On Power-Associative Nilalgebras of Nilindex and Dimension $n$

Sobre nilálgebras de potencia asociativa de nilíndice y dimensión $n$

JUAN C. GUTIERREZ FERNANDEZ$^{1,a,s}$, CLAUDIA I. GARCIA$^1$

MARTY L. R. MONTOYA$^2$

$^1$Universidade de São Paulo, São Paulo, Brazil
$^2$Universidad de Antioquia, Antioquia, Colombia

Abstract. We investigate the structure of commutative power-associative nilalgebras of dimension and nilindex $n$.

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Resumen. Investigamos la estructura de nilálgebras conmutativas de potencia asociativa de dimensión y nilíndice $n$.

Palabras y frases clave. Conmutatividad, potencia asociativa, nilálgebra.

1. Introduction

Commutative power-associative algebras are a natural generalization of associative, alternative and Jordan algebras. An algebra is said to be power-associative if the subalgebra generated by any element is associative. We refer the reader to the paper [1] for more information. In [2] the authors classify Jordan power-associative nilalgebras of nilindex $n$ and dimension $n \geq 4$. In this paper we give the structure constants for power-associative nilalgebras of nilindex $n$ and dimension $n \geq 5$.

Throughout this paper, $\mathfrak{A}$ will be a commutative power-associative nilalgebra of dimension $n$ over a field $F$ of characteristic $\neq 2,3$ and 5. For every $a \in \mathfrak{A}$ we will denote by $\mathfrak{A}_a$ the subalgebra of $\mathfrak{A}$ generated by $a$. We define inductively
the powers of $a \in \mathfrak{A}$ by $a^1 = a$ and $a^k = aa^{k-1}$ for $k > 1$. In a commutative power-associative algebra $\mathfrak{A}$, we have that $a^i a^j = a^{i+j}$ for every $a \in \mathfrak{A}$ and all positive integers $i, j$ and hence $\mathfrak{A}_n$ is spanned, as a vector space, by all the powers $a^k$ with $k$ a positive integer. We remember that in a commutative power-associative algebra, the algebra generated by all right multiplications $R_a : \mathfrak{A} \to \mathfrak{A}$, with $x \in \mathfrak{A}_a$, is in fact generated by $R_a$ and $R_{a^2}$. A commutative algebra is called Engel if every right multiplication of $\mathfrak{A}$ is nilpotent. We will use the process of linearization of identities, which is an important tool in our investigation. Thus, $p(x, y, z, t) = 0$ will be the complete linearization of the fourth power-associative identity $x^4 - (x^2 x^2) = 0$. Next, linearizing the identities $x^2 x^3 = x(x^2 x^2)$ and $x^3 x^3 = (x^2)^3$ we get the following new identities

\[ x^4 y = 2x^3(xy) + x^2(x^2 y) + 2x^2(x(xy)) - 4x(x^2(xy)), \] (1)
\[ x^3(x^2y) + 2x^3(x(xy)) = 2x^2(x^2(xy)) + x^4(xy). \] (2)

For every positive integer $r \geq 3$, the identity $p(a^{r-2}, a, a, b) = 0$ implies the well known multiplication identity

\[ R_{a^r} = \frac{1}{3} (8R_{a^{r-1}}R_a - 2R_a R_{a^{r-1}} + 4R_{a^2}R_{a^{r-2}} - 2R_{a^3}R_{a^{r-2}}) - \]
\[ R_{a^{r-1}}R_{a^2} - 2R_a R_{a^{r-2}}R_a - 2R_{a^r}R_a^2. \] (3)

We observe that each product in a commutative power-associative algebra $\mathfrak{A}$ with $b$, one time, and $a$, $s$ times, can be written as $a^{i_1} (a^{i_2} (\cdots (a^{i_k} b) \cdots ))$, where $i_1, \ldots, i_k$ are positive integers and $i_1 + \cdots + i_k = s$. We get the following relevant facts about the structure of a commutative power-associative algebra $\mathfrak{A}$.

**Lemma 1.** Let $a, b \in \mathfrak{A}$ such that $ba \in \mathfrak{A}_a$. Then

\[ ba^3 = -a(ba^2) + 2a^2(ba), \]
\[ ba^4 = a^2(ba^2), \]
\[ a^3(ba^2) = a^4(ba). \] (4)

Furthermore,

(i) If $ba^2 \in \mathfrak{A}_a$, then $b\mathfrak{A}_a \subset \mathfrak{A}_a$ and $a^{i_1} (a^{i_2} (\cdots (a^{i_k} b) \cdots )) = a^{s-1}(ba)$ for all positive integers $k, i_1, \ldots, i_k$ where $s = i_1 + \cdots + i_k \geq 5$.

(ii) If $ba = 0$ and $b\mathfrak{A}_a^1 \subset \mathfrak{A}_a$, then
\[ ba^3 = -a(ba^2), \]
\[ a^3(ba^2) = 0, \]
\[ ba^5 = -a(ba^4) = 2a^2(ba^3), \]
\[ ba^6 = -a(ba^5) = a^4(ba^2) = a^2(ba^4), \]
for all positive integers \( k, i_1, \ldots, i_k \) where \( i_1 + \cdots + i_k \geq 7 \).

**Proof.** Let \( a, b \in \mathfrak{A} \) such that \( ab \in \mathfrak{A}_a \). From identity \( p(a, a, a, b) = 0 \) we get immediately \( ba^3 = -a(ba^2) + 2a^2(ba) \). Setting \( x = a \) and \( y = b \) in (1) immediately yields relation \( ba^4 = a^2(ba^2) \). Replacing \( x \) by \( a \) and \( y \) by \( b \) in (2) we get \( a^3(ba^2) = a^4(ba) \).

Now we will prove (i). If \( ba^2 \in \mathfrak{A}_a \), then using (3) we can prove inductively on \( r \geq 3 \) that there exist \( \lambda_r, \mu_r \in F \) such that \( \lambda_r + \mu_r = 1 \) and
\[ a^r b = \lambda_r a^{r-2}(ba^2) + \mu_r a^{r-1}(ba). \] (5)

The cases \( r = 3, 4 \) are proved above. For \( r > 4 \), we obtain from (3) and the induction hypothesis that \( ba^r = (1/3)(4a^{r-1}(ba) - a^{r-2}(ba^2) + 2a^2(ba^2) - 2a(ba^2-1)) = (1/3)(4a^{r-1}(ba) - a^{r-2}(ba^2) + 2a^2(ba^2) - 2a(ba^2-1)) \)
\[ = \lambda_r a^{r-3}(ba^2) + \mu_r a^{r-2}(ba) \]
now using (3) we get inductively \( ba^k \in \mathfrak{A}_a \) for all positive integers \( k \geq 3 \). Using (4) we get \( ba^3 = -a(ba^2) \) and \( a^3(ba^2) = 0 \). Now
\[ 0 = p(a, a, a, ba^2)/6 = \]
\[ a^3(ba^2) + a^2(a(ba^2)) + 2a(a(a(ba^2))) = 4a^2(a(ba^2)) = \]
\[ a(ba^4) - 2a(a(ba^3)) + 4a^2(ba^3) = a(ba^4) + 2a^2(ba^3) \]
so that $a(ba^4) = -2a^2(ba^3)$. Next, relation (3) for $r = 5$ forces $ba^5 = (1/3)(-2a(ba^4) + 4a^2(ba^3) - 2a(a(ba^3))) = (1/3)(-2a(ba^4) + 2a^2(ba^3)) = (1/3)(-3a(ba^4)) = -a(ba^4)$. Setting $x = a$ and $y = ba^2$ in (1) immediately yields relation $a^2(ba^2) = a^2(a^2(ba^3))$ and now using second identity of (4) we get $a^4(ba^4) = a^2(ba^4)$. Thus,

$$0 = p(a^2, a^2, a^2, b)/6 = ba^6 + a^2(ba^4) + 2a^2(a^2(ba^3)) - 4a^4(ba^2) = ba^6 - a^4(ba^2).$$

Now, $a(ba^5) = -a(a(ba^4)) = -a^2(ba^4) = -ba^6$.

Taking $x = a$ and $y = ba^2$ in identity (2) we get

$$0 = a^3(ba^4) + 2a^3(a(a(ba^3))) - 2a^2(a^2(a(ba^3))) - a^4(a(ba^2)) = a^3(ba^4) - a^4(a(ba^2)) = a^2(ba^4) + a^4(ba^2) = a(ba^6) + a(a^2(ba^3)) = a(ba^6) + a(a(ba^5))/2 = a(ba^6) - (ba^6)/2 = a(ba^6)/2.$$

Finally, we will prove that $x = a^{i_1}(a^{i_2}(\cdots(a^{i_k}b)\cdots))$ vanishes for all $i_1, i_2, \ldots, i_k$ positive integers with $s = \sum_{l=1}^{k} i_l \geq 7$. Using (3), we can prove, by induction on $s$, that the element $a^{i_1}(a^{i_2}(\cdots(a^{i_k}b)\cdots))$ is spanned by the set of all elements $a^{i_1}(a^{i_2}(\cdots(a^{i_k}b)\cdots))$ with $j_1, \ldots, j_k \in \{1, 2\}$ and $j_1 + \cdots + j_k = s$. Thus, we can assume, without loss of generality, that $i_1, \ldots, i_k \in \{1, 2\}$. If $i_k = 1$, then $x = 0$ since $ba = 0$. If $i_k = 2$ and $i_{k-1} = 1$, then $x = -a^{i_1}(a^{i_2}(\cdots(a^{i_{k-2}}(ba^3)\cdots))) = -a^{s-7}(a(a^2(ba^3))) = a^{k-7}(a(ba^6))/2 = 0$ since $ba^3 \in A$. If $i_k = i_{k-1} = 2$, then $x = a^{i_1}(a^{i_2}(\cdots(a^{i_{k-2}}(ba^3)\cdots))) = a^{s-7}(a(a^2(ba^3))) = a^{s-7}(a(ba^6))/2 = 0$. This complete the proof of the lemma.

2. Nilindex $n$

Throughout this section, $A$ will be a commutative power-associative nilalgebra of dimension and nilindex $n$. Let $a$ be an element in $A$ with maximal nilindex. It is well known that $A^k = A^n$, for all $k \geq 2$ (see [2]). Hence

$$A^j_a \subset A^{j+1}_a,$$

for all $j \geq 1$. Furthermore, $A^n = A^n_0 = 0$ and for each $x \in A$, the power $x^{n-1}$ is in the annihilator of $A$.

For a finite list $S = \{a_1, \ldots, a_n\}$ we write $(a_1, \ldots, a_n)$ for the subspace consisting of all the linear combinations of elements of $S$.

**Lemma 2.** Let $a$ be an element in $A$ with maximal nilindex and $k$ an integer with $1 \leq k \leq n - 1$. Then there exists $b_k \in A \setminus A_a$ such that $b_k a^k = 0$. The annihilator of $a^k$ in $A$ is $\langle b_k, a^{n-k}, a^{n-k+1}, \ldots, a^{n-1} \rangle$.
Proof. Take \( b \in \mathfrak{A} \setminus \mathfrak{A}_a \). Then \( \{ b, a, a^2, \ldots, a^{n-1} \} \) is a basis of \( \mathfrak{A} \). By the above lemma, \( ba^k \in \mathfrak{A}^{k+1}_a \), so that \( ba^k = \lambda_{k+1}a^{k+1} + \cdots + \lambda_{n-1}a^{n-1} \), for \( \lambda_{k+1}, \ldots, \lambda_{n-1} \in F \). Then \( ba^k = 0 \) for \( k = b - \lambda_{k+1}a - \cdots - \lambda_{n-1}a^{n-k-1} \).

Finally, let \( x = \xi_0b_k + \xi_1a + \xi_2a^2 + \cdots + \xi_{n-1}a^{n-1} \) be an arbitrary element in \( \mathfrak{A} \). Then \( x a^k = \xi_0b_k a^k + \xi_1a^{k+1} + \xi_2a^{k+2} + \cdots + \xi_{n-k-1}a^{n-1} = \xi_1a^{k+1} + \xi_2a^{k+2} + \cdots + \xi_{n-k-1}a^{n-1} \). This proves the lemma.

\( \square \)

Corollary 3. Let \( a \in \mathfrak{A} \) be an element in \( \mathfrak{A} \) with maximal nilindex. Then there exists \( b \in \mathfrak{A} \setminus \mathfrak{A}_a \) such that \( ba \in (a^2) \) and \( a^{n-2}b = 0 \). Furthermore, an element \( c \in \mathfrak{A} \) satisfies \( ca \in (a^2) \) and \( ca^{n-2} = 0 \) if and only if \( c \in (b, a^{n-1}) \).

We will denote by \( \mathcal{P}(\mathfrak{A}) \) the set of ordered pairs \( (a, b) \) of elements in \( \mathfrak{A} \) where \( a \) has maximal nilindex, and \( b \in \mathfrak{A} \setminus \mathfrak{A}_a \) with \( ba \in (a^2) \) and \( ba^{n-2} = 0 \). By definition and relation (6), we have that

\[
\begin{align*}
b^2 & \in \mathfrak{A}^2, \\
ba & = \lambda a^2, \\
ba^k & \in (a^{k+1}, \ldots, a^{n-1}) = \mathfrak{A}_a^{k+1} \text{ for } k = 2, \ldots, n-3, \\
ba^{n-2} & = ba^{n-1} = 0,
\end{align*}
\]

for any \( (a, b) \in \mathcal{P}(\mathfrak{A}) \).

For commutative power-associative nilalgebras of dimension 3 and nilindex 3, we have one family of algebras \( A(\alpha) = (b, a, a^2) \), with \( b^2 = \alpha a^2 \), \( ba = 0 \), parametrized by \( F/(F^*)^2 \), that is \( A(\alpha) \) is isomorphic to \( A(\alpha') \) if and only if \( \alpha' = \gamma^2 \alpha \). We denote \( F \sim \{0\} \) by \( F^* \).

M. Gerstenhaber and H. C. Myung [4] showed that commutative, power-associative nilalgebras of dimension 4 over fields of characteristic \( \neq 2 \) are nilpotent and determined the isomorphic classes. They found one family of algebras parametrized by \( F/(F^*)^2 \) and four individual algebras.

Theorem 4. If \( \mathfrak{A} \) is a commutative power-associative nilalgebra over \( F \) with dimension and nilindex 4, then \( \mathfrak{A} \) has a pair \( (a, b) \in \mathcal{P}(\mathfrak{A}) \) where the nontrivial and nonzero product belong to one and only one of the list below:

\[
\begin{align*}
A_1(\alpha) : b^2 & = \alpha a^2 \quad (\alpha \in F) \\
A_2 : b^2 & = a^3 \\
A_3 : \quad ba & = a^2 \\
A_4 : b^2 & = a^3 \quad ba = a^2 \\
A_5 : b^2 & = a^2 \quad ba = a^2
\end{align*}
\]

where \( A_1(\alpha) \) is isomorphic to \( A_1(\alpha') \) if and only if there exists \( \gamma \in F^* \) such that \( \alpha' = \gamma^2 \alpha \).
A description of commutative power-associative nilalgebras of dimension 5 was given by I. Correa and A. Suazo in [2] in the Jordan case, and by L. Elgueta and A. Suazo in [3] for algebras that are not Jordan.

**Lemma 5.** If $\mathfrak{A}$ is a commutative power-associative nilalgebra over the field $F$ with dimension and nilindex 5 and $(a, b) \in \mathcal{P}(\mathfrak{A})$, then $b^2 \in \mathfrak{A}^3$ and $ba^2 - 2a(ba) \in \mathfrak{A}^4$.

In Theorem 6, we will show a classification of such algebras without proof.

**Theorem 6.** If $\mathfrak{A}$ is a commutative power-associative nilalgebra of dimension and nilindex 5, then $\mathfrak{A}$ has a basis $\{b, a, a^2, a^3, a^4\}$ with $(a, b) \in \mathcal{P}(\mathfrak{A})$, and the other nonzero products belong to one and only one of the types listed below.

- $A_1(\alpha) : b^2 = a^3 + \alpha a^4$, $ba = a^2$, $ba^2 = 2a^3$, $(\alpha \in F)$,
- $A_2(\alpha) : b^2 = \alpha a^4$, $ba = a^2$, $ba^2 = 2a^3$, $(\alpha \in F)$,
- $A_3(\alpha) : b^2 = \alpha a^4$, $ba^2 = a^4$, $(\alpha \in F)$,
- $A_4(\alpha) : b^2 = \alpha a^4$, $ba^2 = a^4$, $(\alpha \in F)$,
- $A_5 : b^2 = a^3$, $ba^2 = a^4$,
- $A_6 : b^2 = a^3$.

Furthermore, we have the following conditions for two algebras in such a class to be isomorphic. For $i \in \{2, 4\}$ we have that $A_i(\alpha) \cong A_i(\alpha')$ if and only if there exists $\gamma \in F^* \text{ such that } \alpha' = \gamma^2 \alpha$. Next, $A_3(\alpha) \cong A_3(\alpha')$ if and only if $\alpha = \alpha'$. Finally, we have that $A_1(\alpha) \cong A_1(\alpha')$ if and only if there exists $\gamma \in F^*$ such that

$$\alpha' = \frac{16\alpha - \gamma^4 + 1}{16\gamma^4}.$$ 

We observe that the algebras $A_4(\alpha)$ are associative. The algebras $A_3(\alpha)$, $A_5$ and $A_6$ are Jordan and are not associative. On the other hand, the algebras $A_1(\alpha)$ and $A_2(\alpha)$ are not Jordan.

**Lemma 7.** Let $\mathfrak{A}$ be a commutative power-associative nilalgebra over the field $F$ with dimension and nilindex 6. Take $(a, b) \in \mathcal{P}(\mathfrak{A})$. Then there exist scalars $\alpha, \beta, \lambda, \lambda_1, \lambda_2 \in F$ such that

\begin{align*}
    b^2 &= \alpha a^4 + \beta a^5, \\
    ba &= \lambda a^2, \\
    ba^2 &= \lambda_1 a^4 + \lambda_2 a^5, \\
    ba^3 &= 2\lambda a^4 - \lambda_1 a^5.
\end{align*}

\( (7) \)

Reciprocally, if $\alpha, \beta, \lambda, \lambda_1$ and $\lambda_2$ are scalars and $\mathfrak{A}$ is a commutative algebra with basis $\{b, a, a^2, a^3, a^4, a^5\}$ and products $ba^4 = ba^5 = a^6 = 0$, $a'a^j = a^{i+j}$ for
all \( k \geq 6 \) and for all positive integers \( i, j \) and (7), then \( \mathfrak{B} \) is a power-associative nilalgebra of dimension and nilindex 6. Furthermore, \( \mathfrak{A} \) is Jordan if and only if \( \lambda = 0 = \lambda_1 \).

**Proof.** By (6), we know that \( b^2 \in \mathfrak{A}^2 = \mathfrak{A}_2^2 \) and \( ba^k \in \mathfrak{A}^{k+1} = \mathfrak{A}_a^{k+1} \) for \( k = 2, 3 \).

Because \( (a, b) \in \mathcal{P}(\mathcal{A}) \), we have that \( ba = \lambda a^2 \). By (4) we have that \( a^2(ba^2) = ba^4 = 0 \) so that \( ba^2 \in \mathfrak{A}_a^4 \). Let \( \lambda_1, \lambda_2 \in F \) such that \( ba^2 = \lambda_1 a^4 + \lambda_2 a^5 \). Now Lemma 1 forces \( ba^3 = -a(ba^2) + 2a^2(ba) = -a(ba^2) + 2\lambda a^4 \) and hence \( ba^3 = 2\lambda a^4 - \lambda_1 a^5 \). Next, \( 0 = p(a, a, b, b)/4 = a(ab^2) + b(b^2) + 2a(b(ba)) + 2b(a(ba)) - 4(ba)^2 - 2a^2b^2 = -a^2b^2 \) so that \( b^2 \in \mathfrak{A}_a^4 \). This completes the proof of the first part of the lemma.

Reciprocally, let \( x = \xi b + y \) be an element in \( \mathfrak{B} \), where \( y = \sum_{i=1}^5 \xi_i a^i \). Then

\[
\begin{align*}
x^2 &\equiv y^2 + 2\xi_1 \lambda a^2 \mod (a^4, a^5), \\
x^3 &\equiv y^3 + 2\xi_1 \lambda \lambda^3 + \xi_1 (\xi_1 \lambda + 2\xi_2 \lambda_1 + 6\xi_3 \lambda) a^4 \mod (a^5), \\
x^4 &\equiv (x^2)^2 = y^4 + 4\xi_1 \lambda (\xi_1 + \xi \lambda) a^4 + (8\xi_2 \lambda^2) a^5, \\
x^5 &\equiv \xi_1 (\xi_1 + 2\xi \lambda) a^5,
\end{align*}
\]

and hence \( \mathfrak{B} \) is a power-associative nilalgebra of nilindex 6.

Finally, we observe that \( (a^2b)a - a^2(ba) = \lambda_1 a^5 - \lambda a^4 \) and hence \( \lambda = 0 = \lambda_1 \) if \( \mathfrak{A} \) is Jordan. Reciprocally, if \( \lambda = 0 = \lambda_1 \), then \( (b - \lambda_2 a^3)A^2 = 0 \) and Theorem 2.1 of [3] implies that \( \mathfrak{A} \) is Jordan. This completes the proof of the lemma. \( \Box \)

**Lemma 8.** Let \( \mathfrak{A} \) be a commutative power-associative nilalgebra over the field \( F \) with dimension and nilindex 7. Take \( (a, b) \in \mathcal{P}(\mathcal{A}) \). Then \( ba = 0 \) and there exist scalars \( \alpha, \beta, \lambda, \lambda_1, \lambda_2 \in F \) such that

\[
\begin{align*}
b^2 &= \lambda_2 a^4 + \alpha a^5 + \beta a^6, \\
ba^2 &= \lambda a^4 + \lambda_1 a^5 + \lambda_2 a^6, \\
ba^3 &= -\lambda a^5 - \lambda_1 a^6, \\
ba^4 &= \lambda a^6.
\end{align*}
\]

Reciprocally, if \( \alpha, \beta, \lambda, \lambda_1 \) and \( \lambda_2 \) are scalars and \( \mathfrak{B} \) is a commutative algebra with basis \( \{b, a, a^2, a^3, a^4, a^5, a^6\} \) and products \( ab = 0 = ba^5 = ba^6 = a^6, a^k a^j = a^{k+j} \) for all \( k \geq 7 \) and for all positive integers \( i, j \) and (9), then \( \mathfrak{B} \) is a power-associative nilalgebra of dimension and nilindex 7.

Furthermore, \( \mathfrak{A} \) is Jordan if and only if \( \lambda = 0 = \lambda_1 \).

**Proof.** Because \( (a, b) \in \mathcal{P}(\mathcal{A}) \) and (6), we know that \( b^2 \in \mathfrak{A}^2 = \mathfrak{A}_a^2, ba = \lambda_0 a^2, ba^k \in \mathfrak{A}^{k+1} = \mathfrak{A}_a^{k+1} \) for \( k = 2, 3, 4, ba^5 = 0 \) and \( ba^6 = 0 \).
Combining the above relations and (i) of Lemma 1 we have that \( \lambda_0 a^6 = a^4 (ba) = ba^5 = 0 \) so that \( \lambda_0 = 0 \) and hence \( ba = 0 \). Also, by (i) of Lemma 1 we have \( a^3 (ba^2) = ba^5 = 0 \) and hence \( ba^2 \in \mathbb{F}_2 \). Thus, we have \( ba^2 = \lambda a^4 + \lambda_1 a^5 + \lambda_2 a^6 \), for \( \lambda, \lambda_1, \lambda_2 \in \mathbb{F} \). Using (4) we have that \( ba^3 = -a (ba^2) + a^2 (ba) = -a (ba^2) = -\lambda a^5 - \lambda_1 a^6 \) and \( ba^4 = a^2 (ba^2) = \lambda a^6 \). Now

\[
0 = p(a, a, b, b)/4 = \\
a(ab^2) + b(ba^2) + 2a(b(ba)) + 2b(a(ba)) - 2a^2 b^2 - 4(ab)^2 = \\
a(ab^2) + b(ba^2) - 2a^2 b^2 = -a(ab^2) + b(ba^2) = -a(ab^2) + \lambda^2 a^6,
\]
since \( a(ab^2) = a^2 b^2 \) and \( b(ba^2) = b(\lambda a^4 + \lambda_1 a^5 + \lambda_2 a^6) = \lambda^2 a^6 \). Thus, we have proved that \( a(ab^2) = \lambda^2 a^6 \). This completes the proof of the first part of the lemma.

Reciprocally, let \( x = \xi b + y \) be an element in \( \mathfrak{B} \), where \( y = \sum_{i=1}^6 \xi_i a^i \). Then

\[
x^2 \equiv y^2 + \xi \lambda (\xi \lambda + 2\xi_2) a^4 \mod \langle a^5, a^6 \rangle, \\
x^3 \equiv y^3 + \xi \xi_1^2 \lambda a^4 + \xi \xi_1 (\xi \lambda^2 + \xi_1 \lambda_1) a^5 \mod \langle a^6 \rangle, \\
x^4 = (x^2)^2 = y^4 + 2\xi \xi_1^3 \lambda (\xi \lambda + 2\xi_2) a^6, \\
x^5 = y^5 + \xi \xi_1^2 \lambda a^6, \\
x^6 = y^6 = \xi^6 a^6,
\]
so that \( \mathfrak{B} \) is a power-associative nilalgebra of nilindex 7.

Finally, if \( \mathfrak{A} \) is Jordan, then \( 0 = (a^2 b)a - a^2 (ba) = (a^2 b)a = \lambda a^5 + \lambda_1 a^6 \), so that \( \lambda = 0 = \lambda_1 \). Reciprocally, if \( \lambda = 0 = \lambda_1 \), then \( (b - \lambda_2 a^4) \mathfrak{A}^2 = 0 \) and Theorem 2.1 of [3] implies that \( \mathfrak{A} \) is Jordan. This proves the lemma.

**Theorem 9.** Let \( \mathfrak{A} \) be a commutative power-associative nilalgebra over the field \( \mathbb{F} \) with dimension and nilindex \( n \) and \( n \geq 8 \). Take \( (a, b) \in \mathcal{P}(\mathfrak{A}) \). Then

\[
ba = 0, \\
a^2 b^2 = 0, \\
a^3 (ba^2) = 0, \\
b a^3 = -a (ba^2), \\
b a^4 = a^2 (ba^2), \\
b a^k = 0,
\]
for all \( k \geq 5 \).

Reciprocally, if \( \alpha, \beta, \lambda, \lambda_1 \) and \( \lambda_2 \) are scalars in \( \mathbb{F} \) and \( \mathfrak{B} \) is a commutative algebra with basis \( \{ b, a, a^2, a^3, \ldots, a^{n-1} \} \) and products \( ba = 0, a^6 = 0, a^i a^j = a^{i+j} \), for all positive integers \( i, j \), and
\[ b^2 = \alpha a^{n-2} + \beta a^{n-1}, \]
\[ ba^2 = \lambda a^{n-3} + \lambda_1 a^{n-2} + \lambda_2 a^{n-1}, \]
\[ ba^3 = -\lambda a^{n-2} - \lambda_1 a^{n-1}, \]
\[ ba^4 = \lambda a^{n-1}, \]
\[ ba^k = 0, \quad \forall k \geq 5, \]

then \( \mathfrak{B} \) is a power-associative nilalgebra of dimension and nilindex \( n \).

Furthermore, \( \mathfrak{B} \) is Jordan if and only if \( \lambda = 0 = \lambda_1 \).

**Proof.** Because \((a, b) \in \mathcal{P}(\mathfrak{A})\), we know that \( \mathfrak{A}^2 \subset \mathfrak{A}^2, \) \( ba = \lambda_0 a^2, \) \( ba^k \subset \mathfrak{A}^{k+1} \) for \( k = 2, \ldots, n - 3, \) \( ba^{n-2} = 0 \) and \( ba^{n-1} = 0 \). By (i) of Lemma 1 we have that \( \lambda_0 a^{n-1} = a^{n-3}(ba) = ba^{n-2} = 0 \) so that \( \lambda_0 = 0 \) and hence \( ba = 0 \). Also, by (i) of Lemma 1 we have

\[
a^3(ba^2) = a^4(ba) = 0, \]
\[
ba^k = a^{k-1}(ba) = 0 \quad \text{for} \quad k \geq 5. \]

Using identities of (4) we have that \( ba^3 = -a(ba^2) + a^2(ba) = -a(ba^2) \) and \( a^2(ba^2) = ba^4 \). Now

\[
0 = p(a, a, b, b)/4 = a(ab^2) + b(b(a^2)) + 2a(b(ba)) + 2b(a(ba)) - 2a^2b^2 - 4(ab^2) = a(ab^2) - 2a^2b^2 = -a(ab^2) \]

so that \( a(ab^2) = 0 \). This completes the proof of the first part of the lemma.

Reciprocally, let \( x = \xi b + y \) be an elements in \( \mathfrak{B} \), where \( y = \sum_{i=1}^{n-1} \xi_i a^i \).

Then

\[
x^2 \equiv y^2 + 2 \xi_2 \lambda a^{n-3} \mod \langle a^{n-2}, a^{n-1} \rangle, \]
\[
x^3 \equiv y^3 + \xi_1 (\lambda a^{n-3} + \lambda_1 a^{n-2}) \mod \langle a^{n-1} \rangle, \]
\[
x^4 = (x^2)^2 = y^4 + 4 \xi_2 \lambda a^{n-1}, \]
\[
x^5 = y^5 + \xi_1 \lambda a^{n-1}, \]
\[
x^k = y^k \quad \text{for all} \; \; k > 5, \]
\[
x^{n-1} = y^{n-1} = \xi_1 a^{n-1}, \]

so that \( \mathfrak{B} \) is a power-associative nilalgebra of nilindex \( n - 1 \).

Finally, if \( \mathfrak{B} \) is Jordan, then \( 0 = (a^2b)a - a^2(ba) = (a^2b)a = \lambda a^{n-1} + \lambda_1 a^{n-1}, \)

so that \( \lambda = 0 = \lambda_1 \). Reciprocally, if \( \lambda = 0 = \lambda_1 \), then \( (b - \lambda_2 a^{n-3}) \mathfrak{B}^2 = \{0\} \) and Theorem 2.1 of [3] implies that \( \mathfrak{B} \) is Jordan. This completes the proof of the theorem. \[ \square \]
We therefore have the following result.

**Remark 10.** Let $\mathfrak{A}$ be a commutative nilalgebra of dimension and nilindex $n$. Take $(a, b) \in \mathcal{P}(\mathfrak{A})$ and $\lambda \in F$ such that $ab = \lambda a^2$. If $x = \xi b + \sum_{i=1}^{n-1} \xi_i a^i$ is an element of $\mathfrak{A}$, then:

(i) for $n = 5, 6$ we have that $x$ has nilindex $n$ if and only if $\xi_1(\xi_1 + 2\xi \lambda) \neq 0$;

(ii) for $n \geq 7$, we have that $x$ has nilindex $n$ if and only if $\xi_1 \neq 0$.

**References**


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