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Existence and analyticity of lump solutions for generalized Benney-Luke equations

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ABSTRACT. We prove the existence and analyticity of lump solutions (finite-energy solitary waves) for generalized Benney-Luke equations that arise in the study of the evolution of small amplitude, three-dimensional water waves. The family of generalized Benney-Luke equations reduce formally to the generalized Korteweg-de Vries (GKdV) equation and to the generalized Kadomtsev-Petviashvili (GKP-I or GKP-II) equation in the appropriate limits. Existence lumps is proved via the concentration-compactness method. When surface tension is sufficiently strong (Bond number larger than1/3), we prove that a suitable family of generalized Benney-Luke lump solutions converges to a nontrivial lump solution for the GKP-I equation.

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1. Introduction

The Benney-Luke equation is a model to describe dispersive and weakly nonlinear long water waves with small amplitude (see [1],[4], [5]). For a general nonlinearity $F(u, Q_1, Q_2, v)$ where $u, v \in \mathbb{R}$ and $Q_1, Q_2 \in \mathbb{R}^2$, the Benney-Luke equation has the form:

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \varepsilon F(\Phi_t, \nabla\Phi, \nabla\Phi_t, \Delta\Phi) = 0, \tag{1}$$

where ε, μ, a and b are positive numbers.

For three dimensional water waves, ε is the amplitude parameter (nonlinearity coefficient) and $\mu = (h_0/L)^2$ is the long-wave parameter (dispersion coefficient) when L is long compared with h_0, where L represents the horizontal length of motion and h_0 is the depth at infinity. The parameters a and b should be positive and satisfy $a - b = \sigma - \frac{1}{3}$ where σ is called the Bond number and is defined by $T = h_0^2\rho g\sigma$ with T being the coefficient of surface tension, ρ being the density (assumed constant) and g being the gravitational acceleration. In this particular case the nonlinearity is given by

$$F_1(\Phi_t, \nabla\Phi, \nabla\Phi_t, \Delta\Phi) = \Phi_t\Delta\Phi + (\nabla\Phi)_t^2. \tag{2}$$

and the variable Φ is the nondimensional velocity potential on the bottom $z = 0$, satisfying

$$\phi(x, y, z = 0, t) = \frac{\varepsilon h_0\sqrt{gh_0}}{\sqrt{\mu}}\Phi(\hat{x}, \hat{y}, \hat{t}),$$

where ϕ is the velocity potential and the space-time variables are scaled via

$$(x, y, z, t) = h_0\left(\frac{\hat{x}}{\sqrt{\mu}}, \frac{\hat{y}}{\sqrt{\mu}}, \hat{z}, \frac{\hat{t}}{\sqrt{\mu g h_0}}\right).$$

For three-dimensional water waves, Pego and Quintero derived (1) in the presence of surface tension or Bond number $\sigma \neq 0$ ([5]), Benney and Luke ([1]) derived (1) with $\varepsilon = \mu$, $a = \frac{1}{6}$, $b = \frac{1}{2}$ in the absence of surface tension ($\sigma = 0$). Related work without the long-wave assumption has recently been done by Mileswki and Keller for $\varepsilon = \mu$ and $\sigma = 0$ ([4]).

In this paper we are interested in considering equation (1) when F contains some powers. More exactly, let $p \in \mathbb{N}$ or $p = m_1/m_2 \geq 1$, where m_1, m_2 are relative prime odd numbers, so that we can define ω^p and $\omega^{\frac{p\pm1}{2}}$ for any $\omega \in \mathbb{R}$. Now we consider a generalized gradient and a generalized Laplacian

$$\nabla^p\phi = \left([\partial_x\phi]^p, [\partial_y\phi]^p\right) \quad \text{and} \quad \Delta_p\phi = \nabla\cdot(\nabla^p\phi\ ) = \partial_x[\partial_x\phi]^p + \partial_y[\partial_y\phi]^p.$$

Note that ∇^p and Δ_p are the usual gradient and Laplacian operators for $p = 1$.

We will consider therein nonlinearities containing some powers of the following type

$$F_p = F(\Phi_t, \nabla^p\Phi, \nabla\Phi_t, \Delta_p\Phi) = \Phi_t\Delta_p\Phi + \left(\frac{2}{p+1}\right)|\nabla^{\frac{p+1}{2}}\Phi|_t^2, \tag{3}$$

where for $p \in \mathbb{N}$, $|\nabla^{\frac{p+1}{2}}\Phi|^2 = (\Phi_x)^{p+1} + (\Phi_x)^{p+1}$. Clearly for $p = 1$ we obtain the usual Benney-Luke equation (see [5]).

We note that for $p \geq 1$ the generalization of the Benney-Luke equation obtained by using F_p defined by (3) in equation (1)

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \varepsilon\left(\Phi_t\Delta_p\Phi + \left(\frac{2}{p+1}\right)|\nabla^{\frac{p+1}{2}}\Phi|_t^2\right) = 0, \quad \text{(GBL)}$$

reduces formally to the generalized Kadomtsev-Petviashvili (GKP) equation when we seek waveforms propagating predominantly in one direction, slowly evolving in time and having weak transverse variation. More precisely, when we seek a solution of (1) in the form

$$\Phi(x, y, t) = \gamma f(X, Y, \tau)$$

where $\tau = \frac{\varepsilon^{\frac{2}{p+1}}}{2}t$, $X = x - t$, $Y = \varepsilon^{\frac{1}{p+1}}y$ and $\gamma^p\varepsilon^{\frac{p-1}{p+1}} = 1$. If we substitute $\mu = \varepsilon^{\frac{2}{p+1}}$ and $\eta = f_X + O(\varepsilon^{\frac{2}{p+1}})$ then, after neglecting $O(\varepsilon^{\frac{2}{p+1}})$ terms, we find that η satisfies the generalized Kadomtsev-Petviashvili (GKP) equation

$$\left(\eta_\tau - \left(\sigma - \frac{1}{3}\right)\eta_{XXX} + (p+2)\eta^p\eta_X\right)_X + \eta_{YY} = 0. \quad \text{(GKP)}$$

Pego and Quintero in [5] proved the existence of solitary waves

$$\Phi_{\varepsilon,\mu,c}(t, x, y) = u_{\varepsilon,\mu,c}(x - ct, y)$$

for the Benney-Luke equation, whenever $p = 1$, $\varepsilon > 0$, $\mu > 0$ and the wave speed $c > 0$ satisfies $c^2 < \min\{1, a/b\}$. When the Bond number $\sigma > \frac{1}{3}$, They also showed that physically meaningful finite-energy (lumps) solutions, apparently, corresponds to waves with speed close to one and having weak dependence on y . To obtain this result they proved that in a suitable limit of a renormalized family of the Benney-Luke lump solutions ($\varepsilon = \mu$ and $c^2 = 1 - \varepsilon$), one obtains lump solutions for the KP-I equation as $\varepsilon \to 0^+$.

The paper is organized as follows. In section 2, using the Hamiltonian structure of the generalized Benney-Luke equation we determine the natural finite-energy space for solitary waves solutions. In this space, finite-energy solitary waves (lumps) correspond to critical points of an action functional. Then to prove the existence of solitary waves for the generalized Benney-Luke equation we use the concentration-compactness method, whenever the wave speed $c > 0$ satisfies $c^2 < \min\{1, a/b\}$ and $p \in \mathbb{N}$ or $p = m_1/m_2 \geq 1$ (m_1, m_2 relative prime odd numbers). In section 3 we prove the analyticity of the solitary wave solutions for $p \in \mathbb{N}$. In section 4, when $1 \leq p < 2$ and $\sigma > 1/3$, we show that it is possible to obtain GKP lump solutions through a suitable limit of a renormalized family of the generalized Benney-Luke lump solutions found in section 2.

2. Existence of solitary waves

In this section we are going to prove the existence of a finite-energy solitary wave for the generalized Benney-Luke equation (1) for fixed positive values of the parameters a, b, ε, μ and $p \in \mathbb{N}$ or $p = m_1/m_2 \geq 1$ (m_1, m_2 relative prime odd numbers), when the non dimensional speed c is small enough, satisfying $0 < c^2 < \min\{1, a/b\}$. We will characterize the solitary wave variationally, as a minimizer of a functional, and apply the concentration-compactness method to prove that the minimum is attained.

To determine such functional we will use the fact that the generalized Benney-Luke equation has a Hamiltonian structure. In fact, first note that equation (1) arises as the Euler-Lagrange equation for the action functional

$$S = \int_{t_0}^{t_1} L(\Phi, \Phi_t)\, dt, \tag{4}$$

where the Lagrangian L is given by

$$L(\Phi, \Psi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\Psi^2 + \mu b|\nabla\Psi|^2 - |\nabla\Phi|^2 - \mu a|\Delta\Phi|^2 + \frac{2\varepsilon}{p+1}\Psi|\nabla^{\frac{p+1}{2}}\Phi|^2 \right) dx\, dy. \tag{5}$$

To find a Hamiltonian form for (1), we follow a standard procedure. Introduce the conjugate momentum variable

$$Q = D_2L(\Phi, \Phi_t) = \Phi_t - \mu b\Delta\Phi_t + \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2.$$

Then

$$\Phi_t = B^{-1}\left(Q - \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2 \right),$$

where B denotes the linear operator $B = I - \mu b\Delta$. The Hamiltonian is given via the Legendre transform as

$$\begin{aligned} H &= \int_{\mathbb{R}^2} Q\Phi_t\, dx\, dy - L(\Phi, \Phi_t) \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \Phi_t^2 + \mu b|\nabla\Phi_t|^2 + |\nabla\Phi|^2 + \mu a|\Delta\Phi|^2\, dx\, dy, \end{aligned} \tag{6}$$

or in terms of (Φ, Q) as

$$\begin{aligned} H(\Phi, Q) = \frac{1}{2}\int_{\mathbb{R}^2} &\left(Q - \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2 \right) B^{-1}\left(Q - \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2 \right) \\ &+ |\nabla\Phi|^2 + \mu a|\Delta\Phi|^2\, dx\, dy. \end{aligned} \tag{7}$$

We find that

$$H_Q(\Phi, Q) = B^{-1}\left(Q - \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2 \right) = \Phi_t, \tag{8}$$

$$
\begin{aligned}
H_\Phi(\Phi,Q) &= -\Delta\Phi + \mu a\Delta^2\Phi + \varepsilon\nabla\cdot\left(B^{-1}\left(Q - \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2\right)\nabla^p\Phi\right)\\
&= -\Delta\Phi + \mu a\Delta^2\Phi + \varepsilon\left(\Delta_p\Phi\Phi_t + \left(\frac{1}{p+1}\right)|\nabla^{\frac{p+1}{2}}\Phi|_t^2\right)\\
&= -\Phi_{tt} + \mu b\Delta\Phi_{tt} - \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|_t^2\\
&= -\left(\Phi_t - \mu b\Delta\Phi_t + \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2\right)_t\\
&= -Q_t.
\end{aligned}
\tag{9}
$$

Thus the generalized Benney-Luke equation (1) is equivalent to the system (8)-(9), which is in canonical Hamiltonian form:

$$
\begin{pmatrix}\Phi_t\\ Q_t\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\nabla H(\Phi,Q).
$$

The Hamiltonian in (6) or (7) is formally conserved in time for solutions of (1). Moreover, the Hamiltonian is translation-invariant, so by Noether's theorem there is an associated momentum functional N which is also conserved in time. In fact, consider the functional given by

$$
N(\Phi,Q) = \int_{\mathbb{R}^2} Q\nabla\Phi\,dx\,dy = \int_{\mathbb{R}^2}\begin{pmatrix}(\Phi_t - \mu b\Delta\Phi_t + \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2)\Phi_x\,dx\,dy\\ (\Phi_t - \mu b\Delta\Phi_t + \frac{\varepsilon}{p+1}|\nabla^{\frac{p+1}{2}}\Phi|^2)\Phi_y\,dx\,dy\end{pmatrix}
$$

Before we continue our discussion about the functional and the space, we will unscale the amplitude and the space variables to eliminate μ and ε from the problem. Thus, we look for traveling-wave solutions in the form

$$
\Phi(x,y,t) = \left(\frac{\sqrt{\mu}}{\varepsilon^{\frac{1}{p}}}\right) u\left(\frac{x-ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right). \tag{10}
$$

The traveling-wave profile u should satisfy

$$
\begin{aligned}
(c^2-1)u_{xx} + (a-bc^2)u_{xxxx} - u_{yy} + au_{yyyy} + (2a-bc^2)u_{xxyy}&\\
+ c\left((p+2)u_x^p u_{xx} + pu_x u_y^{p-1}u_{yy} + 2u_y^p u_{xy}\right) &= 0.
\end{aligned}
\tag{11}
$$

We look for traveling-wave solutions with finite energy. In terms of the profile u, the energy from (6) takes the form $H = (\mu/\varepsilon^{\frac{2}{p}})E(u)$, where

$$
E(u) = \frac{1}{2}\int_{\mathbb{R}^2}(1+c^2)u_x^2 + u_y^2 + (a+bc^2)u_{xx}^2 + (2a+bc^2)u_{xy}^2 + au_{yy}^2\,dx\,dy. \tag{12}
$$

Hereafter $p \in \mathbb{N}$ or $p = m_1/m_2 \geq 1$ where m_1 and m_2 are relative prime odd numbers.

Theorem 2.1. *Let a and b be fixed positive numbers. If $c > 0$ and $0 < c^2 < \min\{1, a/b\}$, then equation (11) has a nontrivial weak solution whose* *derivatives of positive order are all square-integrable. Moreover, If $p \in \mathbb{N}$, then any weak solution of equation (11) is already analytic.*

It is natural first to look for a weak solution u in the "finite-energy" space $\mathcal{V}$ defined below. We use the following standard notation for function spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multi-index , $k \geq 0$ be an integer and r be a positive number with $1 \leq p < \infty$. Then

$$C^\infty(\Omega) = \{f : \Omega \to \mathbb{R} |\ D^\alpha f \text{ is continuous on } \Omega \text{ for any } \alpha\},$$

$$C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) |\ \operatorname{supp} f \quad \text{is compact}\},$$

where $D^\alpha : \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}$ with $|\alpha| = \sum_{i=1}^{i=n} \alpha_i$. $\mathcal{D}'(\Omega)$ is the space of distributions on Ω, the continuous linear functionals on $C_0^\infty(\Omega)$.

The Sobolev space $W^{k,r}(\Omega)$ is defined as the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,r}(\Omega)} = \left\{ \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^r \, dx \right\}^{\frac{1}{r}}.$$

We also denote $W^{0,r}(\Omega) = L^r(\Omega)$. For $r = 2$, $W^{k,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{W^{k,2}(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u \cdot D^\alpha v \, dx.$$

In the inequalities below, C denotes a generic constant whose value may change from instance to instance.

Definition 2.1. *Let $\mathcal{V}$ denote the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm given by*

$$\|\psi\|_{\mathcal{V}}^2 := \int_{\mathbb{R}^2} \psi_x^2 + \psi_y^2 + \psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2 \, dx \, dy.$$

Note that $(\mathcal{V}, \|.\|)$ is a Hilbert space with inner product

$$(u, v)_{\mathcal{V}} = (\partial_x u, \partial_x v)_{W^{1,2}(\mathbb{R}^2)} + (\partial_y u, \partial_y v)_{W^{1,2}(\mathbb{R}^2)},$$

Equation (11) can be considered in weak form on the space $\mathcal{V}$, by defining

$$\begin{aligned} A(u, v) =& (1 - c^2)(u_x, v_x)_0 + (u_y, v_y)_0 + (a - bc^2)(u_{xx}, v_{xx})_0 + \\ & (2a - bc^2)(u_{xy}, v_{xy})_0 + a(u_{yy}, v_{yy})_0, \\ B(u, v) =& \frac{c(p+2)}{p+1}(u_x^{p+1}, v_x)_0 + c(u_x u_y^p, v_y)_0 + \frac{c}{p+1}(u_y^{p+1}, v_x)_0 \end{aligned}$$

for all $u, v \in \mathcal{V}$, where $(u, v)_0$ denotes the inner product in $L^2(\mathbb{R}^2)$. We say that $u \in \mathcal{V}$ is a weak solution of (11) if

$$A(u, v) + B(u, v) = 0 \quad \forall v \in \mathcal{V}. \tag{13}$$

Note that a weak solution satisfies (11) in the sense of distributions. Conversely, a distributional solution that lies in $\mathcal{V}$ is a weak solution.

Our goal now is to establish the existence of a weak solution of (11) which will be characterized as a minimizer of a suitable minimization problem. To do this we observe that a variational principle for the traveling wave profile u can be obtained by substituting the form (10) into the action functional in (4) and requiring the resulting functional to be stationary. We find that this means that $I(u) + G_{c,p}(u)$ should be stationary, where the functionals I and $G_{c,p}$ are defined by

$$I(u) = \int_{\mathbb{R}^2} \{(1-c^2)u_x^2 + u_y^2 + (a-bc^2)u_{xx}^2 + (2a-bc^2)u_{xy}^2 + au_{yy}^2\}dV,$$

$$G_{c,p}(u) = c\int_{\mathbb{R}^2} \{u_x^{p+2} + u_y^{p+1}u_x\}dV,$$

where $dV = dx\,dy$. We note that the functionals I and $G_{c,p}$ are smooth maps from $\mathcal{V}$ to $\mathbb{R}$. To show that $G_{c,p}(u)$ is well-defined for all $u \in \mathcal{V}$, note that $u_x, u_y \in W^{1,2}(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for all $q \geq 2$, therefore by applying Young's inequality to the second member of $G(u)$ we obtain

$$\begin{aligned} |G_{c,p}(u)| &\leq c\int_{\mathbb{R}^2} \left(|u_x|^{p+2} + \frac{1}{p+2}\left(|u_x|^{p+2} + (p+1)|u_y|^{p+2}\right)\right) dV \\ &\leq \frac{(p+3)c}{p+2}\|u\|_{\mathcal{V}}^{p+2}. \end{aligned} \tag{14}$$

So it is natural to look for functions $u_0 \in \mathcal{V}$ that are characterized as follows:

$$I(u_0) = \mathcal{I}_p \stackrel{\text{def}}{=} \inf\{I(u) : u \in \mathcal{V} \text{ with } G_{c,p}(u) = 1\}. \tag{15}$$

Note that there exists $v \in \mathcal{V}$ such that $G_{c,p}(v) \neq 0$, so $G_{c,p}(tv) = 1$ for some t. Thus the set in (15) is nonempty, and since $I \geq 0$, the infimum is nonnegative and finite. Moreover, $\mathcal{I}_p > 0$. In fact, the assumptions on c, a and b imply that $I(u) \geq 0$ for all $u \in \mathcal{V}$. On the other hand, the inequality (14) and the definition of the functional I imply that

$$|G_{c,p}(u)| \leq C_1(a,b,c,p)I(u)^{\frac{p+2}{2}}.$$

So, if $G_{c,p}(u) = 1$ then $(C_1)^{\frac{2}{p+2}}I(u) \geq 1$. Therefore $\mathcal{I}_p > 0$.

Note that Theorem 2.1 will follow as a direct consequence of Lemma 3.1, which will be proved at the end of this section, Lemma 2.1 and Proposition 2.1 below.

Lemma 2.1. *If u_0 is a minimizer for problem (15), then $u = -\lambda^{1/p}u_0$ is a weak solution of (11), where $\lambda = \left(\frac{2}{p+2}\right)\mathcal{I}_p > 0$.*

Proof. Since u_0 is characterized as the minimizer for $\mathcal{I}_p$, by the Lagrange theorem there is a Lagrange multiplier λ such that

$$I'(u_0)(w) - \lambda G'_{c,p}(u_0)(w) = 0 \quad \forall w \in \mathcal{V}.$$

But, we know that

$$I'(u_0)(w) = 2A(u_0, w) \quad \text{and} \quad G'_{c,p}(u_0)(w) = (p+1)B(u_0, w).$$

That is, for all $w \in \mathcal{V}$, $A(u_0, w) - \lambda B(u_0, w) = 0$. If we put $w = u_0$ and use the following facts

$$I'(u_0)(u_0) = 2I(u_0) \quad \text{and} \quad G'_{c,p}(u_0)(u_0) = (p+2)G_{c,p}(u_0) = p+2,$$

we conclude that $\lambda = \left(\frac{2}{p+2}\right)\mathcal{I}_p > 0$. Then $u = -\lambda^{1/p}u_0$ is a nontrivial weak solution of (11). ☑

Proposition 2.1. *Assume $a, b, c > 0$ and $c^2 < \min\{1, a/b\}$. If $\{u_m\}_{m\geq 1} \subset C_0^\infty(\mathbb{R}^2)$ is a minimizing sequence for (15), then there is a subsequence (denoted the same), a sequence of points $(x_m, y_m) \in \mathbb{R}^2$, and a minimizer $u_0 \in \mathcal{V}$ of (15), such that the translated functions $v_m = u_m(\cdot + x_m, \cdot + y_m)$ converge strongly in $\mathcal{V}$ to u_0.*

Before beginning the proof, we want to discuss the main tool used in order to prove our theorem.

Let $\{u_m\} \subset C_0^\infty(\mathbb{R}^2)$ be a minimizing sequence for $\mathcal{I}_p$. Define

$$\begin{aligned}\rho_m(x,y) = (1-c^2)(u_m)_x^2 + (u_m)_y^2 + (a-bc^2)(u_m)_{xx}^2 \\ + (2a-bc^2)(u_m)_{xy}^2 + a(u_m)_{yy}^2.\end{aligned} \tag{16}$$

Then we have

$$\lim_{m\to\infty} I(u_m) = \int_{\mathbb{R}^2} \rho_m(x,y)\, dV = \mathcal{I}_p \quad \text{and} \quad G_{c,p}(u_m) = 1.$$

Consider the positive measures $\nu_m = \rho_m(x,y)\, dV$ given by (16). By the concentration-compactness lemma [7, Lemma 4.3, p 37] there exists a subsequence of $\{\nu_m\}$ (which we denote the same) such that one of the following three conditions holds:

(i) (Vanishing) For all $R > 0$ there holds

$$\lim_{m\to\infty} \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_R(x,y)} d\nu_m \right) = 0.$$

(ii) (Dichotomy) There exists $\theta \in (0, \mathcal{I}_p)$ such that for any $\gamma > 0$, there exist a positive number R and a sequence $\{(x_m, y_m)\} \subset \mathbb{R}^2$ with the following property: Given $R' > R$ there are nonnegative measures ν_m^1, ν_m^2 such that

(a) $0 \leq \nu_m^1 + \nu_m^2 \leq \nu_m,$

(b) $\text{supp}(\nu_m^1) \subset B_R(x_m, y_m), \quad \text{supp}(\nu_m^2) \subset \mathbb{R}^2 \setminus B_{R'}(x_m, y_m),$

(c) $\limsup_{m\to\infty} \left(|\theta - \int_{\mathbb{R}^2} d\nu_m^1| + |(\mathcal{I}_p - \theta) - \int_{\mathbb{R}^2} d\nu_m^2|\right) \leq \gamma.$

(iii) (Compactness) There exists a sequence $\{(x_m, y_m)\} \subset \mathbb{R}^2$ such that for any $\gamma > 0$, there is a radius $R > 0$ with the property that

$$\int_{B_R(x_m,y_m)} d\nu_m \geq \mathcal{I}_p - \gamma, \qquad \text{for all } m.$$

The general strategy is to show that Vanishing (i) and Dichotomy (ii) are impossible. Then we have Compactness (iii) to prove the proposition, in which case convergence will follow in a fashion typical for the direct minimization method. The basic result needed to rule out (i) and (ii) are to use the scaling properties of the functionals I and $G_{c,p}$ and the Sobolev inequality.

Proposition 2.2 (A Sobolev inequality). *There exists a positive constant C_1 such that for all $u \in C^\infty(\mathbb{R}^n)$ and $q \geq 2$*

$$\left(\int_{A(R,x_0)} |u(x) - a(R,x_0)|^q \, dV\right)^{1/q} \leq C_1 R^{n\left(\frac{1}{q}-\frac{1}{2}\right)+1} \left(\int_{A(R,x_0)} |\nabla u(x)|^2 \, dV\right)^{1/2},$$

where

$$a(R,x_0) = \frac{1}{Vol(A(R,x_0))}\left(\int_{A(R,x_0)} u(x)\, dV\right).$$

and $A(R,x_0) = B(2R,x_0)\setminus B(R,x_0)$ with $B(R,x_0)$ denoting the R-ball around $x_0 \in \mathbb{R}^n$.

We will first establish some basic results.

Lemma 2.2. *Vanishing (i) is not possible.*

Proof. Suppose that we have vanishing (i). Let (x, y) be any point in $\mathbb{R}^2$ and let B_1 denote the ball of radius 1 around the point (x, y). Since $W^{1,2}(B_1)$ is continuously embedded into $L^q(B_1)$ for $q \geq 2$, we have

$$\begin{aligned}\int_{B_1(x,y)} &|(u_m)_x|^q + |(u_m)_y|^q dV \\ &\leq C_2 \left(\|(u_m)_x\|^2_{W^{1,2}(B_1)} + \|(u_m)_y\|^2_{W^{1,2}(B_1)} \right)^{\frac{q}{2}} \\ &\leq C_3 \left(\|(u_m)_x\|^2_{W^{1,2}(B_1)} + \|(u_m)_y\|^2_{W^{1,2}(B_1)} \right) \left(\int_{B_1} \rho_m(x,y)\, dV \right)^{\frac{q-2}{2}}.\end{aligned}$$

We can cover $\mathbb{R}^2$ by balls of radius 1 such that any point of $\mathbb{R}^2$ is contained in at most 3 balls. Summing up, using the inequality above and the fact that $\|\nabla u\|^2_{W^{1,2}(\mathbb{R}^2)} \leq CI(u)$, we find that

$$\begin{aligned}\int_{\mathbb{R}^2} &|(u_m)_x|^q + |(u_m)_y|^q dV \\ &\leq 3C_4(q, C_3) I(u_m) \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_1(x,y)} \rho_m(x,y)\, dV \right)^{\frac{q-2}{2}}.\end{aligned}$$

Taking $q = p+2$ and using the inequality (14), it follows

$$\begin{aligned}1 = G_{c,p}(u_m) &\leq \frac{(p+3)c}{p+2} \int_{R^2} |(u_m)_x|^{p+2} + |(u_m)_y|^{p+2}\, dV \\ &\leq \frac{3c(p+3)}{p+2} C_4 I(u_m) \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_1(x,y)} \rho_m(x,y)\, dV \right)^{\frac{q-2}{2}}.\end{aligned}$$

Then using the vanishing condition, we get the contradiction

$$1 = \lim_{m\to\infty} G_{c,p}(u_m) = 0.$$

This rules out vanishing. ☑

To rule out dichotomy (ii), we shall obtain a contradiction by a standard sub-additivity argument, after showing that a minimizing sequence u_m splits appropriately into two sequences u^1_m and u^2_m with gradients having disjoint supports. Because of the nature of the finite-energy space $\mathcal{V}$, the construction of this splitting is non-standard. First note that if dichotomy (ii) holds, we can choose a sequence $\gamma_m \to 0$ and corresponding sequence $R_m \to \infty$, such that, passing to a subsequence if necessary, we can assume

(*d*) $\operatorname{supp}(\nu^1_m) \subset B_{R_m}(x_m, y_m), \quad \operatorname{supp}(\nu^2_m) \subset \mathbb{R}^2 \setminus B_{2R_m}(x_m, y_m)$,

(*e*) $\limsup_{m\to\infty} \left(|\theta - \int_{R^2} d\nu^1_m| + |(\mathcal{I}_p - \theta) - \int_{\mathbb{R}^2} d\nu^2_m| \right) = 0.$

Note that in particular, conditions (d) and (e) imply that

$$\limsup_{m\to\infty}\left(\int_{A(m)}\rho_m(x,y)\,dV\right)=0,$$

where $A(m)$ is the annulus

$$A(m)=B_{2R_m}(x_m,y_m)\setminus B_{R_m}(x_m,y_m).$$

In fact,

$$\begin{aligned}\int_{A(m)}\rho_m(x,y)\,dV&=\left\{\int_{\mathbb{R}^2}-\int_{B_{R_m}(x_m,y_m)}-\int_{\mathbb{R}^2\setminus B_{2R_m}(x_m,y_m)}\right\}\rho_m(x,y)\,dV\\&\le\left(\int_{\mathbb{R}^2}\rho_m(x,y)\,dV-\mathcal{I}_p\right)+\left|\theta-\int_{\mathbb{R}^2}d\nu_m^1\right|\\&\quad+\left|(\mathcal{I}_p-\theta)-\int_{\mathbb{R}^2}d\nu_m^2\right|.\end{aligned}$$

In consequence,

$$\int_{A(m)}|\partial_x u_m|^2+|\partial_y u_m|^2+|\partial_{xx}u_m|^2+2|\partial_{xy}u_m|^2+|\partial_{yy}u_m|^2\,dV\to 0$$

as $m\to\infty$.

Next, we will prove a result which is a generalization of the Splitting Theorem proved by Pego and Quintero in [5].

Lemma 2.3. *(Splitting of a minimizing sequence). Fix* $\phi\in C_0^\infty(\mathbb{R}^2,\mathbb{R}^+)$ *such that* $\operatorname{supp}(\phi)\subset B_2(0,0)$ *and* $\phi\equiv 1$ *in* $B_1(0,0)$, *and let*

$$a_m=\frac{1}{\operatorname{vol}(A_m)}\int_{A(m)}u_m\,dV,\quad \phi_m(x,y)=\phi\left(\frac{x-x_m}{R_m},\frac{y-y_m}{R_m}\right).$$

Define

$$u_m^1=(u_m-a_m)\phi_m,\quad u_m^2=(u_m-a_m)(1-\phi_m)+a_m.$$

Then as $m\to\infty$ *we have*

(1) $I(u_m)=I(u_m^1)+I(u_m^2)+o(1)$;
(2) $G_{c,p}(u_m)=G_{c,p}(u_m^1)+G_{c,p}(u_m^2)+o(1)$.

Proof. We use the notation $\partial_1=\partial_x$, $\partial_2=\partial_y$, etc. The first part is already proved in the Splitting result by Pego and Quintero [5]. To prove part (2) note that the Hölder's inequality and an estimate as in (14) imply that

$$
\begin{aligned}
&|G(u_m) - G(u_m^1) - G(u_m^2)| \\
&= \left| \int_{A(m)} (\partial_x u_m)|\nabla u_m|^{p+1} - (\partial_x u_m^1)|\nabla u_m^1|^{p+1} - (\partial_x u_m^2)|\nabla u_m^2|^{p+1}\, dV \right| \\
&\le C_{12} \int\limits_{A(m)} (|\partial_x u_m| + |u_m - a_m||\partial_x \phi_m|)\left(|\nabla u_m|^{p+1} + |u_m - a_m|^{p+1}|\nabla \phi_m|^{p+1}\right) dV \\
&\le C_{13} \int\limits_{A(m)} |\nabla u_m|^{p+2} + \frac{|u_m - a_m|^{p+2}}{R_m^{p+2}}\, dV \le C\left(\int_{A(m)} \rho_m(x,y)\, dV\right)^{\frac{p+2}{2}} \to 0
\end{aligned}
$$

as $m \to \infty$. This proves part (2), finishing the proof of the Lemma. ☑

We can now rule out dichotomy using a standard sub-additivity argument.

Lemma 2.4. *Dichotomy is not possible.*

Proof. Let $\lambda_{m,1} = G_{c,p}(u_m^1)$ and $\lambda_{m,2} = G_{c,p}(u_m^2)$. Suppose that $\lim_{m\to\infty} \lambda_{m,1} = 0$. Then $\lim_{m\to\infty} \lambda_{m,2} = 1$. So, for m sufficiently large, $\lambda_{m,2} > 0$. Define $w_m = \lambda_{m,2}^{-\frac{1}{p+2}} u_m^2$. Then $w_m \in \mathcal{V}$ and $G_{c,p}(w_m) = 1$. So by Lemma 2.3,

$$
\begin{aligned}
I(u_m) &= I(u_m^1) + I(u_m^2) + \delta_m \\
&\ge \int_{B_{R_m}(x_m,y_m)} \rho_m(x,y)\, dV + (\lambda_{m,2})^{\frac{2}{3}} \mathcal{I}_p + \delta_m \\
&\ge \int_{\mathbb{R}^2} d\nu_m^1 + (\lambda_{m,2})^{\frac{2}{3}} \mathcal{I}_p + \delta_m
\end{aligned}
$$

(because $(u_m - a_m)\phi_m = u_m - a_m$ in $B_{R_m}(x_m, y_m)$). Then as $m \to \infty$, it follows $\mathcal{I}_p \ge \theta + \mathcal{I}_p$. This is a contradiction.

Hence we may assume $\lim_{m\to\infty} |\lambda_{m,i}| = |\lambda_i| > 0$, $i = 1, 2$. Then for m sufficiently large, $|\lambda_{m,i}| > 0$. Now, we define functions $w_{m,1}$ and $w_{m,2}$ by

$$
w_{m,1} = \lambda_{m,1}^{-\frac{1}{p+2}} u_m^1, \quad w_{m,2} = \lambda_{m,2}^{-\frac{1}{p+2}} u_m^2.
$$

Then $w_{m,i} \in \mathcal{V}$ and $G_{c,p}(w_{m,i}) = 1$. So,

$$
\begin{aligned}
I(u_m) &= |\lambda_{m,1}|^{\frac{2}{p+2}} I(w_{m,1}) + |\lambda_{m,2}|^{\frac{2}{p+2}} I(w_{m,2}) + o(1) \\
&\ge \mathcal{I}_p\left(|\lambda_{m,1}|^{\frac{2}{p+2}} + |\lambda_{m,2}|^{\frac{2}{p+2}}\right) + o(1).
\end{aligned}
$$

Taking the limit as $m \to \infty$, we conclude that

$$
1 \ge |\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}} \ge (|\lambda_1| + |\lambda_2|)^{\frac{2}{p+2}} \ge 1.
$$

Thus $|\lambda_1| + |\lambda_2| = 1$. Since we have $\lambda_1 + \lambda_2 = 1$ then $\lambda_i \ge 0$. So by the first case we have to have $\lambda_i > 0$. Since the function $f(t) = t^{\frac{2}{p+2}}$ is strictly concave for $t > 0$, then the inequality above gives us again a contradiction. In other words, we have ruled out dichotomy. ☑

Proof of Proposition 2.2. By the previous results, we have compactness (iii). I.e., there exists a sequence $\{(x_m, y_m)\} \subset \mathbb{R}^2$ such that for any $\gamma > 0$, there is a radius $R > 0$ with the property

$$\int_{B_R(x_m,y_m)} d\nu_m \geq \mathcal{I}_p - \gamma \quad \forall m.$$

Define $\hat{\rho}_m(x,y) = \rho_m(x + x_m, y + y_m)$ and $v_m(x,y) = u_m(x + x_m, y + y_m)$. Then we have

$$G_{c,p}(v_m) = 1, \quad \lim_{m\to\infty} I(v_m) = \mathcal{I}_p,$$

and

$$\int_{B_R(0,0)} \hat{\rho}_m(x,y)\, dV \geq \mathcal{I}_p - \gamma, \quad \forall m. \tag{17}$$

In particular, $\|v_m\|_{\mathcal{V}}$ is bounded. Since $(v_m)_x, (v_m)_y \in W^{1,2}(B_R(0,0))$ and this space is compactly embedded in $L^q(B_R(0,0)$ for $q \geq 2$, we conclude that there exist a subsequence of $\{v_m\}$ (denoted the same) and $v_0 \in \mathcal{V}$ such that

$$\begin{array}{lcll}
v_m & \rightharpoonup & v_0 & \text{in } \mathcal{V}, \\
\partial_i v_m & \rightharpoonup & \partial_i v_0 & \text{in } L^2(\mathbb{R}^2), \quad i = 1,2, \\
\partial_{ij} v_m & \rightharpoonup & \partial_{ij} v_0 & \text{in } L^2(\mathbb{R}^2), \quad i,j = 1,2, \\
\partial_i v_m & \to & \partial_i v_0 & \text{in } L^2_{\text{loc}}(\mathbb{R}^2), \quad i = 1,2, \\
\partial_i v_m & \to & \partial_i v_0 & \text{a.e. in } \mathbb{R}^2, \quad i = 1,2.
\end{array}$$

We claim that actually for some further subsequence (denoted the same),

$$\partial_i v_m \to \partial_i v_0 \quad \text{in } L^2(\mathbb{R}^2), \quad i = 1,2. \tag{18}$$

By the compactness condition (17) we have that given any $\gamma > 0$, there exists $R > 0$ such that for m large enough,

$$\int_{B_R((0,0))} |\partial_i v_m|^2 dV \geq \int_{\mathbb{R}^2} |\partial_i v_m|^2 dV - 2\gamma.$$

Since the inclusion $W^{1,2}(B_R(0,0)) \subset L^q(B_R(0,0))$ for $q \geq 2$ is compact, this inequality implies the strong convergence of $\partial_i v_m$ to $\partial_i v_0$ in $L^2(\mathbb{R}^2)$. In fact,

$$\int_{\mathbb{R}^2} |\partial_i v_0|^2 dV \leq \liminf_{m\to\infty} \int_{\mathbb{R}^2} |\partial_i v_m|^2 dV \leq \liminf_{m\to\infty} \int_{B_R(0,0)} |\partial_i v_m|^2 dV + 2\gamma$$

$$= \int_{B_R((0,0))} |\partial_i v_0|^2 dV + 2\gamma \leq \int_{\mathbb{R}^2} |\partial_i v_0|^2 dV + 2\gamma.$$

Consequently,

$$\int_{\mathbb{R}^2} |\partial_i v_0|^2 dV = \liminf_{m\to\infty} \int_{\mathbb{R}^2} |\partial_i v_m|^2 dV,$$

and the claim (18) follows.

Now we want to prove that

$$G_{c,p}(v_0) = 1. \tag{19}$$

First, note that Hölder's inequality and the inclusion $W^{1,2}(\mathbb{R}^2) \subset L^{2(p+1)}(\mathbb{R}^2)$ give us that

$$\int_{\mathbb{R}^2} |\partial_i(v_m - v_0)|^{p+2} dV \leq \|\partial_i(v_m - v_0)\|_{L^2} \cdot \|\partial_i(v_m - v_0)\|^{p+1}_{L^{2(p+1)}}$$
$$\leq C_{14}\|\partial_i(v_m - v_0)\|_{L^{2(p+1)}} \cdot [I(v_m) + I(v_0)]^{\frac{p+1}{2}} = o(1)$$

since $I(v_m)$ is bounded. Hence $\int_{\mathbb{R}^2}(\partial_i(v_m - v_0))^{p+2} dV = o(1)$, and using Hölder's inequality we find that

$$\left| \int_{\mathbb{R}^2} \partial_x(v_m - v_0)(\partial_y(v_m - v_0))^{p+1} dV \right| = o(1),$$

thus $G_{c,p}(v_m - v_0) = o(1)$. On the other hand, it is not hard to show that

$$G_{c,p}(v_m - v_0) = G_{c,p}(v_m) - G_{c,p}(v_0) + o(1).$$

Then we conclude that $G_{c,p}(v_0) = \lim_{m\to\infty} G_{c,p}(v_m) = 1$.

In particular, (19) implies $v_0 \neq 0$ and $I(v_0) \geq \mathcal{I}_p$. Now a direct computation using weak convergence gives us that

$$I(v_m - v_0) = I(v_m) - I(v_0) + o(1) = \mathcal{I}_p - I(v_0) + o(1) \leq o(1).$$

This implies

$$\lim_{m\to\infty} I(v_m) = I(v_0) = \mathcal{I}_p.$$

In other words, v_0 is a minimizer for $\mathcal{I}_p$. Moreover, this also proves that the subsequence $\{v_m\}$ converges to v_0 in $\mathcal{V}$. This finishes the proof of the proposition. ☑

3. Analyticity

In this section we will establish that weak solutions of (11) are analytic for any $p \in \mathbb{N}$. The proof of this result is based on a Proposition (3.1) below[1].

Proposition 3.1. (1) *If $f : \mathbb{R} \to \mathbb{R}$ is a $C^\infty(\mathbb{R})$ and $\phi \in W^{k,2}(\mathbb{R}^2)$ for all $k \geq 1$, then*

$$\partial^\alpha (f(\phi)) = \sum_{j=1}^{|\alpha|} \frac{f^{(k)}(\phi)}{j!} \sum_{A(\alpha,j)} \frac{\alpha!}{\alpha_1!\alpha_2!\cdots\alpha_j!} \partial^{\alpha_1}\phi \partial^{\alpha_2}\phi \cdots \partial^{\alpha_j}\phi ,$$

[1] The author wants to thank professor F. Soriano for pointing out about this result, which was used to prove analyticity of solitary wave solutions for a K.P.-Boussinesq type system ([6]).

where $A(\alpha, j) = \{(\alpha_1, \cdots, \alpha_j) : \alpha_1 + \alpha_2 + \cdots + \alpha_j = \alpha,\ |\alpha_i| \geq 1,\ 1 \leq i \leq j\}$.

(2) *For each* $(n_1, n_2, \cdots, n_j) \in \mathbb{N}^j$ *we have*

$$|\alpha|! = \sum_{A(\alpha,j), |\alpha_i| = n_i} \frac{\alpha! |\alpha_1|! |\alpha_2|! \cdots |\alpha_j|!}{\alpha_1! \alpha_2! \cdots \alpha_j!}.$$

(3) *There exists* C_1 *such that for all* $j, k \in \mathbb{N}$,

$$\sum_{k_1 + k_2 \cdots + k_j = k} \frac{1}{(k_1+1)^2 + (k_2+1)^2 \cdots + (k_j+1)^2} \leq \frac{C_1^{j-1}}{(k+1)^2}$$

Lemma 3.1. *If* $p \in \mathbb{N}$, *then any weak solution* $u \in \mathcal{V}$ *of (11) is already analytic.*

Proof. First we will establish that for any weak solution $u \in \mathcal{V}$ of (11) we have that $u_x, u_y \in W^{k,2}(\mathbb{R}^2)$ for any $k \geq 1$. Note that $u_x,\ u_y \in W^{1,2}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for $q \geq 2$. Let

$$g_1 = \frac{p+2}{p+1} u_x^{p+1} + \frac{1}{p+1} u_y^{p+1}, \quad g_2 = u_x u_y^p,$$

$$P(\zeta, \eta) = (1 - c^2)\zeta^2 + \eta^2 + (a - bc^2)\zeta^4 + (2a - bc^2)\zeta^2\eta^2 + a\eta^4. \tag{20}$$

Then $g_1,\ g_2 \in L^2(\mathbb{R}^2)$, and by taking the Fourier transform of (11) we find

$$\begin{aligned} \widehat{u_x}(\zeta, \eta) &= -\frac{c}{P(\zeta, \eta)} \left(\zeta^2 \widehat{g_1} + \zeta\eta \widehat{g_2}\right), \\ \widehat{u_y}(\zeta, \eta) &= -\frac{c}{P(\zeta, \eta)} \left(\eta\zeta \widehat{g_1} + \eta^2 \widehat{g_2}\right). \end{aligned} \tag{21}$$

Since the coefficients in (20) are all positive, it is evident that for some constant $M > 0$ we have

$$P(\zeta, \eta) \geq M(\zeta^2 + \eta^2)(1 + \zeta^2 + \eta^2)$$

for all real $\zeta,\ \eta$. From (21) it follows that $u_x,\ u_y \in W^{2,2}(\mathbb{R}^2)$. A simple bootstrapping argument then yields that $u_x,\ u_y \in W^{k,2}(\mathbb{R}^2)$ for all $k \geq 1$. In other words, u is smooth.

The main step to prove the analyticity of u is the following result:

Claim 1. There exists $R > 0$ such that for all $\alpha \in \mathbb{N}^2$, with $|\alpha| \geq 1$,

$$\|\partial^\alpha u\|_{W^{2,2}(\mathbb{R}^2)} \leq C \frac{(|\alpha| - 1)!}{|\alpha| + 1} R^{|\alpha| - 1}. \tag{22}$$

Proof of Claim 1. Case $|\alpha| = 1$ follows since $u_x, u_y \in W^{k,2}(\mathbb{R}^2)$ for all $k \geq 1$. Now suppose that (22) holds for $|\alpha| = 1, 2, \cdots n$ and R (which will be choosen later). First we obtain an estimate for $\|\partial^\alpha u\|_{W^{1,2}(\mathbb{R}^2)}$. To do this, we apply operator ∂^α to equation (11) and compute the L^2- inner product with $\partial^\alpha u$. Then we get

$$\begin{aligned}(1-c^2)||\partial^\alpha\partial_x u||_2 + ||\partial^\alpha\partial_y u||_2 + (a-bc^2)||\partial^\alpha\partial_x^2 u||_2 \\ + (2a-bc^2)||\partial^\alpha\partial_{xy}^2 u||_2 + a||\partial^\alpha\partial_y^2 u||_2 \\ = \frac{c(p+2)}{p+1}\left\langle \partial^\alpha(u_x^{p+1}), \partial^\alpha\partial_x u\right\rangle + c\left\langle \partial^\alpha(u_x u_y^p), \partial^\alpha\partial_y u\right\rangle \\ + \frac{c}{p+1}\left\langle \partial^\alpha(u_y^p), \partial^\alpha\partial_x u\right\rangle. \quad (23)\end{aligned}$$

Using the formula given below in the right hand side of (23)

$$\partial^\alpha(u_x u_y^p) = \partial^\alpha(u_x)u_y^p + \sum_{|\beta|=k\geq 1}^{|\alpha|-1}\binom{|\alpha|}{k}\partial^{\alpha-\beta}(u_x)\partial^\beta(u_y^p) + u_x\partial^\alpha(u_y^p)$$

and applying the Hölder inequality, we conclude that there exists a positive constant C_3 such that

$$\begin{aligned}||\partial^\alpha\nabla u||_{W^{1,2}(\mathbb{R}^2)} \leq C_3\Big(||u_y^p||_2 + ||\partial^\alpha(u_x^{p+1})||_2 + ||\partial^\alpha(u_y^{p+1})||_2 + ||u_x||_2||\partial^\alpha(u_y^p)||_2 \\ + \sum_{|\beta|=k\geq 1}^{|\alpha|-1}\binom{|\alpha|}{k}||\partial^{\alpha-\beta}(u_x)||_2||\partial^\beta(u_y^p)||_2\Big) \quad (24)\end{aligned}$$

Now in Proposition (3.1), we consider $f(t)=t^q$ for $q=p+1$ and $\phi=u_x$ or $\phi=u_y$ (or $q=p$ and $\phi=u_y$). We then get

$$||\partial^\alpha f(\phi)||_2 \leq C_2\sum_{j=1}^{q}\sum_{A(\alpha,j)}\frac{\alpha!}{\alpha_1!\alpha_2!\cdots\alpha_j!}||\partial^{\alpha_1}\phi\partial^{\alpha_2}\phi\cdots\partial^{\alpha_j}\phi||_2.$$

But

$$||\partial^{\alpha_1}\phi\partial^{\alpha_2}\phi\cdots\partial^{\alpha_j}\phi||_2 \leq ||\partial^{\alpha_1}\phi||_{2j}||\partial^{\alpha_2}\phi||_{2j}\cdots||\partial^{\alpha_j}\phi||_{2j}.$$

We also know that for any $\beta\in\mathbb{N}^2$,

$$||\partial^\beta\phi||_{2j} \leq C(j)||\partial^\beta\phi||_{W^{1,2}(\mathbb{R}^2)} \leq C_3||\partial^\beta\phi||_{W^{1,2}(\mathbb{R}^2)},$$

where $C_3=\max\{C(j): 1\leq j\leq q\}$. In consequence, for $i=1,2$

$$\begin{aligned}&||\partial^{\alpha_1}\partial_i u\partial^{\alpha_2}\partial_i u\cdots\partial^{\alpha_j}\partial_i u||_2 \\ &\leq C_3^j||\partial^{\alpha_1}\partial_i u||_{W^{1,2}(\mathbb{R}^2)}||\partial^{\alpha_2}\partial_i u||_{W^{1,2}(\mathbb{R}^2)}\cdots||\partial^{\alpha_j}\partial_i u||_{W^{1,2}(\mathbb{R}^2)} \\ &\leq C_3^j||\partial^{\alpha_1}u||_{W^{2,2}(\mathbb{R}^2)}||\partial^{\alpha_2}u||_{W^{2,2}(\mathbb{R}^2)}\cdots||\partial^{\alpha_j}u||_{W^{2,2}(\mathbb{R}^2)}.\end{aligned}$$

Since $|\alpha_k|+1\leq|\alpha|$ for $1\leq k\leq j$, we use the induction hypothesis to get

$$||\partial^{\alpha_1}\partial_i u\partial^{\alpha_2}\partial_i u\cdots\partial^{\alpha_j}\partial_i u||_2 \leq C_3^j C^j\frac{(|\alpha_1|-1)!(|\alpha_2|-1)!\cdots(|\alpha_j|-1)!}{(|\alpha_1|+1)(|\alpha_2|+1)\cdots(|\alpha_j|+1)}R^{|\alpha|-j}$$

This implies that,

$$\begin{aligned}
&||\partial^\alpha f(\phi)||_2 \\
&\le C_2 \sum_{j=1}^{q} C_3^j C^j \sum_{A(\alpha,j)} \frac{\alpha!(\,|\alpha_1|-1)!(|\alpha_2|-1)!\cdots(|\alpha_j|-1)!R^{|\alpha|-j}}{\alpha_1!\alpha_2!\cdots\alpha_j!(|\alpha_1|+1)(|\alpha_2|+1)\cdots(|\alpha_j|+1)} \\
&\le C_2 \sum_{j=1}^{q} \sum_{B(\alpha,j)} \sum_{A(\alpha,j),|\alpha_i|=n_i} \frac{(C_3CC_4)^j \alpha!\,|\alpha_1|!|\alpha_2|!\cdots|\alpha_j|!R^{|\alpha|-j}}{\alpha_1!\alpha_2!\cdot\cdot\alpha_j!(|\alpha_1|+1)^2(|\alpha_2|+1)^2\cdot\cdot(|\alpha_j|+1)^2} \\
&\le C_2 R^{|\alpha|} \sum_{j=1}^{q} \sum_{B(\alpha,j)} \frac{(C_3CC_4)^j|\alpha|!}{(|n_1|+1)^2(|n_2|+1)^2\cdots(|n_j|+1)^2} R^{-j} \\
&\le C_2 R^{|\alpha|} \frac{|\alpha|!}{(|\alpha|+2)^2} \sum_{j=1}^{q} C_3^j C^j C_1^{j-1} C_4^j R^{-j},
\end{aligned}$$

where $B(\alpha,j) = \{(n_1,\cdot,\cdot,\cdot,n_j) : n_1 + \cdot,\cdot,\cdot + n_j = |\alpha|, n_i \ge 1\}$ and C_k does not depend on α for $1 \le k \le 4$. Thus for R large enough,

$$C_2 \sum_{j=1}^{q} C_3^j C^{j-1} C_1^{j-1} C_4^j R^{-j} < 1,$$

which proves that

$$||\partial^\alpha f(\phi)||_2 \le CR^{|\alpha|} \frac{|\alpha|!}{(|\alpha|+2)^2}. \tag{25}$$

On the other hand, observing that $|\alpha| - |\beta| < |\alpha|$ and using previous inequality,

$$\begin{aligned}
&\sum_{|\beta|=k\ge1}^{|\alpha|-1} \binom{|\alpha|}{k} ||\partial^{\alpha-\beta}(u_x)||_2 ||\partial^\beta(u_y^p)||_2 \\
&\le \sum_{|\beta|=k\ge1}^{|\alpha|-1} \binom{|\alpha|}{k} ||\partial^{\alpha-\beta}u||_{W^{1,2}(\mathbb{R}^2)} ||\partial^\beta(u_y^p)||_2 \\
&\le C^2 \sum_{k\ge1}^{|\alpha|-1} \binom{|\alpha|}{k} \frac{(|\alpha|-k-1)!\,k!}{(|\alpha|-k+1)(k+2)^2} R^{|\alpha|-1} \\
&\le C^2 R^{|\alpha|-1} \sum_{k\ge1}^{|\alpha|-1} \frac{|\alpha|!}{(|\alpha|-k)!\,k!} \frac{(|\alpha|-k-1)!\,k!}{(|\alpha|-k+1)(k+2)^2} \\
&\le C^2 R^{|\alpha|-1} |\alpha|! \sum_{k\ge1}^{|\alpha|-1} \frac{1}{(|\alpha|-k)^2(k+2)^2} \\
&\le C^2 C_5 R^{|\alpha|-1} \frac{|\alpha|!}{|\alpha|-1} \le CR^{|\alpha|} \frac{|\alpha|!}{|\alpha|+2} CC_5C_6R^{-1}\left(\frac{|\alpha|+2}{|\alpha|-1}\right).
\end{aligned}$$

As above, for R large enough,

$$CC_5C_6R^{-1}\left(\frac{|\alpha|+2}{|\alpha|-1}\right)\le 4CC_5C_6R^{-1}<1.$$

Thus, we have proved that for R large enough,

$$\sum_{|\beta|=k\ge 1}^{|\alpha|-1}\binom{|\alpha|}{k}||\partial^{\alpha-\beta}(u_x)||_2||\partial^{\beta}(u_y^p)||_2\le CR^{|\alpha|}\frac{|\alpha|!}{|\alpha|+2}.$$

Finally,

$$||u_x||_2||\partial^{\alpha}(u_y^p)||_2\le C_1CR^{|\alpha|}\frac{|\alpha|!}{|\alpha|+2}.$$

In other words,

$$||\partial^{\alpha}\nabla u||_{W^{1,2}(\mathbb{R}^2)}\le C\frac{|\alpha|!}{|\alpha|+2}R^{|\alpha|}.$$

Now we have to estimate terms of the form:

$$||\partial^{\alpha}\partial^3_{ijk}u||_2 \quad \text{for} \quad 0\le i,j,k\le 3,\ i+j+k=3.$$

To do this, we apply operator $\partial^{\alpha}\partial_x$ $(\partial^{\alpha}\partial_y)$ to equation (11) and compute the L^2- inner product with $\partial^{\alpha}\partial_x u$ $(\partial^{\alpha}\partial_y u)$. Then we get, after doing similar calculations as above, that

$$\begin{aligned}||\partial^{\alpha}\partial_x\nabla u||_{W^{1,2}(\mathbb{R}^2)}+||\partial^{\alpha}\partial_y\nabla u||_{W^{1,2}(\mathbb{R}^2)}\le\\ C_3\Big(||u_y^p||_2+||\partial^{\alpha}(u_x^{p+1})||_2+||\partial^{\alpha}(u_y^{p+1})||_2+||u_x||_2||\partial^{\alpha}(u_y^p)||_2\\ +\sum_{|\beta|=k\ge 1}^{|\alpha|-1}\binom{|\alpha|}{k}||\partial^{\alpha-\beta}(u_x)||_2||\partial^{\beta}(u_y^p)||_2\Big).\end{aligned}$$

Putting previous estimates together, we conclude that for R large enough

$$||\partial^{\alpha}\nabla u||_{W^{2,2}(\mathbb{R}^2)}\le C\frac{|\alpha|!}{|\alpha|+2}R^{|\alpha|}.$$

Claim 2. Given $(x_0,y_0)\in\mathbb{R}^2$, there exists $r>0$ such that for all $(x,y)\in B((x_0,y_0),r)$

$$u(x,y)=\sum_{\alpha}\frac{\partial^{\alpha}u(x_0,y_0)}{\alpha!}(x-x_0,y-y_0)^{\alpha}.$$

Proof of Claim 2. By Taylor's Theorem

$$u(x,y)=\sum_{k=0}^{N-1}\sum_{|\alpha|=k,\alpha\in\mathbb{N}^2}\frac{\partial^{\alpha}u(x_0,y_0)}{\alpha!}(x-x_0,y-y_0)^{\alpha}+R_N(x,y),$$

where

$$R_N(x,y) = \sum_{|\alpha|=N, \alpha\in\mathbb{N}^2} \frac{\partial^\alpha u(x_0 + t(x-x_0), y_0 + t(y-y_0))}{\alpha!}(x-x_0, y-y_0)^\alpha.$$

Since $2^k = (1+1)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}$, we conclude that $\frac{|\alpha|!}{\alpha!} \le 2^{|\alpha|}$. On the other hand, for $|\alpha| \ge 1$

$$|\partial^\alpha u(x,y)| \le ||\partial^\alpha u||_{W^{2,2}(\mathbb{R}^2)} \le CR^{|\alpha|-1}\frac{(|\alpha|-1)!}{|\alpha|+1}.$$

In consequence if we take $r > 0$ such that $2^3 r^2 R < 1$, then

$$\begin{aligned} |R_N(x,y)| &\le C \sum_{|\alpha|=N, \alpha\in\mathbb{N}^2} \frac{(N-1)!R^{|\alpha|-1}}{(N+1)\alpha!}(|x-x_0||y-y_0|)^{|\alpha|} \\ &\le C\frac{2^N(N-1)!R^N}{(N+1)\alpha!} r^{2N} \le C2^N 2^N (Rr^2)^N \\ &\le C(2^2Rr^2)^N \le C2^{-N} \to 0, \text{ as } N \to \infty\,. \end{aligned}$$

In other words, the Taylor's series converges in $B((x_0,y_0),r)$. ☑

4. From Benney-Luke lumps to a KP-I lump

As it was shown by Pego and Quintero in [5], solitary wave with some sort of physical sense can be obatined when the wave speed c is not taken fixed. To see this, let $\mu = \varepsilon^{\frac{2}{p+1}}$ suppose for ε small the existence of a traveling-wave solution $\Phi(x,y,t) = \varphi(x-ct, y)$ with order one derivatives, as given by Theorem 2.1, so that $0 < c^2 < \min\{1, a/b\}$. Then φ should satisfy

$$(1-c^2)\varphi_{xx} + \varphi_{yy} = O(\varepsilon^{\frac{2}{p+1}}).$$

If $1 - c^2$ remains bounded away from zero, we expect that no solutions with order-one derivatives will exist. This suggests that if $1 - c^2 = O(\varepsilon^{2/p+1})$, then $\varphi_{yy} = O(\varepsilon^{2/p+1})$, so the waves should travel with speed close to 1 and have weak dependence on y. This further suggests introducing scaled variables similar to those used to derive the GKP equation in section 2. Namely, we let $Y = \varepsilon^{\frac{1}{p+1}} y$, $X = x - ct$ where $1 - c^2 = \varepsilon^{2/p+1}$ and look for a solution of GBL of the form $\Phi(x,y,t) = \gamma v(X,Y)$, where $\gamma^p \varepsilon^{\frac{p-1}{p+1}} = 1$. Then v should satisfy

$$\begin{aligned} -v_{XX} - v_{YY} + (a-bc^2)v_{XXXX} + \varepsilon^{2/p+1}(2a-bc^2)v_{XXYY} + \varepsilon^{4/p+1} a v_{YYYY} \\ - c\Big((p+2)v_{XX}v_X^p + \varepsilon\left(pv_Y^{p-1}v_{YY}v_X + 2v_{XY}v_Y^p\right)\Big) = 0. \quad (26) \end{aligned}$$

Formally this means

$$-v_{XX}-v_{YY}+\left(\sigma-\frac{1}{3}\right)v_{XXXX}-(p+2)v_X^p v_{XX}=O(\varepsilon^{2/p+1}). \quad (27)$$

If we differentiate with respect to x and neglected the $O(\varepsilon^{2/p+1})$, then $w=v_X$ satisfies

$$\left(w_X-\left(\sigma-\frac{1}{3}\right)w_{XXX}+(p+2)w^p w_X\right)_X+w_{YY}=0. \quad (28)$$

This is the equation for a traveling wave solution of the generalized GKP equation.

This remark shows us that appropriate order one solutions can be constructed for arbitrarily small values of the parameters, if the wave speed c is chosen in an appropriate way, mainly close to 1. We will prove that it is possible to obtain a GKP-I lump solution as a limit in $\mathcal{V}$ of a sequence of GBL lum solutions when the wave speed c is not taken fixed but close to 1.

The scaling leading to (26) is related to that of the previous section as follows. Let $I_{\varepsilon,p}$ and $G_{\varepsilon,p}$ be the functionals defined in $\mathcal{V}$ by

$$I_{\varepsilon,p}(u)=\int_{\mathbb{R}^2}\{\varepsilon^{2/p+1}u_x^2+u_y^2+(a-b+b\varepsilon^{2/p+1})u_{xx}^2+(2a-b+b\varepsilon^{2/p+1})u_{xy}^2+au_{yy}^2\}dV,$$

$$G_{\varepsilon,p}(u)=c\int_{\mathbb{R}}\{u_x^{p+2}+u_xu_y^{p+1}\}dV.$$

Since $a-b=\sigma-\frac{1}{3}>0$, $I_{\varepsilon,p}$ and $G_{\varepsilon,p}$ are just the functionals I and G from section 2 with wave speed satisfying $c^2=1-\varepsilon^{2/p+1}$. Given an arbitrary function u in $\mathcal{V}$, let v be the function defined by

$$u(x,y)=\varepsilon^{\frac{1-p}{(p+1)(p+2)}}v(\varepsilon^{1/p+1}x,\varepsilon^{2/p+1}y).$$

Then

$$I_{\varepsilon,p}(u)=\varepsilon^{\frac{4-p}{(p+1)(p+2)}}J_{\varepsilon,p}(v)\quad\text{and}\quad G_{\varepsilon,p}(u)=K_{\varepsilon,p}(v).$$

where

$$J_{\varepsilon,p}(v)=\int_{\mathbb{R}^2}\{v_x^2+v_y^2+(a-b+b\varepsilon^{2/p+1})v_{xx}^2+\varepsilon^{2/p+1}(2a-b+b\varepsilon^{2/p+1})v_{xy}^2+\varepsilon^{4/p+1}av_{yy}^2\}dV,$$

$$K_{\varepsilon,p}(v)=c\int_{\mathbb{R}^2}\{v_x^3+\varepsilon v_xv_y^{p+1}\}dV.$$

In particular, we have that

$$\varepsilon^{\frac{p-4}{(p+1)(p+2)}}\mathcal{I}_\varepsilon = \mathcal{J}_\varepsilon$$

where

$$\mathcal{I}_{\varepsilon,p} = \inf\{I_{\varepsilon,p}(u) : u \in \mathcal{V}, \quad G_{\varepsilon,p}(u) = 1\},$$
$$\mathcal{J}_{\varepsilon,p} = \inf\{J_{\varepsilon,p}(v) : v \in \mathcal{V}, \quad K_{\varepsilon,p}(v) = 1\}.$$

Now by Theorem 2.1, we know the existence of a family $\{u^\varepsilon\}_{\varepsilon>0} \subseteq \mathcal{V}$ such that

$$I_{\varepsilon,p}(u^\varepsilon) = \mathcal{I}_{\varepsilon,p} \quad \text{and} \quad G_{\varepsilon,p}(u^\varepsilon) = 1.$$

This implies that the members of the corresponding family $\{v^\varepsilon\}$ defined by

$$u^\varepsilon(x,y) = \varepsilon^{\frac{1-p}{(p+1)(p+2)}} v^\varepsilon(\varepsilon^{1/p+1}x, \varepsilon^{2/p+1}y).$$

satisfy

$$J_{\varepsilon,p}(v^\varepsilon) = \mathcal{J}_{\varepsilon,p} \quad \text{and} \quad K_{\varepsilon,p}(v^\varepsilon) = 1.$$

In particular, $v = -\left(\frac{p+1}{p+2}\mathcal{J}_{\varepsilon,p}\right)^{1/p} v^\varepsilon$ is a solution of equation (26) in the sense of distributions.

Now we are in position to state the main theorem of this section.

Theorem 4.1. *Assume $\sigma > \frac{1}{3}$ and $1 \le p < 2$. For any sequence $\varepsilon_j \to 0$, there is a subsequence (denoted the same) and there exists a nontrivial distribution $v_0 \in \mathcal{D}'(\mathbb{R}^2)$ with $\partial_x v_0$, $\partial_{xx} v_0$ and $\partial_y v_0$ belonging to $L^2(\mathbb{R}^2)$ such that, as $j \to \infty$*

$$\partial_x v^{\varepsilon_j} \to \partial_x v_0, \quad \partial_{xx} v^{\varepsilon_j} \to \partial_{xx} v_0, \quad \partial_y v^{\varepsilon_j} \to \partial_y v_0 \quad \textit{in} \quad L^2(\mathbb{R}^2).$$

Moreover, $w = -\left(\frac{p+1}{p+2}\mathcal{J}_{0,p}\right)^{\frac{1}{p}} \partial_x v_0$ is a nontrivial lump solution of the GKP traveling wave equation (28) in the sense of distributions, where

$$\mathcal{J}_{0,p} := \{J_{0,p} : v \in \mathcal{V}, K_{0,p}(v) = 1\}.$$

In order to prove this result, we are going to discuss first an important property of the family $\{v^\varepsilon\}$.

Lemma 4.1. *Let $1 \le p < 2$. Then*

$$\lim_{\varepsilon \to 0^+} \mathcal{J}_{\varepsilon,p} = \mathcal{J}_{0,p} \quad \textit{and} \quad \lim_{\varepsilon \to 0^+} K_{0,p}(v^\varepsilon) = 1.$$

In particular, for any sequence $\varepsilon_j \to 0$, the sequence $\{K_{0,p}(v^{\varepsilon_j})^{-\frac{1}{p+2}} v^{\varepsilon_j}\}$ is a minimizing sequence for $J_{0,p}$.

Proof. Let $v \in \mathcal{V}$ be such that $\int_{\mathbb{R}} v_x^{p+2}\, dV = 1$. Then for ε small enough, we get

$$K_{\varepsilon,p}(v) = c\int_{\mathbb{R}^2}\{v_x^{p+2} + \varepsilon v_x v_y^{p+1}\}dV \neq 0.$$

Thus

$$J_{\varepsilon,p}\left(K_{\varepsilon,p}(v)^{-\frac{1}{p+2}}v\right) = K_{\varepsilon,p}(v)^{-\frac{2}{p+2}}J_{\varepsilon,p}(v) \geq \mathcal{J}_{\varepsilon,p}.$$

But, we note that $J_{\varepsilon,p}(v) \to J_{0,p}(v)$ and $K_{\varepsilon,p}(v) \to 1$ as $\varepsilon \to 0$. Thus, we conclude that $J_{0,p}(v) \geq \limsup_{\varepsilon\to 0^+}\mathcal{J}_{\varepsilon,p}$ for all $v \in \mathcal{V}$ with $\int_{\mathbb{R}^2}(v_x)^{p+2}\, dV = 1$. In consequence,

$$\mathcal{J}_{0,p} \geq \limsup_{\varepsilon\to 0^+}\mathcal{J}_{\varepsilon,p}.$$

In particular, for ε small enough,

$$\varepsilon^{\frac{p-4}{(p+1)(p+2)}}\mathcal{I}_{\varepsilon,p} = \mathcal{J}_{\varepsilon,p} \leq 2\cdot\mathcal{J}_{0,p}.$$

This fact implies that $\varepsilon^{\frac{3p}{2(p+1)(p+2)}}u_x^\varepsilon$ is uniformly bounded in $W^{1,2}(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$, $q \geq 2$ and $\varepsilon^{\frac{p-4}{2(p+1)(p+2)}}u_y^\varepsilon$ is uniformly bounded in $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{2(p+1)}(\mathbb{R}^2)$. Hence, we have the estimate

$$\|u_y^\varepsilon\|_{L^{2(p+1)}(\mathbb{R}^2)} \leq C_1\|u_y^\varepsilon\|_{W^{1,2}(\mathbb{R}^2)} \leq C_2\varepsilon^{\frac{4-p}{2(p+1)(p+2)}},$$

where C_2 is a constant independent of ε. Now, by Hölder's inequality,

$$\begin{aligned}\left|\int_{\mathbb{R}^2}u_x^\varepsilon(u_y^\varepsilon)^{p+1}dV\right| &\leq \left(\int_{\mathbb{R}^2}(u_x^\varepsilon)^2dV\right)^{\frac{1}{2}}\cdot\left(\int_{\mathbb{R}^2}(u_y^\varepsilon)^{2(p+1)}dV\right)^{\frac{1}{2}}\\ &\leq C_2^{p+1}\varepsilon^{\frac{4-p}{2(p+2)}}\|u_x^\varepsilon\|_{L^2(\mathbb{R}^2)}\\ &\leq C_2^{p+1}\varepsilon^{\frac{2-p}{2(p+2)}}\|\varepsilon^{\frac{3p}{2(p+1)(p+2)}}u_x^\varepsilon\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Thus, as $\varepsilon \to 0^+$ we have

$$\int_{\mathbb{R}^2}u_x^\varepsilon(u_y^\varepsilon)^{p+1}dV = \varepsilon\int_{\mathbb{R}^2}v_x^\varepsilon(v_y^\varepsilon)^{p+1}dV \to 0.$$

This implies that $K_{0,p}(v^\varepsilon) = \int_{\mathbb{R}^2}(v_x^\varepsilon)^{p+2}dV \to 1$.

Hence for ε small enough, $K_{0,p}(v^\varepsilon) \neq 0$ and we have

$$J_{0,p}\left(\frac{v^\varepsilon}{K_{0,p}(v^\varepsilon)^{\frac{1}{p+2}}}\right) = \frac{J_{0,p}(v^\varepsilon)}{K_{0,p}(v^\varepsilon)^{\frac{2}{p+2}}} \geq \mathcal{J}_{0,p}.$$

But note that $J_{\varepsilon,p}(v) \geq J_{0,p}(v)$ for all $v \in \mathcal{V}$. It follows that

$$\frac{J_{\varepsilon,p}(v^\varepsilon)}{K_{0,p}(v^\varepsilon)^{\frac{2}{p+2}}} \geq J_{0,p}\left(\frac{v^\varepsilon}{K_{0,p}(v^\varepsilon)^{\frac{1}{p+2}}}\right) \geq \mathcal{J}_{0,p}.$$

Since $\lim_{\varepsilon\to 0^+} K_{0,p}(v^\varepsilon) = 1$, we conclude that

$$\liminf_{\varepsilon\to 0^+} \mathcal{J}_\varepsilon \geq \mathcal{J}_{0,p} \geq \limsup_{\varepsilon\to 0^+} \mathcal{J}_{\varepsilon,p}.$$

This implies that $\lim_{\varepsilon\to 0^+} \mathcal{J}_{\varepsilon,p} = \mathcal{J}_{0,p}$, finishing the proof. ☑

In addittion to the last result, we have to use a de Bouard and Saut result in [2, 3] which is related with solitary waves for a generalized KP equation GKP or simply with the case $\varepsilon = 0$.

Theorem 4.2. [2, 3] *Let $1 \leq p < 4$ be a rational number with odd denimominator and $\sigma > \frac{1}{3}$. If $\{v_m\}_{m\geq 1}$ is a minimizing sequence for $\mathcal{J}_{0,p}$, then there exists a subsequence (denoted the same) and there exists a nonzero $v_0 \in \mathcal{D}'(\mathbb{R}^2)$ such that $\partial_x v_0$, $\partial_y v_0$ $\partial_{xx} v_0 \in L^2(\mathbb{R}^2)$ and*

$$J_{0,p}(v_0) = \mathcal{J}_{0,p} > 0,$$

and there exists a sequence of points $\{\zeta_m\}_{m\geq 1}$ in $\mathbb{R}^2$ such that

$$\begin{aligned}\partial_x v_m(\cdot + \zeta_m) &\to \partial_x v_0 \quad in \quad L^2(\mathbb{R}),\\ \partial_y v_m(\cdot + \zeta_m) &\to \partial_y v_0 \quad in \quad L^2(\mathbb{R}),\\ \partial_{xx} v_m(\cdot + \zeta_m) &\to \partial_{xx} v_0 \quad in \quad L^2(\mathbb{R}).\end{aligned}$$

Moreover, v_0 is a solution in the sense of distributions of the equation

$$-v_{XX} - v_{YY} + \left(\sigma - \frac{1}{3}\right) v_{XXXX} + (p+1)\mathcal{J}_{0,p}(v_X)^p v_{XX} = 0. \tag{29}$$

If $p = 1, 2$ or 3 and $i = 1, 2$, $\partial_i v_0 \in \bigcap_{n\in\mathbb{N}} H^n(\mathbb{R}^2)$, where $H^n(\mathbb{R}^2)$ denotes the Sobolev space of distributions whose derivatives up to order n are in $L^2(\mathbb{R}^2)$.

Proof of Theorem 4.1. First note that v^ε satisfies in the sense of distributions the following equation,

$$\begin{aligned}&-v_{xx} - v_{yy} + (a - b - b\varepsilon^{2/p+1})v_{xxxx} + \varepsilon(2a - b - b\varepsilon^{2/p+1})v_{xxyy} + \varepsilon^{4/p+1} a v_{yyyy}\\ &\quad + \left(\frac{p+1}{p+2}\mathcal{J}_{\varepsilon,p}\right)^{1/p} \cdot c\left((p+2)v_x^p v_{xx} + \varepsilon v_x v_y^{p-1} v_{yy} + 2\varepsilon v_{xy} v_y^p\right) = 0.\end{aligned}$$

Since $0 < \mathcal{J}_{\varepsilon,p} \leq 2\mathcal{J}_{0,p}$ for ε small enough, the family $\{\varepsilon^{\frac{2}{p+1}} v^\varepsilon\}$ is bounded in $\mathcal{V}$. We also have that $\{v_x^\varepsilon\}$, $\{v_y^\varepsilon\}$, $\{v_{xx}^\varepsilon\}$ are bounded families in $L^2(\mathbb{R}^2)$ for ε small enough. Then for any sequence $\varepsilon_j \to 0$ as $j \to \infty$, there exists a subsequence (denoted the same) and functions $W, Z \in L^2(\mathbb{R}^2)$ with $W_x \in L^2(\mathbb{R}^2)$ such that

$$v_x^{\varepsilon_j} \rightharpoonup W, \quad v_{xx}^{\varepsilon_j} \rightharpoonup W_x, \quad v_y^{\varepsilon_j} \rightharpoonup Z \quad \text{in} \quad L^2(\mathbb{R}^2).$$

But from Lemma 4.1 we have that

$$\lim_{j\to\infty} J_{0,p}(v^{\varepsilon_j}) = \mathcal{J}_{0,p}.$$

Then from Theorem 4.2, there exists a nontrivial distribution $v_0 \in \mathcal{D}'(\mathbb{R}^2)$ such that $\partial_x v_0 = W$, $\partial_y v_0 = Z$ and

$$v_x^{\varepsilon_j} \to \partial_x v_0, \quad v_{xx}^{\varepsilon_j} \to \partial_{xx} v_0, \quad v_y^{\varepsilon_j} \to \partial_y v_0, \quad \text{in } L^2(\mathbb{R}^2)$$

and

$$\int_{\mathbb{R}^2} (\partial_x v_0)^{p+2} dV = 1.$$

In particular, we have that $\partial_x v_0 \neq 0$. On the other hand, $(\partial_x v^{\varepsilon_j})^{p+1}$ is bounded in $L^{\frac{p+2}{p+1}}(\mathbb{R}^2)$. In consequence, as $j \to \infty$

$$\int_{\mathbb{R}^2} (\partial_x v^{\varepsilon_j})^{p+1} \psi \to \int_{\mathbb{R}^2} (\partial_x v_0)^{p+1} \psi,$$

for all $\psi \in L^2(\mathbb{R}^2)$. This implies that

$$(\partial_x v^{\varepsilon_j})^p \partial_{xx} v^{\varepsilon_j} \to (\partial_x v_0)^p \partial_{xx} v_0.$$

Now for any test function $\psi \in C_0^\infty(\mathbb{R}^2)$, if we denote evaluation in $\mathcal{D}'(\mathbb{R}^2)$ by $(\cdot,\cdot)$, as $j \to \infty$ we have

$$\begin{aligned}
&|\varepsilon_j^{2/p+1} \left(p v_x^{\varepsilon_j} (v_y^{\varepsilon_j})^{p-1} v_{yy}^{\varepsilon_j} + 2 v_{xy}^{\varepsilon_j} (v_y^{\varepsilon_j})^p, \psi\right)| \\
&\quad = \left| \varepsilon_j^{2/p+1} \left(p (v_x^{\varepsilon_j} (v_y^{\varepsilon_j})^p, \psi_y) + \frac{1}{p+1} ((v_y^{\varepsilon_j})^{p+1}, \psi_x) \right) \right| \\
&\quad \le \varepsilon_j^{2/p+1} \left(p \| v_x^{\varepsilon_j} (v_y^{\varepsilon_j})^p \|_{L^1} + \| v_y^{\varepsilon_j} \|_{L^{p+1}}^{p+1} \right) \cdot \| \nabla \psi \|_{L^\infty} \\
&\quad \le \varepsilon_j (\| v_x^{\varepsilon_j} \|_{L^{p+1}} \| v_y^{\varepsilon_j} \|_{L^{p+1}}^p + \| v_y^{\varepsilon_j} \|_{L^{p+1}}^{p+1}) \cdot \| \nabla \psi \|_{L^\infty} \to 0.
\end{aligned}$$

We also have that

$$\left(\varepsilon_j^{2/p+1} b v_{xxxx}^{\varepsilon_j} + \varepsilon_j^{2/p+1} (2a - b + b \varepsilon_j^{2/p+1}) v_{xxyy}^{\varepsilon_j} + a \varepsilon_j^{4/p+1} v_{yyyy}^{\varepsilon_j}, \psi \right) =$$
$$\varepsilon_j^{2/p+1} b (v_{xx}^{\varepsilon_j}, \psi_{xx}) + \varepsilon_j^{2/p+1} (2a - b - b\varepsilon^{2/p+1}) (v_{xx}^{\varepsilon_j}, \psi_{yy}) - a \varepsilon_j^{4/p+1} (v_y^{\varepsilon_j}, \psi_{yyy}) \to 0.$$

Furthermore we have

$$\lim_{j\to\infty} \left(-v_{xx}^{\varepsilon_j} - v_{yy}^{\varepsilon_j} + \left(\sigma - \frac{1}{3}\right) v_{xxxx}^{\varepsilon_j}, \psi \right)$$
$$= \left(-(v_0)_x - (v_0)_y + \left(\sigma - \frac{1}{3}\right) (v_0)_{xxxx}, \psi \right).$$

Since $\sigma > \frac{1}{3}$, the nonzero distribution v_0 is a nontrivial solution of the equation

$$-v_{XX} - v_{YY} + \left(\sigma - \frac{1}{3}\right) v_{XXXX} + \left(\frac{p+1}{p+2} \mathcal{J}_{0,p} \right)^{1/p} (3 v_X v_{XX}) = 0.$$

In particular, $w = -\left(\frac{p+1}{p+2}\mathcal{J}_{0,p}\right)^{1/p} \partial_x v_0$ is a nontrivial lump solution for the GKP traveling wave equation (28) in the sense of distributions. ☑

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