	0.14594	0.70708	1.75950
30	0.50076	0.59901	1.09411	2.04107
90	0.69308	0.84360	1.59451	2.82390
300	0.76825	0.93500	1.78310	3.26516
500	0.78038	0.94968	1.81428	3.34077
1000	0.78928	0.96045	1.83712	3.39549

TABLE 2

	$\beta = 1.01$	$\beta = 1.1$	$\beta = 1.5$	$\beta = 2.0$
δ_*	0.01005	0.10517	0.64872	1.71828

Let us note some properties of the critical elongations δ_n distribution. With growth of n, values δ_n monotonically grow. With growth of n, since $n = 1$, the difference $\delta_{n+1} - \delta_n$ increases initially and then begins to decrease becoming very small for n being large enough. Critical values δ_n corresponding to different instability modes are close to each other. That is why the real shape of the neck is formed as a result of the superposition of many sinusoidal instability modes.

The numerical calculations show that with growth of n, the number of zeros of eigenfunctions $U_n(r)$, $W_n(r)$ of the boundary value problem (3.9)–(3.11) increases. This means that with the increase of the mode number, the oscillation of solution along the radial coordinate increases. For example at $\beta = 1.1$ and $n < 10$, functions $W_n(r)$ do not change their sign in the interval $(0, r_0)$; at $40 < n < 55$ they have one zero; at $75 < n < 90$ two zeroes and so on. At $n = 300$, function $W_n(r)$ has eight zeros.

Character of solution oscillation depends on the hardening parameter β. With growth of β, the oscillation decreases. It is found that, at stability loss, for the modes of higher orders deformation at stability loss is localized at the lateral surface of the cylinder.

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