Existence of Weak Entropy Solution for a Symmetric System of Keyfitz-Kranzer Type

Existencia de una solución débil entrópica para un sistema de tipo Keyfitz-Kranzer simétrico

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Abstract. We consider the Cauchy problem for a 2×2 symmetric system of Keyfitz-Kranzer type with bounded measurable initial data. The existence of a weak entropy solution to this system is proved by using classical viscosity, an estimate in $L^1(\mathbb{R})$ related to one of the Riemann invariants and the div-curl lemma, but avoiding the use of Young measures.

Key words and phrases. System of Keyfitz-Kranzer type, Existence, Weak entropy solution.

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Resumen. Consideramos el problema de Cauchy para un sistema 2×2 de tipo Keyfitz-Kranzer simétrico con valor inicial acotado y medible. La existencia de una solución débil entrópica para este sistema es probada mediante el uso de viscosidad clásica, una $L^1(\mathbb{R})$ estimativa relacionada con uno de los invariantes de Riemann y el lema del divergente-rotacional, pero evitando el uso de medidas de Young.

Palabras y frases clave. Sistema de tipo Keyfitz-Kranzer, existencia, solución débil entrópica.

1. Introduction

A $n \times n$ system of Keyfitz-Kranzer type is a $n \times n$ system of partial differential equations of the following form

$$(u_i)_t + \big(u_i \phi(u_1, \ldots, u_n)\big)_x = 0, \qquad i = 1, \ldots, n. \tag{1}$$

This type of system was first introduced for two equations by Barbara L. Keyfitz and Herbert C. Kranzer in [5] as a model of an elastic string in the plane and was almost one of the first examples of nonstrictly hyperbolic systems. Systems of the form (1) appear in areas as elasticity theory [5] and magnetohydrodynamics [3].

The symmetric system of Keyfitz-Kranzer type where the function ϕ is of the form

$$\phi(u_1, \ldots, u_n) = \phi(r), \qquad r = \sum_{i=1}^{n} u_i^2,$$

has been studied by different authors, see for example [1, 2, 4, 5, 7, 9]. When the symmetric function ϕ is given by

$$\phi(u_1, \ldots, u_n) = \phi(r), \qquad r = \sum_{i=1}^{n} |u_i|,$$

the system (1) is the system of multicomponent chromatography studied in [3].

In connection with the Keyfitz-Kranzer system of two equations, the existence and uniqueness of entropy solution in L^∞ of its Cauchy problem is shown in [14], where besides, an explicit expression of its entropy solution is also provided.

We must underline that the main results presented in this paper were established earlier in more general setting in [12] by E. Yu. Panov, who proved the existence of entropy solution (the existence of a unique strong entropy solution) to the Cauchy problem for multidimensional system (1) with bounded measurable initial data and where $\phi(u_1, \ldots, u_n) = \varphi\big(|(u_1, \ldots, u_n)|\big)$, being $|\bullet|$ the Euclidean norm. The theory for the above problem can be generalized to the case of an arbitrary norm (in place of the Euclidean norm $|\bullet|$), see Remark 4 in [12]. Moreover, The Cauchy problem for (1) with initial function $u_0(x) \in L^\infty(\mathbb{R}; X)$, where X is an arbitrary Banach space, was treated in [13].

Nevertheless, the present paper contains a different approach for justification of the strong convergence of some subsequences of the viscosity solutions based on application of div-curl lemma together with some estimates related with the Riemann invariant $z = \frac{v}{u}$.

This paper is devoted to the study of the Cauchy problem for a 2×2 symmetric system of Keyfitz-Kranzer type, namely

$$\begin{cases} u_t + \big(u\phi(r)\big)_x = 0, \\ v_t + \big(v\phi(r)\big)_x = 0; \end{cases} \tag{2}$$

with initial data

$$\big(u(x,0), v(x,0)\big) = \big(u_0(x), v_0(x)\big), \tag{3}$$

where $u_0(x), v_0(x) \in L^\infty(\mathbb{R})$, ϕ is a nonlinear smooth function and r is given by

$$r = |u|^\alpha + |v|^\alpha, \tag{4}$$

for any $\alpha > 1$ fixed.

We shall give a proof of the global existence of bounded weak solution for the Cauchy problem (2)–(3), using the vanishing viscosity method with the help of the theory of compensated compactness but without involving Young measures.

For the system (2) we have that the Jacobian matrix of the flux functions

$$dF(u,v) = \begin{bmatrix} \phi(r) + \alpha |u|^\alpha \phi'(r) & \alpha |v|^{\alpha-2} uv\phi'(r) \\ \alpha |u|^{\alpha-2} uv\phi'(r) & \phi(r) + \alpha |v|^\alpha \phi'(r) \end{bmatrix}$$

has the two real eigenvalues

$$\lambda_1 = \phi(r) + \alpha r\phi'(r), \qquad \lambda_2 = \phi(r),$$

with corresponding right eigenvectors

$$r_{\lambda_1} = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad r_{\lambda_2} = \begin{bmatrix} -v|v|^{\alpha-2} \\ u|u|^{\alpha-2} \end{bmatrix},$$

the functions

$$z(u,v) = \frac{v}{u}, \qquad w(u,v) = \phi(r), \tag{5}$$

are Riemann invariants for system (2), i.e., they satisfy the equations

$$\left\langle \nabla z, r_{\lambda_1} \right\rangle = 0, \qquad \left\langle \nabla w, r_{\lambda_2} \right\rangle = 0$$

(here $\langle \bullet, \bullet \rangle$ denotes the usual inner product in $\mathbb{R}^2$).

2. Existence of Viscosity Solutions

We now consider the following Cauchy problem for the diffusion system associated with the system (2)

$$\begin{cases} u_t^\epsilon + \left(u^\epsilon \phi(r^\epsilon) \right)_x = \epsilon u_{xx}^\epsilon, \\ v_t^\epsilon + \left(v^\epsilon \phi(r^\epsilon) \right)_x = \epsilon v_{xx}^\epsilon; \end{cases} \tag{6}$$

with initial data (3).

We initially prove the existence of a sequence $\left(u^\epsilon(x,t), v^\epsilon(x,t) \right)$, solutions of problem (6)–(3) on $\mathbb{R} \times \mathbb{R}^+$, uniformly bounded with respect to ϵ. Hereafter we write the functions u, v with an index u^ϵ, v^ϵ only when it avoids confusion with the system (2).

Lemma 1. *For any $\epsilon > 0$, the following a-priori bounds hold for the Cauchy problem* (6)–(3)

$$|u^\epsilon| \le M, \qquad |v^\epsilon| \le M, \tag{7}$$

for a positive constant M independent of ϵ.

Proof. We multiply the first and second equations of system (6) respectively by $\alpha|u|^{\alpha-2}u$ and $\alpha|v|^{\alpha-2}v$, and adding the results, we obtain

$$r_t + \lambda_1 r_x = \epsilon r_{xx} - \epsilon\alpha(\alpha-1)\big(|u|^{\alpha-2}u_x^2 + |v|^{\alpha-2}v_x^2\big). \tag{8}$$

We have from (8) the following inequality

$$r_t + \lambda_1 r_x \le \epsilon r_{xx}. \tag{9}$$

Applying the maximum principle to (9) we get the estimate $r^\epsilon \le N$, where N is a positive constant, being independent of ϵ. Then for u^ϵ and v^ϵ we have the a-priori bounds in (7), which implies the existence of viscosity solutions for the Cauchy problem (6)–(3). ☑

We present a lemma that provides one condition on the initial datum $u_0(x)$ to get the positivity of $u^\epsilon(x,t)$, we give below a simpler direct proof of this fact, proof that we borrow from Bereux and Sainsaulieu [8], [16]. This result will be required in Section 3.

Lemma 2. *If the initial data $u_0(x)$ is such that $u_0(x) \ge c_1 > 0$ for a constant c_1, then we have*

$$u^\epsilon(x,t) \ge c(t,\epsilon,c_1) > 0, \tag{10}$$

where $c(t,\epsilon,c_1)$ could tend to 0 as $t \to +\infty$ or $\epsilon \to 0$.

Proof. Algebraic manipulations on the system (6) give the equalities

$$(\ln u)_t + \phi(r)_x + \phi(r)(\ln u)_x = \epsilon\frac{1}{u}u_{xx}$$
$$= \epsilon(\ln u)_{xx} + \epsilon\big((\ln u)_x\big)^2.$$

We set $\nu = -\ln u$ and deduce from the above equalities that

$$\nu_t - \epsilon\nu_{xx} = -\epsilon(\nu_x)^2 + \phi(r)_x - \phi(r)\nu_x$$
$$= -\epsilon\left(\nu_x + \frac{\phi(r)}{2\epsilon}\right)^2 + \frac{\big(\phi(r)\big)^2}{4\epsilon} + \phi(r)_x$$
$$\le \frac{\big(\phi(r)\big)^2}{\epsilon} + \phi(r)_x.$$

Thus,

$$\nu(x,t) \leq \nu_0(x) * k_\epsilon(x,t) + \int_0^t \left(\tfrac{1}{\epsilon}\big(\phi(r)\big)^2 + \phi(r)_x \right) *_x k_\epsilon(x,t-s)\, ds,$$

where $\nu_0(x) = -\ln u_0^\epsilon(x)$ and here

$$k_\epsilon(x,t) = \frac{1}{\sqrt{4\epsilon\pi t}}\exp\left(-\frac{x^2}{4\epsilon t}\right), \qquad (11)$$

denotes the heat kernel for $\nu_t - \epsilon\nu_{xx}$. Hence

$$\nu(x,t) \leq \nu_0(x) * k_\epsilon(x,t) + \frac{N_1}{\epsilon}t + \int_0^t \phi(r) *_x \big(k_\epsilon(x,s)\big)_x\, ds$$

$$\leq -\ln c_1 + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}},$$

because $u_0^\epsilon(x) \geq c_1 > 0$. Whence

$$u(x,t) \geq c_1 \exp-\left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\right) \geq c(t,\epsilon,c_1) > 0.$$

This proves (10).   ☑

3. Estimates for $z(x,t)$ and $z_x(\cdot,t)$

Let z be the Riemann invariant given in (5). From now on, $z(x,t) = z(u^\epsilon, v^\epsilon)$ and $z_x(\bullet,t)$ denotes the function of x, defined by $z_x(\bullet,t) = z_x(x,t)$. In this section we prove bounds for $z(x,t)$ in $L^\infty(\mathbb{R}\times\mathbb{R}^+)$ and for $z_x(\bullet,t)$ in $L^1(\mathbb{R})$.

Before this, we want to show that u_x is bounded on $\mathbb{R}\times[0,t]$ for all $t > 0$. Remembering that the function u satisfies the integral equation

$$u(x,t) = \int_{-\infty}^{+\infty} k_\epsilon(x-\xi,t)u_0(\xi)\, d\xi\ +$$

$$\int_0^t \int_{-\infty}^{+\infty} u(\xi,\tau)\phi\big(|u(\xi,\tau)|^\alpha + |v(\xi,\tau)|^\alpha\big)k_{\epsilon\xi}(x-\xi,t-\tau)\, d\xi\, d\tau$$

where $k_\epsilon(x,t)$ is the function given by (11), and setting

$$\Delta_h u(x,t) = u(x+h,t) - u(x,t),$$

we then conclude that

$$m_h(t) \leq a + b\int_0^t \frac{m_h(\tau)}{\sqrt{t-\tau}}\, d\tau + c\sqrt{t}, \qquad (12)$$

for $m_h(t) = \sup\limits_{x \in \mathbb{R}} |\Delta_h u(x,t)|$, $a = m_h(0)$ and some positive constants b, c.

For $\beta > 0$, from the inequality (12) we get

$$m_h(t)\exp(-\beta t) \le a + b\int_0^t m_h(\tau)\exp(-\beta\tau)\frac{\exp\big(\beta(\tau - t)\big)}{\sqrt{t-\tau}}\,d\tau + c\sqrt{t}\exp(-\beta t)$$

$$\le A + b\int_0^t m_h(\tau)\exp(-\beta\tau)\frac{\exp\big(\beta(\tau - t)\big)}{\sqrt{t-\tau}}\,d\tau, \qquad (13)$$

which can easily be verified by using the inequality $t^{1/2}\exp(-\beta t) \le C$ for $0 < t < \infty$ (β is any positive number and the constant C depends only on β), here A is a positive constant.

Let $M(t) = \sup\limits_{0 \le \tau \le t} m_h(\tau)\exp(-\beta\tau)$. Then it follows from (13) that

$$M(t) \le A + M(t)b\int_0^t \frac{\exp\big(\beta(\tau - t)\big)}{\sqrt{t-\tau}}\,d\tau = A + M(t)b\int_0^t \frac{\exp\big(-\beta r\big)}{\sqrt{r}}\,dr$$

$$\le A + M(t)b\int_0^{+\infty} \frac{\exp\big(-\beta r\big)}{\sqrt{r}}\,dr. \qquad (14)$$

We derive from (14) that $M(t) \le 2A$ if β is chosen such that

$$b\int_0^{+\infty} \frac{\exp\big(r\big)}{\sqrt{r}}\,dr \le \frac{1}{2}.$$

Therefore

$$m_h(t) \le 2A\exp(\beta t) \le B|h|\exp(\beta t),$$

where B is also constant. Passing to the limit as $h \to 0$, we can derive the estimate

$$|u_x| \le B\exp(\beta t). \qquad (15)$$

Lemma 3. *When the hypotheses of Lemma 2 are satisfied, then the total variation of $z(\bullet, t)$ is a decreasing function of t.*

Proof. Multiplying the first equation of system (6) by $-z$ and adding the result to the second equation, we have

$$z_t + \phi(r)z_x = \epsilon z_{xx} + 2\epsilon\frac{u_x}{u}z_x. \qquad (16)$$

Now we differentiate (16) with respect to x and then we set $\theta = z_x$ to get

$$\theta_t + \big(\phi(r)\theta\big)_x = \epsilon\theta_{xx} + \big(2\epsilon u^{-1}u_x\theta\big)_x.$$

Multiplying this equation by the sequence of smooth functions $g'(\theta, \alpha)$, where α is a parameter, we obtain

$$g(\theta,\alpha)_t + \big(\phi(r)g(\theta,\alpha)\big)_x + \phi(r)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big)$$
$$= \epsilon g(\theta,\alpha)_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon u^{-1}u_x g(\theta,\alpha)\big)_x +$$
$$\big(2\epsilon u^{-1}u_x\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big). \quad (17)$$

If we choose $g(\theta,\alpha)$ such that $g''(\theta,\alpha) \geq 0$, $g'(\theta,\alpha) \to \text{sign}(\theta)$ and $g(\theta,\alpha) \to |\theta|$ as $\alpha \to 0$, we have from (17) that

$$|\theta|_t + \big(\phi(r)|\theta|\big)_x \leq \epsilon|\theta|_{xx} + \big(2\epsilon u^{-1}u_x|\theta|\big)_x. \quad (18)$$

Using (15) we find that the function $u^{-1}u_x$ is bounded on $\mathbb{R} \times [0,t]$ for any $t > 0$. Then by integrating (18) on $\mathbb{R}$, we obtain

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\big|z_x(x,t)\big|\,dx \leq 0, \quad (19)$$

which concludes the proof. ☑

As a corollary of this Lemma we can prove the result announced at the beginning of this section. Also as in [18], the key is that the total variation of the Riemann invariant $z(\bullet,t)$ is decreasing in t.

Corollary 4. *If in addition to the assumption of Lemma 3, the total variation of $z_0(x) = z(x,0)$ is bounded, then $z(x,t) \in L^\infty\big(\mathbb{R} \times \mathbb{R}^+\big)$ and $z_x(\bullet,t) \in L^1(\mathbb{R})$; moreover*

$$TV\big(z(\bullet,t)\big) = \int_{-\infty}^{+\infty}\left|\left(\frac{v^\epsilon}{u^\epsilon}\right)_x(x,t)\right|dx$$
$$\leq \int_{-\infty}^{+\infty}\left|\left(\frac{v_0}{u_0}\right)'(x)\right|dx = TV\big(z_0(x)\big), \quad (20)$$

where TV is the total variation.

Proof. Applying the maximum principle to (16), we find that $z(x,t) \in L^\infty(\mathbb{R}\times \mathbb{R}^+)$. To conclude (20) we use inequality (19), which we integrate from 0 to t. ☑

4. Two Pairs of Entropy-Entropy Flux

A function $\eta = \eta(u,v)$ is called an entropy for the system (2), with entropy flux $q = q(u,v)$ if

$$\nabla q(u,v) = \nabla\eta(u,v)\begin{bmatrix}\phi(r) + \alpha|u|^\alpha\phi'(r) & \alpha|v|^{\alpha-2}uv\phi'(r) \\ \alpha|u|^{\alpha-2}uv\phi'(r) & \phi(r) + \alpha|v|^\alpha\phi'(r)\end{bmatrix}.$$

A pair of functions η, q satisfying the above equation is called an entropy-entropy flux pair and we denote it (η, q).

Two entropy-entropy flux pairs of system (2) are given by

$$(\eta, q) = \left(r, \int^{r} \big(\phi(s) + \alpha s \phi'(s) \big)\, ds \right), \tag{21}$$

$$(\eta, q) = \left(\int^{r} \big(\phi(s) + \alpha s \phi'(s) \big)\, ds, \int^{r} \big(\phi(s) + \alpha s \phi'(s) \big)^2 ds \right), \tag{22}$$

By means of these pairs, we shall obtain the pointwise convergence of a subsequence of $\{r^{\epsilon}(x, t)\}$.

5. H_{loc}^{-1} Compactness

Throughout this section we establish the results related to compactness in H_{loc}^{-1} that allow us to apply the div-curl Lemma in the next section in order to prove for each of the sequences $\{r^{\epsilon}\}$, $\{u^{\epsilon}\}$ and $\{v^{\epsilon}\}$ the pointwise convergence of a subsequence. The first two lemmas given here refer to the two pairs of entropy-entropy flux (21) and (22). The results of the Lemma 7, Lemma 8 and Lemma 10 are possible thanks to the estimate (20) obtained in Corollary 4 of the previous section.

Lemma 5. *We have that*

$$r_t^{\epsilon} + \left(\int^{r^{\epsilon}} \big(\phi(s) + \alpha s \phi'(s) \big)\, ds \right)_x \tag{23}$$

is compact in $H_{loc}^{-1}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. The equation (8) may be written as

$$r_t + \left(\int^{r} \big(\phi(s) + \alpha s \phi'(s) \big)\, ds \right)_x =$$
$$\epsilon r_{xx} - \epsilon \alpha (\alpha - 1) \big(|u|^{\alpha-2} u_x^2 + |v|^{\alpha-2} v_x^2 \big). \tag{24}$$

From (24) it follows that

$$\epsilon \big| u^{\epsilon} \big|^{\alpha-2} \big(u_x^{\epsilon} \big)^2 \quad \text{and} \quad \epsilon \big| v^{\epsilon} \big|^{\alpha-2} \big(v_x^{\epsilon} \big)^2 \quad \text{are bounded in} \quad L_{loc}^1(\mathbb{R} \times \mathbb{R}^+). \tag{25}$$

Thus, $-\epsilon \alpha (\alpha - 1) \big(|u|^{\alpha-2} u_x^2 + |v|^{\alpha-2} v_x^2 \big)$ is bounded in $\mathcal{M}(\mathbb{R} \times \mathbb{R}^+)$ (the space of Radon measures). The bounds in (25) imply that the term ϵr_{xx} is $H_{loc}^{-1}(\mathbb{R} \times \mathbb{R}^+)$ compact, by using Cauchy-Schwarz inequality. The left-hand side of (24) is bounded in $W_{loc}^{-1,\infty}(\mathbb{R} \times \mathbb{R}^+)$. Hence by Murat's Lemma [10], [11], (23) is compact in $H_{loc}^{-1}(\mathbb{R} \times \mathbb{R}^+)$.   ☑

Lemma 6. *The sequence*

$$\left(\int^{r^\epsilon} \big(\phi(s) + \alpha s\phi'(s)\big)\, ds\right)_t + \left(\int^{r^\epsilon} \big(\phi(s) + \alpha s\phi'(s)\big)^2\, ds\right)_x \qquad (26)$$

is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. Multiplying the equation (8) by $\phi(r) + \alpha r\phi'(r)$ we obtain

$$\left(\int^{r^\epsilon} \big(\phi(s) + \alpha s\phi'(s)\big)\, ds\right)_t + \left(\int^{r^\epsilon} \big(\phi(s) + \alpha s\phi'(s)\big)^2\, ds\right)_x =$$
$$\epsilon r_{xx}\big(\phi(r) + \alpha r\phi'(r)\big) -$$
$$\epsilon\alpha(\alpha-1)\Big(|u|^{\alpha-2}u_x^2 + |v|^{\alpha-2}v_x^2\Big)\big(\phi(r) + \alpha r\phi'(r)\big). \quad (27)$$

The left-hand side of the above equation is bounded in $W^{-1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$. Making use of (25), we get that the second term on the right-hand side of (27) is bounded in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$, then this is bounded in $\mathcal{M}(\mathbb{R} \times \mathbb{R}^+)$. It follows from the estimates (25) and from the Cauchy-Schwarz inequality that $\epsilon r_{xx}\big(\phi(r) + \alpha r\phi'(r)\big)$ is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$. We can apply Murat's Lemma and thus conclude that (26) is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$. ☑

Lemma 7. *Suppose the conditions of corollary 4 holds. Then*

$$\big((u^\epsilon)^\alpha\big)_t + \left(\frac{(u^\epsilon)^\alpha}{r^\epsilon} \int^{r^\epsilon} \big(\phi(s) + \alpha s\phi'(s)\big)\, ds\right)_x \qquad (28)$$

is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. Introducing the function $\varphi(x,t)$ defined by

$$\varphi = 1 + \left|\frac{v}{u}\right|^\alpha, \qquad (29)$$

we can write the first equation of system (6) as

$$u_t + u_x\Big(\phi\big(|u|^\alpha\varphi\big) + \alpha|u|^\alpha\varphi\phi'\big(|u|^\alpha\varphi\big)\Big) = \epsilon u_{xx} - |u|^\alpha u\varphi_x\phi'\big(|u|^\alpha\varphi\big). \qquad (30)$$

Multiplying both sides of (30) by $\alpha|u|^{\alpha-2}u$, we obtain

$$\big(|u|^\alpha\big)_t + \big(|u|^\alpha\big)_x\Big(\phi\big(|u|^\alpha\varphi\big) + \alpha|u|^\alpha\varphi\phi'\big(|u|^\alpha\varphi\big)\Big) =$$
$$\epsilon\alpha|u|^{\alpha-2}uu_{xx} - \alpha\big(|u|^\alpha\big)^2\varphi_x\phi'\big(|u|^\alpha\varphi\big), \qquad (31)$$

this equation is equivalent to

$$\left(u^{\alpha}\right)_{t} + \left(\frac{u^{\alpha}}{r}\int^{r}\left(\phi(s) + \alpha s \phi'(s)\right)ds\right)_{x} =$$
$$\epsilon\left(u^{\alpha}\right)_{xx} - \epsilon\alpha(\alpha-1)u^{\alpha-2}u_{x}^{2} - \alpha\left(u^{\alpha}\right)^{2}\varphi_{x}\phi'\left(u^{\alpha}\varphi\right), \quad (32)$$

where we have used that $u^{\epsilon} > 0$ and that

$$\int^{u^{\alpha}}\left(\phi(s\varphi) + \alpha s \varphi \phi'(s\varphi)\right)ds = \frac{u^{\alpha}}{r}\int^{r}\left(\phi(s) + \alpha s \phi'(s)\right)ds.$$

By Corollary 4, $\varphi(\bullet,t)_{x}$ is bounded in $L^{1}(\mathbb{R})$; this implies that the third term on the right-hand side of (32) is bounded in $L_{\text{loc}}^{1}(\mathbb{R}\times\mathbb{R}^{+})$. It follows from (32) that

$$\epsilon\left(u^{\epsilon}\right)^{\alpha-2}\left(u_{x}^{\epsilon}\right)^{2} \quad \text{is bounded in} \quad L_{\text{loc}}^{1}(\mathbb{R}\times\mathbb{R}^{+}), \quad (33)$$

since $\varphi(\bullet,t)_{x}$ is bounded in $L^{1}(\mathbb{R})$. Thus, these last two terms are bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^{+})$. Using the estimate (33), we can prove that the first term on the right is $H_{\text{loc}}^{-1}(\mathbb{R}\times\mathbb{R}^{+})$ compact. In addition, the left-hand side of (32) is bounded in $W_{\text{loc}}^{-1,\infty}(\mathbb{R}\times\mathbb{R}^{+})$. So, we are in a position to apply Murat's lemma to see that (28) is compact in $H_{\text{loc}}^{-1}(\mathbb{R}\times\mathbb{R}^{+})$.   ☑

Lemma 8. *If conditions in corollary 4 are satisfied, then*

$$u_{t}^{\epsilon} + \left(u^{\epsilon}\phi\left(r^{\epsilon}\right)\right)_{x} \quad (34)$$

is compact in $H_{\text{loc}}^{-1}(\mathbb{R}\times\mathbb{R}^{+})$.

Proof. Multiply (30) by $2u$ to obtain

$$\left(u^{2}\right)_{t} + \left(u^{2}\right)_{x}\left(\phi\left(|u|^{\alpha}\varphi\right) + \alpha|u|^{\alpha}\varphi\phi'\left(|u|^{\alpha}\varphi\right)\right) =$$
$$\epsilon\left(u^{2}\right)_{xx} - 2\epsilon u_{x}^{2} - 2|u|^{\alpha+2}\varphi_{x}\phi'\left(|u|^{\alpha}\varphi\right). \quad (35)$$

The equation (35) shows that

$$\epsilon\left(u_{x}^{\epsilon}\right)^{2} \quad \text{is bounded in} \quad L_{\text{loc}}^{1}(\mathbb{R}\times\mathbb{R}^{+}). \quad (36)$$

Indeed, here one makes use of the two estimates given in Corollary 4.

According to first equation of system (6), we have

$$u_{t} + \left(u\phi(r)\right)_{x} = \epsilon u_{xx}. \quad (37)$$

One can show that ϵu_{xx} is compact in $H_{\text{loc}}^{-1}(\mathbb{R}\times\mathbb{R}^{+})$. To this end we use Cauchy-Schwarz inequality together with the estimate (36). Hence, in view of the Murat's Lemma, we conclude that (34) is compact in $H_{\text{loc}}^{-1}(\mathbb{R}\times\mathbb{R}^{+})$.   ☑

As a corollary of Lemma 8 one can prove the following result.

Corollary 9. *With the assumptions given in Corollary 4, it follows that*

$$u_t^\epsilon + \left(u^\epsilon \phi\big(r^\epsilon\big) + \frac{v^\epsilon}{u^\epsilon} \right)_x \tag{38}$$

is compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. Due to the compactness of (34) in $H_{\mathrm{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$, it is sufficient to show that $\left(\frac{v^\epsilon}{u^\epsilon}\right)_x$ is also compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$, which is true since $\left(\frac{v^\epsilon}{u^\epsilon}\right)_x$ is bounded in $L_{\mathrm{loc}}^1(\mathbb{R}\times\mathbb{R}^+)$ and in $W_{\mathrm{loc}}^{-1,\infty}(\mathbb{R}\times\mathbb{R}^+)$. This concludes the proof. ☑

Lemma 10. *Assuming the hypotheses as in Corollary 4, then*

$$v_t^\epsilon + \big(v^\epsilon \phi\big(r^\epsilon\big)\big)_x \tag{39}$$

is compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. We begin by multiplying the first equation in system (6) by $\left(\frac{v}{u}\right)^2$ and (16) by $2u\left(\frac{v}{u}\right)$. Adding the above results, we obtain

$$\left(u\left(\frac{v}{u}\right)^2 \right)_t + \left(u\left(\frac{v}{u}\right)^2 \phi \right)_x = \epsilon\left(u\left(\frac{v}{u}\right)^2 \right)_{xx} - 2\epsilon u\left(\frac{v}{u}\right)_x^2. \tag{40}$$

From (40) it would follow that

$$\epsilon u^\epsilon \left(\frac{v^\epsilon}{u^\epsilon} \right)_x^2 \quad \text{is bounded in} \quad L_{\mathrm{loc}}^1(\mathbb{R} \times \mathbb{R}^+). \tag{41}$$

The second equation of system (6) may written in the form

$$v_t + \big(v\phi(r)\big)_x = \epsilon\left(u\frac{v}{u} \right)_{xx} = \epsilon\left(u_x\left(\frac{v}{u}\right) + u\left(\frac{v}{u}\right)_x \right)_x. \tag{42}$$

We now claim that

$$\epsilon\left(u_x\left(\frac{v}{u}\right) \right)_x \quad \text{and} \quad \epsilon\left(u\left(\frac{v}{u}\right)_x \right)_x \quad \text{are compact in} \quad H_{\mathrm{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+).$$

To see this, we use respectively the estimates (36) and (41), together with the Cauchy-Schwarz inequality. Hence by Murat's Lemma, the sequence $v_t^\epsilon + \big(v^\epsilon \phi(r^\epsilon)\big)_x$ is compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$. ☑

A consequence of the previous Lemma is the next Corollary.

Corollary 11. *Suppose the conditions of Corollary 4. Then we have that*

$$v_t^\epsilon + \left(v^\epsilon \phi(r^\epsilon) + \left(\frac{v^\epsilon}{u^\epsilon} \right)^2 \right)_x \tag{43}$$

is compact in $H_{\text{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. Being $\left(\left(\frac{v^\epsilon}{u^\epsilon} \right)^2 \right)_x$ bounded in both $L_{\text{loc}}^1(\mathbb{R} \times \mathbb{R}^+)$ and $W_{\text{loc}}^{-1,\infty}(\mathbb{R} \times \mathbb{R}^+)$, we deduce that $\left(\left(\frac{v^\epsilon}{u^\epsilon} \right)^2 \right)_x$ is compact in $H_{\text{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$. So by Lemma 10 the conclusion in this Corollary holds. ☑

6. Results on Convergence

We will omit subscripts on subsequences and let any subsequence of the sequence $\{u^\epsilon\}$ be denoted by $\{u^\epsilon\}$. In particular, whenever speaking of convergence of $\{u^\epsilon\}$, we really mean convergence of some subsequence.

In order to show that the sequence $\{r^\epsilon(x,t)\}$ converges pointwise, we use only the two pairs of entropy-entropy flux (21) and (22) together with the div-curl Lemma.

Lemma 12. *When the function $\phi(r) \in C^2(\mathbb{R}^+)$ and*

$$meas\{r : (\alpha + 1)\phi'(r) + \alpha r \phi''(r) = 0\} = 0, \tag{44}$$

then there exists a subsequence of $\{r^\epsilon(x,t)\}$ which converges pointwisely.

Proof. Due to Lemmas 5 and 6, the div-curl Lemma can be applied to (21) and (22); this yields to the equation

$$\overline{r^\epsilon \int_k^{r^\epsilon} f'^2(s)\,ds - f^2(r^\epsilon)} = \overline{r^\epsilon} \, \overline{\int_k^{r^\epsilon} f'^2(s)\,ds} - \overline{f(r^\epsilon)}^2, \tag{45}$$

where k is real constant and $f(r^\epsilon) = \int^{r^\epsilon} \left(\phi(s) + \alpha s \phi'(s) \right) ds$. Here the overline denotes the weak-star limit (i.e. $\overline{r^\epsilon} = w^\star - \lim r^\epsilon$). Let $\overline{r^\epsilon} = r$, we first notice that

$$\overline{r^\epsilon \int_k^{r^\epsilon} f'^2(s)\,ds - f^2(r^\epsilon)} = \overline{r^\epsilon \int_r^{r^\epsilon} f'^2(s)\,ds - \left(f(r^\epsilon) - f(r) \right)^2}$$
$$+ \overline{r^\epsilon} \int_k^r f'^2(s)\,ds + \left(\overline{f(r^\epsilon)} - f(r) \right)^2 - \overline{f(r^\epsilon)}^2, \quad (46)$$

and

$$\overline{r^\epsilon} \overline{\int_k^{r^\epsilon} f'^2(s)\,ds} = \overline{r^\epsilon} \overline{\int_r^{r^\epsilon} f'^2(s)\,ds} + \overline{r^\epsilon} \int_k^r f'^2(s)\,ds. \tag{47}$$

Using the above equalities in (45), we obtain

$$\overline{(r^\epsilon - r) \int_r^{r^\epsilon} f'^2(s)\, ds - \big(f(r^\epsilon) - f(r)\big)^2} + \Big(\overline{f(r^\epsilon)} - f(r)\Big)^2 = 0. \qquad (48)$$

But on the other hand, by Cauchy-Schwarz inequality

$$\big(f(r^\epsilon) - f(r)\big)^2 = \left(\int_r^{r^\epsilon} f'(s)\, ds\right)^2 \le (r^\epsilon - r) \int_r^{r^\epsilon} f'^2(s)\, ds.$$

Therefore, the weak-star limit of $(r^\epsilon - r) \int_r^{r^\epsilon} f'^2(s)\, ds - \big(f(r^\epsilon) - f(r)\big)^2$ is nonnegative. Since both terms in the left-hand of Equation (45) are nonnegative, we deduce

$$\overline{(r^\epsilon - r) \int_r^{r^\epsilon} f'^2(s)\, ds - \big(f(r^\epsilon) - f(r)\big)^2} = 0. \qquad (49)$$

From (49) we get the conclusion in this Lemma (see the argument used in the proof of Theorem 3.1.1 in [8]). ☑

On the basis of the Lemmas 7 and 8, we can now establish the following result.

Lemma 13. *Assume that in addition to the hypotheses of the Lemmas 4 and 12, $\phi(r)$ is strictly increasing or decreasing for positive r. Then there is a subsequence of $\{u^\epsilon(x,t)\}$ which converges pointwisely.*

Proof. Applying again the div-curl Lemma to the functions (28) and (34), we get

$$\overline{(u^\epsilon)^{\alpha+1}\left(\frac{1}{r^\epsilon} \int_0^{r^\epsilon} \big(\phi(s) + \alpha s \phi'(s)\big)\, ds - \phi(r^\epsilon)\right)} =$$
$$\overline{u^\epsilon}\, \overline{\frac{(u^\epsilon)^\alpha}{r^\epsilon} \int_0^{r^\epsilon} \big(\phi(s) + \alpha s \phi'(s)\big)\, ds} - \overline{(u^\epsilon)^\alpha}\, \overline{u^\epsilon \phi(r^\epsilon)}. \qquad (50)$$

By Lemma 12 we may extract a subsequence of $\{r^\epsilon(x,t)\}$ (still denoted $\{r^\epsilon(x,t)\}$) which converges pointwisely. Let $r^\epsilon(x,t) \to R(x,t)$ (strong). Using this fact in (50) it follows that

$$\left(\frac{1}{R} \int_0^R \big(\phi(s) + \alpha s \phi'(s)\big)\, ds - \phi(R)\right)\left(\overline{(u^\epsilon)^{\alpha+1}} - \overline{u^\epsilon}\, \overline{(u^\epsilon)^\alpha}\right) = 0. \qquad (51)$$

By the condition on ϕ we conclude that $\frac{1}{r} \int_0^r \big(\phi(s) + \alpha s \phi'(s)\big)\, ds - \phi(r) = 0$ only on $r = 0$. According to (51) we obtain that

$$\overline{(u^\epsilon)^{\alpha+1}} - \overline{u^\epsilon}\, \overline{(u^\epsilon)^\alpha} = 0, \qquad (52)$$

equation from which we get the pointwise convergence of $\{u^\epsilon\}$ in the region $r > 0$.   ☑

Lemma 14. *If the conditions of Lemma 13 are fulfilled, then there is a subsequence of $\{v^\epsilon\}$ such that it converges pointwise.*

Proof. It now follows from the div-curl Lemma applied to the functions (38) and (43) that

$$\overline{u^\epsilon\, u^\epsilon \frac{v^\epsilon}{u^\epsilon}\phi(r^\epsilon) + \left(\frac{v^\epsilon}{u^\epsilon}\right)^2} \;-\; \overline{u^\epsilon\phi(r^\epsilon) + \frac{v^\epsilon}{u^\epsilon}u^\epsilon\frac{v^\epsilon}{u^\epsilon}} \;=\; 0. \tag{53}$$

Combining (53) and the strong convergence of the sequences $\{\phi(r^\epsilon)\}$ and $\{u^\epsilon\}$ we find that

$$\overline{u^\epsilon}\left(\overline{\left(\frac{v^\epsilon}{u^\epsilon}\right)^2} - \overline{\left(\frac{v^\epsilon}{u^\epsilon}\right)}^2\right) = 0, \tag{54}$$

which implies the pointwise convergence of $\left\{\frac{v^\epsilon}{u^\epsilon}\right\}$ on the region $u > 0$, and therefore the convergence of $\{v^\epsilon\}$.   ☑

7. Existence of Weak Entropy Solution

The weak entropy solutions we address are defined in the following sense.

We say that a pair of functions $\big(u(x,t), v(x,t)\big) \in L^\infty \times L^\infty$ is a weak entropy solution of (2)–(3) if (u,v) is a weak solution, i.e.,

$$\int_{\mathbb{R}} \int_0^{+\infty} \big(u\varphi_t + u\phi(r)\varphi_x\big)\, dt\, dx + \int_{\mathbb{R}} u_0\varphi(x,0)\, dx$$

$$+ \int_{\mathbb{R}} \int_0^{+\infty} \big(v\psi_t + v\phi(r)\psi_x\big)\, dt\, dx + \int_{\mathbb{R}} v_0\psi(x,0)\, dx = 0, \quad (55)$$

for any $\varphi, \psi \in C_0^\infty\big(\mathbb{R} \times [0,\infty)\big)$ and the entropy inequality

$$\int_{\mathbb{R}} \int_0^{+\infty} \big(\eta(u,v)\varphi_t + q(u,v)\varphi_x\big)\, dt\, dx \geq 0, \tag{56}$$

holds for any nonnegative function $\varphi \in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$ and any convex entropy-entropy flux pair $\big(\eta(u,v), q(u,v)\big)$.

Lax [6] proves that if the solution u^ϵ of the Cauchy problem for the parabolic system $(\epsilon > 0)$

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \quad u(x,t) \in \mathbb{R}^n$$

satisfies certain a-priori estimates and converges almost everywhere to a limit u, then the function u must satisfy the inequality

$$\int_{\mathbb{R}} \int_0^{+\infty} \big(\eta(u)\varphi_t + q(u)\varphi_x\big)\, dt\, dx \geq 0, \tag{57}$$

for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ and any entropy-entropy flux pair $(\eta, q) \in C^2$, with η convex.

For a system of two strictly hyperbolic conservation laws, the rigourous proof of this derivation has been given by Di Perna [15] (see also Tartar [19], Rascle [17]).

As a consequence of this fact we may now arrive at the next result.

Theorem 15. *Let $\phi(r) \in C^2(\mathbb{R}^+)$ such that $\phi(r)$ is strictly increasing or decreasing for positive r and $\text{meas}\{r : (2n+1)\phi'(r) + 2nr\phi''(r) = 0\} = 0$. We assume that $u_0(x) \geq c > 0$ and that $\left(\frac{v_0}{u_0}\right)'(x) \in L^1(\mathbb{R})$, where c is a constant. Then the Cauchy problem (2)-(3) has a weak entropy solution.*

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