Revista Colombiana de Matemáticas
Volumen 36 (2002), páginas 97–106

Stable minimal cones in $\mathbb{R}^8$ and $\mathbb{R}^9$ with constant scalar curvature

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ABSTRACT. In this paper we prove that if $M \subset \mathbb{R}^n$, $n = 8$ or $n = 9$, is a $n-1$ dimensional stable minimal complete cone such that its scalar curvature varies radially, then M must be either a hyperplane or a Clifford minimal cone. By Gauss' formula, the condition on the scalar curvature is equivalent to the condition that the function $\kappa_1(m)^2 + \cdots + \kappa_{n-1}(m)^2$ varies radially. Here the κ_i are the principal curvatures at $m \in M$. Under the same hypothesis, for $M \subset \mathbb{R}^{10}$ we prove that if not only $\kappa_1(m)^2 + \cdots + \kappa_{n-1}(m)^2$ varies radially but either $\kappa_1(m)^3 + \cdots + \kappa_{n-1}(m)^3$ varies radially or $\kappa_1(m)^4 + \cdots + \kappa_{n-1}(m)^4$ varies radially, then M must be either a hyperplane or a Clifford minimal cone.

Keywords and phrases. Clifford hypersurfaces, minimal hypersurfaces, shape operator.

2000 Mathematics Subject Classification. Primary: 53C42. Secondary: 58J50.

1. Introduction

Let M be an n-dimensional Riemannian manifold. A natural problem in geometry is that of finding k-dimensional submanifolds $N \subset M$ with the property that for any bounded open set U in M, the k-volume of $N \cap U$ is less than or equal to the volume of any other submanifold in M with boundary equal to $\partial(N \cap U)$. The submanifolds of M with the above property are called *area-minimizing*. Notice that when $k = 1$, area-minimizing submanifolds are

*The author is grateful to *Colciencias* for its financial support.

geodesics. Locally, the problem reduces to the one of finding minimal submanifolds, manifolds for which the mean curvature vector vanishes; globally, the problem of finding *complete* area-minimizing submanifolds is a difficult one, even in the case when M is the Euclidian space $\mathbb{R}^n$. It is clear that planes in $\mathbb{R}^3$ are area-minimizing. In general, hyperplanes in $\mathbb{R}^n$ are area-minimizing hypersurfaces.

A family of hypersurface of $\mathbb{R}^n$ which is important in the study of area-minimizing hypersurfaces are the cones: $N \subset \mathbb{R}^n$ is a cone if for every $a > 0$, $ap \in N$ any time $p \in N$. The study of cones is important for two reasons, the first one is that if $p \in N$ is a singular point of a area-minimizing hypersurface S, then there is an area-minimizing tangent cone, which makes the role of tangent space at p, with the property that $p \in S$ is a removable singularity if and only if this tangent cone is a hyperplane. The second reason area-minimizing cones are important is that if S is a complete area-minimizing hypersurface, then there is an area-minimizing cone that makes the role of tangent cone at infinity; this cone is a hyperplane if and only if S is a hyperplane.

A giant step toward this problem of classifying area-minimizing hypersurfaces in Euclidean spaces was made by James Simons in 1968 [S]. He showed that the only area-minimizing complete hypersurfaces in $\mathbb{R}^n$, with $n \leq 7$, are the hyperplanes. On the other hand, Bombieri-De Giorgi-Guisti showed that the hypercone

$$C_{4,4} = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\} \subset \mathbb{R}^8$$

is area-minimizing. They also found a family of complete, smooth, area-minimizing hypersurfaces in $\mathbb{R}^8$ (notice that, in general, cones are not smooth at the origin). These area-minimizing hypersurfaces found in [B-DG-G] converges at infinity to $C_{4,4}$. An open and important question in this direction is the one of classifying all area-minimizing hypersurfaces in $\mathbb{R}^8$. A reasonable conjecture is the claim that the only complete area-minimizing hypersurfaces in $\mathbb{R}^8$ are the ones found in [B-DG-G].

In this paper we prove, for $n = 8$ and $n = 9$, that if M is an area-minimizing cone and the scalar curvature of M varies radially, then M is isometric to a Clifford cone, i.e. a cone of the form

$$C_{l,k} = \{(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{l+1} : l|x|^2 = k|y|^2\}$$

where k and l are positive integers with $k + l = n - 1$. These cones are called *Clifford minimal cones.*

2. Preliminaries

Let $M \subset \mathbb{R}^n$ be a smooth hypersurface, i.e. a immersion with codimension 1. For any $p \in M$ we will denote by T_pM the tangent space of M at p. We will think of this space as a $n - 1$ dimensional subspace of $\mathbb{R}^n$. Since the codimension of $T_pM \subset \mathbb{R}^n$ is one then we can find a unit vector $\nu(p) \in \mathbb{R}^n$

such that $\nu(p)$ is perpendicular to T_pM; it is not difficult to see that there are just two possibilities for this vector. M is orientable if and only if we can pick the vector $\nu(p)$ in a continuous way over all M. In this case, the map $\nu : M \to S^{n-1} \subset \mathbb{R}^n$, where $S^k = \{x \in \mathbb{R}^{k+1} : |x| = 1\}$, turns out to be not only continuous but differentiable. Since every manifold is locally orientable, then we can always define the map ν locally. The map ν is called the Gauss map.

Let $\overline{\nabla}$ be the Levi Civita connection in $\mathbb{R}^n$, defined by the directional derivative, i.e. if $X, Y : \mathbb{R}^n \to \mathbb{R}^n$ are vector fields on $\mathbb{R}^n$ then $\overline{\nabla}_X Y(p)$ is the derivative in the direction $X(p)$ of the function Y at p. Recall that we can compute this derivative either by multiplying the Jacobian matrix of Y at p with the vector $X(p)$ or by taking any curve $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ with $\alpha(0) = p$ and $\alpha'(0) = X(p)$ and then computing the derivative at $t = 0$ of the curve $\beta(t) = Y(\alpha(t))$. An important fact about the Levi Civita connection is that in order to compute $\overline{\nabla}_X Y(p)$ it is enough to know $X(p)$ and to know Y along any curve passing through p with velocity $X(p)$ at p.

The *shape operator* of the manifold M at p is the linear map $A : T_pM \to \mathbb{R}^n$ defined by $A(v) = -\overline{\nabla}_v \nu$ for any $v \in T_pM$. Since the norm of ν is always 1, it is not difficult to show that the image of A is a subspace of T_pM, therefore A is actually a map from T_pM to itself. The map A turns out to be a symmetry linear transformation [D], i.e. if $\langle\ .\ ,\ .\ \rangle$ denotes the inner product in $\mathbb{R}^n$ then for any pair of vectors $v, w \in T_pM$ we have that $\langle A(v), w\rangle = \langle v, A(w)\rangle$. By linear algebra, A has $n-1$ real eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$. This eigenvalues are known as the *principal curvatures* of M at p. We define the functions *mean curvature*, $H : M \to \mathbb{R}$, and *the norm of the shape operator* $|A| : M \to \mathbb{R}$ by

$$H(p) = \frac{\kappa_1 + \cdots + \kappa_{n-1}}{n-1}$$

and

$$|A|(p) = \sqrt{\kappa_1^2 + \cdots + \kappa_{n-1}^2}$$

for any $p \in M$.

Let us denote by $C_0^\infty(M)$ the set of smooth functions with compact support. Given any function $f : M \longrightarrow \mathbb{R}^1$ we can form the 1-parameter variational family defined by

$$M_t = \{p + tf(p)\nu(p) : p \in M\}.$$

Notice that $M_0 = M$ and that M_t agrees with M outside a compact set. By using the implicit function theorem we have that there exists $\varepsilon > 0$ such that the sets M_t are hypersurfaces for every $t \in (-\varepsilon, \varepsilon)$. Let $W \subset M$ be an open set with finite n-dimensional area that contains the support of f. Let $V_f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be the real function that assigning the $n-1$ dimensional volume of $M_t \cap W$ to any $t \in (-\varepsilon, \varepsilon)$. It is well known (see e.g. [S]) that the function V satisfies:

$$\frac{1}{2}\frac{dV_f}{dt}\Big|_{t=0} = -\int_M fH. \tag{1}$$

Notice that the mean curvature function H of a manifold M vanishes at every point if and only if V_f has a critical point at $t = 0$ for every $f \in C_0^\infty$. Submanifolds whose mean curvature function vanishes identically are called *minimal submanifolds.* Equation (1) tell us that minimal submanifold are critical points of the area functional. Notice that if M is an area-minimizing hypersurface then for any $f \in C_0^\infty(M)$ the volume of M_t is greater than or equal to the volume of $M = M_0$, i.e., $V_f(t) \geq V_f(0)$. Therefore, if M is area-minimizing we have that:

(i) $V_f'(0) = 0$ for any function f i.e M is minimal.
(ii) $V_f''(0) \geq 0$ for any function f.

Minimal submanifolds satisfying condition (ii) above are called minimal *stable* submanifolds. The formula for $V_f''(0)$ is given by the following equation; its proof can be found in [S].

$$\frac{d^2V_f}{dt^2}(0) = \int_M J(f)f \qquad \text{(second variation formula)} \tag{2}$$

where J is the stability operator on M, given by

$$J = -\Delta - \|A\|^2.$$

The operator Δ is the Laplacian of M, which can be defined as follows: Let $p_0 \in M$ be a point in M and let $f : M \to \mathbb{R}$ be a smooth function. Let $\{e_1, \ldots, e_{n-1}\}$ be vector fields defined in an open neighborhood U in $\mathbb{R}^n$ of p_0 such that $\langle e_i, e_j\rangle = \delta_{ij}$ for every point in U and such that $\{e_1(p), \ldots, e_{n-1}(p)\}$ form a base for T_pM for every point $p \in M \cap U$.

The *gradient of* f at p_0 is given by

$$\nabla f(p_0) = e_1(f)(p_0)e_1(p_0) + \cdots + e_{n-1}(f)(p_0). \tag{3}$$

Here, $e_i(f)(p)$ is the directional derivative of f at p in the direction $e_i(p)$. Notice that ∇f defines a vector field on $M \cap U$.

The *Laplacian* of f is given by

$$\Delta(f)(p_0) = \langle \overline{\nabla}_{e_1}\nabla f, e_1\rangle(p_0) + \cdots + \langle \overline{\nabla}_{e_{n-1}}\nabla f, e_{n-1}\rangle(p_0). \tag{4}$$

From now on we will assume that M is a cone in $\mathbb{R}^n$ such that M is a smooth minimal hypersurface without boundary and $M \cup \{\mathbf{0}\}$ is topologically complete. We will refer to these sets just as *minimal complete cones with codimension 1.* We will state some facts about M. Notice that we can build back M just by knowing the set $M \cap S^{n-1} = N$. Let $p \in M$ be any point in M, since M is a cone, we have that T_pM is equal to $T_{\frac{p}{|p|}}M$; recall that we are viewing these tangent spaces as vector subspaces of $\mathbb{R}^n$.

The following lemma gives a relation between the shape operator at $p \in M$ and the shape operator at ap for any $a > 0$.

Lemma 2.1. *If M is smooth hypersurface which is a cone, then for any $p \in M$ we have that*

$$|A|^2(ap) = \frac{1}{a^2}|A|^2(\frac{p}{|p|}) \quad \text{for any } a > 0.$$

Proof. Let ν be the Gauss map defined in a neighborhood of $\frac{p}{|p|}$. If $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal bases of $T_{\frac{p}{|p|}}M$ then $A(e_i)(\frac{p}{|p|}) = \beta'(0)$, where $\beta(t) = \nu(\alpha(t))$ and $\alpha(t)$ is a curve on M with $\alpha(0) = \frac{p}{|p|}$ and $\alpha'(0) = e_i$. Since M is a cone then the curve $\bar{\alpha}(t) = a\alpha(t)$ is a curve on M. Notice that $\bar{\alpha}(0) = p$ and $\bar{\alpha}'(0) = ae_i$. This fact gives us what we have pointed out before: $T_pM = T_{\frac{p}{|p|}}M$, hence $\nu(p) = \nu(\frac{p}{|p|})$ and,

$$A(p)(ae_i) = \frac{d\nu(\alpha(\bar{t}))}{dt}\Big|_{t=0} = \frac{d\nu(\alpha(t))}{dt}\Big|_{t=0} = A\Big(\frac{p}{|p|}\Big)(e_i).$$

Using the above equation we get

$$|A|^2(p) = \sum_{i=1}^{n-1}\big|A(e_i)(p)\big|^2 = \sum_{i=1}^{n-1}\frac{1}{a^2}\Big|A(e_i)\Big(\frac{p}{|p|}\Big)\Big|^2 = \frac{1}{a^2}|A|^2\Big(\frac{p}{|p|}\Big).$$

This completes the proof of the lemma. ☑

Given a complete minimal cone M, let us define $N = S^{n-1} \cap M$. Under the conditions we have imposed on M we can deduce that N is a complete smooth manifold on S^{n-1}. The following lemma gives us a formula for the integral of a function over M in terms of integrals over N.

Lemma 2.2. *Let $N \subset S^{n-1}$ be a smooth manifold of dimension $n-2$. For any $0 < \varepsilon < 1$, let us define $M_\varepsilon = \{tp : p \in N \text{ and } t \in [\varepsilon, 1]\}$. If $f : M_\varepsilon \to \mathbb{R}$ is a smooth function then*

$$\int_{M_\varepsilon} f = \int_\varepsilon^1 \int_N t^{n-2} f_t(p)$$

where $f_t : N \to \mathbb{R}$ is defined by $f_t(p) = f(tp)$.

Proof. Without loss of generality (otherwise consider a partition of the unit of N) we may assume that $N = \phi(U)$ where U is an open set of $\mathbb{R}^{n-2}$ and $\phi : U \to \mathbb{R}^n$ is a parametrization of N that induces coordinates $y_1, \ldots, y_{n-2}$ on N. Let us define $b_{ij} = \langle \frac{\partial\phi}{\partial y_i}, \frac{\partial\phi}{\partial y_j}\rangle$ for $i, j \in \{1, \ldots, n-2\}$. Since ϕ is a parametrization, we have that the matrix $B = \{b_{ij}\}$ is a symmetric positive defined matrix. Moreover,

$$\int_N g = \int_U g(\phi(y))\sqrt{\det(B)}\, dy, \quad \text{for any } g : N \to \mathbb{R}.$$

Now, if we define $\rho : (\varepsilon, 1) \times U \to M_\varepsilon$ by $\rho(t, y) = t\phi(y)$, then it is clear that ρ defines a parametrization on M_ε. We define $c_{ij} = \langle \frac{\partial\rho}{\partial y_i}, \frac{\partial\rho}{\partial y_j}\rangle$ where $i, j \in \{0, 1, \ldots, n-2\}$; here we are identifying the y_0 coordinate with the t

coordinate. Since $N\subset S^{n-1}$, $c_{00}(t\phi(y))=\langle\phi(y),\phi(y)\rangle=1$ and for any $j\neq 0$, $c_{0j}(t\phi(y))=c_{j0}(t\phi(y))=0$; moreover, if $i,j\in\{1,\dots,n-2\}$, $c_{ij}(t\phi(y))=t^2b_{ij}$.

Therefore, if $C=\{c_{ij}\}$ we get

$$\det(C)(t\phi(y))=t^{2(n-2)}\det(B)(y).$$

Using the above equation, we have that

$$\int_{M_\varepsilon} f=\int_0^1\int_U f(\rho(t,y))\sqrt{\det C}\,dydt=\int_0^1\int_U t^{n-2}f_t\sqrt{\det B}\,dydt$$
$$=\int_0^1\int_N t^{n-2}f_t\,dydt.$$

This equation completes the proof of the lemma. ☑

Lemma 2.3. *Under the same hypothesis of the previous lemma we have that if $f:M_\varepsilon\to\mathbb{R}$ satisfies that $f(tp)=h(t)$ for every $t\in[\varepsilon,1]$ and $p\in N$, then*

$$\Delta f(tp)=h''(t)+\frac{(n-2)}{t}h'(t),\qquad \textit{for every } t\in(\varepsilon,1) \textit{ and } p\in N.$$

Proof. Let $\{e_0,e_1,\dots,e_{n-2}\}$ be an orthonormal frame defined in a neighborhood of tp such that $e_0(x)=\frac{x}{|x|}$ and $e_i(tp)=e_i(p)$ for every $p\in N$, $t\in[\varepsilon,1]$ and $i=1,\dots,n-2$. We have that

$$\nabla f(x)=\{e_0(f)e_0+e_1(f)+\dots+e_{n-1}(f)e_{n-1}\}\big|_x.$$

If we take $\alpha(s)=|x|\frac{x}{|x|}+s\frac{x}{|x|}$, then $\alpha(0)=x$ and $\alpha'(0)=e_0(x)$. Therefore $e_0(f)(x)=\beta'(0)$ where $\beta(s)=f(\alpha(s))=h(|x|+s)$ and $e_0(f)(x)=h'(|x|)$. On the other hand, since the frame is orthonormal, for every $i=1,\dots,n-2$ we may choose curves $\alpha_i(t)$ such that $\alpha_i(0)=x$, $\alpha_i'(0)=e_i(x)$ and $|\alpha_i(t)|=|x|$ for all t. Under this choice of curves, we have, by using the hypothesis on the function f, that $e_i(f)(x)=0$ for $i\geq 1$. Therefore,

$$\nabla f(x)=h'(|x|)\frac{x}{|x|}. \tag{5}$$

We will use the same curves α_i's to compute $\overline{\nabla}_{e_i}(\nabla f)$. Notice that $\overline{\nabla}_{e_i}e_0(x)=\frac{e_i}{|x|}$ for every $i\in\{1,\dots,n-2\}$. If we make $tp=p_0$ then,

$$\begin{aligned}\Delta(f)(p_0)&=\langle\overline{\nabla}_{e_0}\nabla f,e_0\rangle(p_0)+\dots+\langle\overline{\nabla}_{e_{n-2}}\nabla f,e_{n-1}\rangle(p_0)\\&=\langle\overline{\nabla}_{e_0}(h'(|x|)e_0(x)),e_0\rangle(p_0)+\dots+\langle\overline{\nabla}_{e_{n-2}}(h'(|x|)e_0(x)),e_{n-1}\rangle(p_0)\\&=e_0(h'(|x|))+h'(t)\langle\overline{\nabla}_{e_0}e_0,e_0\rangle(p_0)+e_1(h'(|x|))+h'(t)\langle\overline{\nabla}_{e_1}e_0,e_1\rangle(p_0)\\&\quad+\dots+e_{n-2}(h'(|x|))+h'(t)\langle\overline{\nabla}_{e_{n-2}}e_0,e_{n-2}\rangle(p_0)\\&=h''(t)+\frac{1}{t}h'(t)+\dots+\frac{1}{t}h'(t)=h''(t)+\frac{(n-2)}{t}h'(t).\end{aligned}$$ ☑

We will also need the following results.

Theorem 2.1. *Let $M \subset \mathbb{R}^n$ be a complete minimal cone and let $N = M \cap S^{n-1}$.*

(a) *([C-D-K], [L]) $|A|^2(p) = n-2$ for any $p \in N$ if and only if M is isometric to a Clifford minimal cone.*

(b) *[Y-C] If the function $|A|^2(p)$ is constant for all $p \in N$ and this constant is smaller than $\frac{4(n-2)}{3}$ on N, then M is either part of a hyperplane or M is a Clifford cone.*

(c) *[Y-C] Let $\{\kappa_1(m), \ldots, \kappa_{n-1}(m)$ be the principal curvatures at $m \in M$, i.e. they are the eigenvalues of the shape operator $A(m) : T_mM \to T_mM$. If (i) the function $|A|^2(p)$ is constant for all $p \in N$ and this constant is smaller than $\frac{5(n-2)}{3}$ on N and (ii) $\kappa_1(m)^3 + \cdots + \kappa_{n-1}(m)^3$ is constant for all $m \in N$ or $\kappa_1(m)^4 + \cdots + \kappa_{n-1}(m)^4$ is constant for all $m \in N$, then M is either part of a hyperplane or M is a Clifford cone.*

3. Main result

In this section we will state and prove the main results of this paper. The idea in the proof of these theorems is the one used by James Simons in [S].

Theorem 3.1. *Let $M \subset \mathbb{R}^n$, with $n = 8$ or $n = 9$, be a complete minimal cone with codimension 1. If (i) the norm of the shape operator is constant on the points of M with norm 1, i.e $|A|^2(m) = c$ for every $m \in M \cap S^{n-1} = N$; and (ii) M is stable, then M must be either a hyperplane or a Clifford minimal cone.*

Proof. Let us assume that M is not a hyperplane or a Clifford minimal cone; we will show that M can not be stable. We will do this by showing a function with compact support such that $V_f''(0) < 0$. Let us define $f : M \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } |x| \leq \varepsilon \text{ or } |x| \geq 1, \\ h(x), & \text{for } \varepsilon < |x| < 1, \end{cases}$$

where $h : [\varepsilon, 1] \to \mathbb{R}$ is a smooth function such that $h(\varepsilon) = h(1) = 0$. We will define the function h later on. By using Lemma 2.1 we get that the stability operator J is given by

$$J(f)(tp) = -(\Delta f)(tp) - \frac{1}{t^2} c f(tp), \quad \text{for every } t \in (0, 1) \text{ and } p \in N.$$

By using equation (2), Lemma 2.3 and Lemma 2.2, we obtain that:

$$
\begin{aligned}
\frac{d^2V_f}{dt^2}(0) &= \int_M J(f)f \\
&= \int_{M_\varepsilon} (-(\Delta f)(tp) - \frac{1}{t^2}cf(tp))f(tp) \\
&= \int_{M\varepsilon} (-h''(t) - \frac{(n-2)}{t}h'(t) - \frac{1}{t^2}ch(t))h(t) \\
&= -\int_\varepsilon^1\int_N t^{n-4}(t^2h''(t) + t(n-2)h'(t) + ch(t))h(t).
\end{aligned} \tag{6}
$$

Since we are assuming that M is neither an equator nor a Clifford hypersurface, Theorem 2.1 part (b) gives us that $c > 4\frac{(n-1)}{3}$. For $n = 8$ let us take $d = 4(c - \frac{1}{3}) - 25$ and $\varepsilon = \exp\frac{-2\pi}{\sqrt{d}}$. Notice that $d > 0$ because $c > 8$. Let us also define $h(t) = t^{\frac{-5}{2}}\sin(-\frac{\sqrt{d}}{2}\ln(t))$. Notice that $h(\varepsilon) = 0 = h(1)$ and $h(t) > 0$ for every $t \in (\varepsilon, 1)$. A direct verification shows that

$$t^2h''(t) + 6th'(t) + ch(t) = \frac{1}{3}h(t).$$

Replacing the above equation in (6) we obtain:

$$\frac{d^2V_f}{dt^2}(0) = \int_M J(f)f = \frac{-1}{3}\int_N\int_\varepsilon^1 t^4h(t)^2 < 0.$$

Therefore M is not stable. For $n = 9$ we define $d = 4(c - \frac{1}{3}) - 36$ and $\varepsilon = \exp\frac{-2\pi}{\sqrt{d}}$. Notice that $d > 0$ because $c > \frac{28}{3}$. Let us also define $h(t) = t^{-3}\sin(-\frac{\sqrt{d}}{2}\ln(t))$. Notice that $h(\varepsilon) = 0 = h(1)$ and $h(t) > 0$ for every $t \in (\varepsilon, 1)$. A direct verification shows that

$$t^2h''(t) + 7th'(t) + ch(t) = \frac{1}{3}h(t).$$

Replacing the above equation in (6) results in

$$\frac{d^2V_f}{dt^2}(0) = \int_M J(f)f = \frac{-1}{3}\int_N\int_\varepsilon^1 t^5h(t)^2 < 0.$$

Therefore M is not stable. ☑

Corollary 3.1. *Let $M \subset \mathbb{R}^n$, with $n = 8$ or $n = 9$, be a complete minimal cone with codimension 1. If*

(i) *The scalar curvature R is constant on the points of M with norm 1, i.e $R(m) = R_0$ for every $m \in M \cap S^{n-1} = N$, and*
(ii) *M is stable.*

Then M must be either a hyperplane or a Clifford minimal cone.

Proof. Let $m \in M$ and let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal base of T_mM such that $A(m)(e_i) = \kappa_i e_i$. By Gauss' Theorem, the sectional curvature $K(e_i, e_j)$ of the plane spanned by the vectors e_i, e_j is the product $\kappa_i\kappa_j$ for every $i \neq j$. We have that,

$$
\begin{aligned}
(n-1)(n-2)R(m) &= \sum_{i,i=1,i\neq j}^{n-1} K(e_i, e_j) \\
&= \sum_{i,i=1,i\neq j}^{n-1} \kappa_i\kappa_j \\
&= \sum_{i,i=1}^{n-1} \kappa_i\kappa_j - \sum_{i=1}^{n-1} \kappa_i^2 \\
&= (\kappa_1 + \cdots + \kappa_{n-1})^2 - |A|^2 \\
&= -|A|^2.
\end{aligned} \tag{7}
$$

In the last equality we have used that $0 = (n-1)H = \kappa_1 + \cdots + \kappa_{n-1}$ because M is minimal. By equation (7), R is constant on $M \cap S^{n-1}$ if and only if $|A|^2$ is constant on $M \cap S^{n-1}$. The corollary now follows from Theorem 2.1. ☑

For the next theorem we will consider stable minimal cones in $\mathbb{R}^{10}$ with codimension 1. We will get the same result but with the additional condition that either the function $\kappa_1(m)^3 + \cdots + \kappa_{n-1}(m)^3$ varies radially or the function $\kappa_1(m)^4 + \cdots + \kappa_{n-1}(m)^4$ varies radially, namely we will prove:

Theorem 3.2. *Let $M \subset \mathbb{R}^{10}$ be a complete minimal cone with codimension 1. If*

(i) *The norm of the shape operator is constant on the points of M with norm 1, i.e $|A|^2(m) = \kappa_1(m)^2 + \cdots + \kappa_{n-1}(m)^2 = c$ for every $m \in M \cap S^{n-1} = N$;*
(ii) *Either $\kappa_1(m)^3+\cdots+\kappa_{n-1}(m)^3$ varies radially or $\kappa_1(m)^4+\cdots+\kappa_{n-1}(m)^4$ varies radially, and*
(iii) *M is stable.*

Then M must be either a hyperplane or a Clifford minimal cone.

Proof. Let us take $d = 4(c - \frac{1}{3}) - 49$ and $\varepsilon = \exp \frac{-2\pi}{\sqrt{d}}$. Notice that $d > 0$ because $c > \frac{40}{3}$ by Theorem 2.1 part (c). Let us also define $h(t) = t^{\frac{-7}{2}} \sin\left(-\frac{\sqrt{d}}{2} \ln(t)\right)$. Notice that $h(\varepsilon) = 0 = h(1)$ and $h(t) > 0$ for every $t \in (\varepsilon, 1)$. A direct verification shows that

$$t^2h''(t) + 6th'(t) + ch(t) = \frac{1}{3}h(t).$$

Replacing the above equation in (6) we get

$$\frac{d^2V_f}{dt^2}(0) = \int_M J(f)f = \frac{-1}{3}\int_N \int_\varepsilon^1 t^4 h(t)^2 < 0.$$

Therefore M is not stable. ☑

Remarks.

(a) The isoperimetric hypercone in $\mathbb{R}^{13}$ with 3 non zero principal curvatures gives an example of a stable complete minimal cone with codimension 1 which scalar curvature varies radially.

(b) Chern's conjecture states that if $M \subset \mathbb{R}^n$ is a $n-1$ dimensional complete minimal cone and $|A|^2(m) = c$, with c a constant less than $2(n-1)$, for every $m \in M \cap S^{n-1} = N$, then M must be either a hyperplane or a Clifford minimal cone. Using the same technique we used in the proof of our theorems we have that the veracity of Chern's conjecture implies the veracity of Theorem 2.1 for $n = 10$ and $n = 11$.

References

[B-DG-G] Bombieri, E., De Giorgi E., & Guisti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.

[C-Y] Cheng, Q-M & Yang, H, *Chern's conjecture on minimal hypersurfaces*, Math Z. **227** (1998), 377–390.

[D] Do Carmo, M., *Riemannian Geometry*, Birkhauser, 1992.

[S] Simons, J., *Minimal varieties in Riemannian manifolds* Ann. of Math. **88** (1968), 62–105.

(Recibido en octubre de 2002)

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