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Gelfand-Kirillov Dimension of Skew *PBW* Extensions

Dimensión de Gelfand-Kirillov de las extensiones *PBW* torcidas

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Dedicated to my dear professor Alexander Zavadskij

Abstract. Gelfand-Kirillov dimension of Poincaré-Birkhoff-Witt (PBW for short) extensions was established by Matczuk ([15], Theorem A). Since PBW extensions are a particular example of skew PBW extensions (also called $\sigma - PBW$ extensions), the aim of this paper is to compute this dimension for these extensions and hence generalize Matczuk's results for several algebras which can not be classified as PBW extensions.

Key words and phrases. Non-commutative algebras, Filtered and graded rings, PBW extensions, Skew quantum polynomials, Gelfand Kirillov dimension.

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Resumen. La dimensión de Gelfand-Kirillov de las extensiones de Poincaré-Birkhoff-Witt (abreviadas PBW) fue establecida por Matczuk ([15] Theorem A). Dado que las extensiones PBW son un ejemplo particular de las extensiones PBW torcidas (también llamadas extensiones $\sigma - PBW$), el objetivo de este artículo es calcular esta dimensión para dichas extensiones y así generalizar los resultados de Matczuk para varias álgebras que no pueden ser clasificadas como extensiones PBW.

Palabras y frases clave. Álgebras no conmutativas, anillos filtrado graduados, extensiones PBW, polinomios cuánticos torcidos, dimensión de Gelfand-Kirillov.

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1. Introduction

Originated in 2011 in the work of Gallego and Lezama [5], skew PBW extensions are a generalization of PBW extensions introduced by Bell and Goodearl [2] in 1988. These extensions defined in algebraic terms by generators and a list of commutation relations allow to study a considerable number of non-commutative rings of polynomial type. Skew PBW extensions include PBW extensions and many other algebras of interest for modern mathematical physicists which are not PBW extensions. Some of these algebras are group rings of polycyclic-by-finite groups, Ore algebras, operator algebras, diffusion algebras, quantum algebras, quadratic algebras in 3 variables, Clifford algebras among many others. For some remarkable examples of skew PBW extensions probably its Gelfand-Kirillov dimension have not been computed before. Indeed, for some particular non-commutative rings considered in this work several properties are probably known.

In this Section we recall the definition of skew PBW extensions presented in [5] and we establish some key properties of this kind of non-commutative rings. The content and proofs of this introductory Section can be found in [5] and [13].

Definition 1. Let R and A be rings. We say that A *is a skew* PBW *extension of* R (also called a $\sigma - PBW$ extension of R) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finite elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis

$$\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case we say that A *is a left polynomial ring over* R with respect to $\{x_1, \dots, x_n\}$ and $\mathrm{Mon}(A)$ is the set of standard monomials of A. In addition, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n\rangle$.

The following Proposition justifies the notation and the alternative name given for the skew PBW extensions.

Proposition 2. ([5, Proposition 3]) *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$ for each $r \in R$.*

Proof. We follow the proof presented in [5]. For each $1 \leq i \leq n$ and all $r \in R$, we have elements $c_{i,r}, r_i \in R$ with $x_i r = c_{i,r}x_i + r_i$. Since $\mathrm{Mon}(A)$ is a left R-basis of A, it follows that $c_{i,r}$ and r_i are unique for r. Hence we define $\sigma_i, \delta_i : R \to R$ by $\sigma_i(r) := c_{i,r}$, $\delta_i(r) := r_i$. We can check that σ_i is an endomorphism and δ_i is a σ_i-derivation of R, i.e., $\delta_i(r+r') = \delta_i(r) + \delta_i(r')$ and $\delta_i(rr') = \sigma_i(r)\delta_i(r') + \delta_i(r)r'$, for any elements $r, r' \in R$. By Definition 1 (iii), $c_{i,r} \neq 0$ for $r \neq 0$, which shows that σ_i is injective for all i. ☑

A particular case of skew PBW extension is considered when all derivations δ_i are zero. If all σ_i are bijective another interesting case is presented. We recall the following definition (cf. [5].)

Definition 3. Let A be a skew PBW extension.

(a) A is *quasi-commutative* if the conditions (iii) and (iv) in Definition 1 are replaced by

 (iii') For every $1 \leq i \leq n$ and $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that
$$x_i r = c_{i,r}x_i.$$

 (iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that
$$x_j x_i = c_{i,j}x_i x_j.$$

(b) A is *bijective* if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Skew PBW extensions can be characterized in a similar way as left PBW rings in [3, Proposition 2.4].

Theorem 4. ([5, Theorem 7]) *Let A be a left polynomial ring over R with respect to $\{x_1, \ldots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions hold:*

(a) *For every $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$ such that*
$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{3}$$
where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, then r_α is left invertible.

(a) *For every* $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$ *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that*

$$x^{\alpha}x^{\beta} = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{4}$$

where $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$ *or* $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ *if* $p_{\alpha,\beta} \neq 0$.

We remember also the following facts from [5, Remark 8].

***Remark* 5.**

(i) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c'_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c'_{\alpha,\beta} = 1$.

(ii) Let $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$. Then we have the following identities:

$$\sigma^{\theta}(c_{\gamma,\beta})c_{\theta,\gamma+\beta} = c_{\theta,\gamma}c_{\theta+\gamma,\beta},$$
$$\sigma^{\theta}\big(\sigma^{\gamma}(c)\big)c_{\theta,\gamma} = c_{\theta,\gamma}\sigma^{\theta+\gamma}(c).$$

(iii) We observe that if A is quasi-commutative then from the proof of Theorem 4 we conclude that $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$ for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}^n$.

(iv) From the proof of Theorem 4 we get also that if A is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

Next we present some key results proved in [13]. We start with a proposition that establishes that one can construct a quasi-commutative skew PBW extension from a given skew PBW extension of a ring R.

Proposition 6. *Let A be a skew PBW extension of R. Then there exists a quasi-commutative skew PBW extension A^{σ} of R in n variables $z_1, \dots, z_n$ defined by*

$$z_j r = c_{j,r}z_j, \qquad z_j z_i = c_{i,j}z_i z_j, \qquad 1 \leq i, j \leq n,$$

where $c_{j,r}$, $c_{i,j}$ are the same constants that define A. Moreover, if A is bijective then A^{σ} is also bijective.

***Proof*.** We follow the proof presented in [13]. Consider variables $z_1, \dots, z_n$ and the set of standard monomials $\mathcal{M} := \{z_1^{\alpha_1} \cdots z_n^{\alpha_n} : \alpha_i \in \mathbb{N}^n,\ 1 \leq i \leq n\}$. Let A^{σ} be the free R-module with basis $\mathcal{M}$ (i.e., A and A^{σ} are isomorphic R-modules). We define the product in A^{σ} by the distributive law and the rules

$$rz^{\alpha}sz^{\beta} := r\sigma^{\alpha}(s)c_{\alpha,\beta}z^{\alpha+\beta},$$

where the σ's and the constants c's are as in Theorem 4. The identities of Remark 5 show that this product is associative. Moreover, note that $R \subseteq A^{\sigma}$ since for $r \in R$, $r = rz_1^0 \cdots z_n^0$. Thus, A^{σ} is a quasi-commutative skew

PBW extension of R, and also, each element f^σ of A^σ corresponds to a unique element $f \in A$, when the variables x's are replaced by the variables z's. The last assertion of the proposition is obvious. Therefore, $A^\sigma \cong R[z_1;\sigma_1]\cdots[z_n;\sigma_n]$ where $\sigma_j(r) = c_{j,r}$, $\sigma_j(z_i) = c_{i,j}z_i$ for $r \in R$ and $1 \leq i < j \leq n$. ☑

An important fact for this work is that skew PBW extensions are filtered rings. We recall the definition of these rings.

Definition 7. A *filtered ring* is a ring B with a family $FB = \{F_nB : n \in \mathbb{Z}\}$ of additive subgroups of B where we have the ascending chain $\cdots \subset F_{n-1}B \subset F_nB \subset \cdots$ such that $1 \in F_0B$ and $F_nBF_mB \subseteq F_{n+m}B$ for all $n, m \in \mathbb{Z}$.

From a filtered ring B it is posible to construct its associated graded ring $G(B)$ taking $G(B)_n := F_nB/F_{n-1}B$. It is sufficient to consider the multiplication in $G(B)$ on homogeneous elements. If $a \in F_nB/F_{n-1}B$, it says that a has *degree* n, and $\overline{a} = a + F_{n-1}B \in G(B)_n$ is the *leading term* of a. If c has degree m, then $\overline{a}\,\overline{c}$ is defined as $ac + F_{m+n-1}B \in G_{m+n}B$. This multiplication is well defined and hence $G(S)$ is effectively a ring, which is known in the literature as the *associated graded ring* of B.

The first key theorem establishes the graduation of a general skew PBW extension of a ring R.

Theorem 8. *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by*

$$F_mA := \begin{cases} R, & \text{if} \quad m = 0; \\ \{f \in A : \deg(f) \leq m\}, & \text{if} \quad m \geq 1; \end{cases} \tag{5}$$

and the corresponding graded ring $G(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $G(A)$ is a quasi-commutative bijective skew PBW extension of R.

The next theorem characterizes the quasi-commutative skew PBW extensions.

Theorem 9. *Let A be a quasi-commutative skew PBW extension of a ring R. Then,*

(i) A is isomorphic to an iterated skew polynomial ring of endomorphism type.

(ii) If A is bijective, then each endomorphism is bijective.

2. Gelfand-Kirillov Dimension

For finitely generated $\Bbbk$-algebras B, there exists the Gelfand-Kirillov dimension denoted by $\mathrm{GKdim}(B)$, which is an invariant and coincides with the Krull dimension in the commutative case. Algebras with Gelfand-Kirillov dimension zero are precisely those finite dimensional. Since this dimension applies only to algebras over a field $\Bbbk$, throughout this section, R is affine, that is, R is finitely generated as $\Bbbk$-algebra and all automorphisms and derivations are $\Bbbk$-linear. We recall that a filtration $FB = \{F_nB : n \in \mathbb{Z}\}$ of a $\Bbbk$-algebra B is said to be *finite* if each F_iB is a finite dimensional $\Bbbk$-subspace.

It is known that if δ is a derivation of an $\Bbbk$-algebra R for a field $\Bbbk$, then the Gelfand-Kirillov dimension GKdim of the ring of derivation type $R[x;\delta]$ is equal to $\mathrm{GKdim}(R)+1$, provided that R is finitely generated [10]. Generalization of this result was established by Matczuk [15, Theorem A] for Poincaré-Birkhoff-Witt extensions introduced by Bell and Goodearl [2] over finitely generated algebras. More exactly, Matczuk showed that if R is an affine $\Bbbk$-algebra and A is a PBW extension of R, then $\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + n$. This result generalizes [16, Proposition 8.2.10]. In this Section we generalize the Matczuk's result for skew PBW extensions of a $\Bbbk$-algebra R being R finitely generated or with locally algebraic automorphisms. We start recalling the definition of Gelfand-Kirillov dimension.

Definition 10. Let B be an affine $\Bbbk$-algebra with finite generating set given by $\{b_1,\dots,b_n\}$. Let V be a finite dimensional subspace of B. V is called a *finite dimensional generating subspace* for B if we can express every element of B as a linear combination of monomials formed by elements of V.

An example is the case where V is the subspace of B spanned by the generators $b_1,\dots,b_n$. If we set $V^0 := \Bbbk$ and $V^n :=$ the subspace spanned by monomials of the form $b_{i_1}^{l_1}\cdots b_{i_m}^{l_m}$, $b_{i_j}\in\{b_1,\cdots,b_m\}$ and $\sum_{i=1}^m l_i = n$, we have $B_n = \sum_{i=0}^n V^i$ and $B = \bigcup_{n=0}^\infty B_n$. Define $d_V(n) := \dim_{\Bbbk}(B_n)$. GKdim is a measure of the rate of growth of the algebra in terms of any generating set. More exactly

Definition 11. The *Gelfand-Kirillov* dimension of B is

$$\mathrm{GKdim}(B) := \overline{\lim}\left(\frac{\log d_V(n)}{\log(n)}\right)$$

for a finite dimensional generating subspace V of B.

The Gelfand-Kirillov dimension of the algebra B is independent of the choice of V. For details about Gelfand-Kirillov dimension see [10] or [16].

We need two preliminary results.

Proposition 12. ([10, Proposition 6.6]) *Let B be a $\Bbbk$-algebra with a finite filtration $\{B_i\}_{i\in\mathbb{Z}}$ such that $G(B)$ is finitely generated. Then*

$$\mathrm{GKdim}\left(G(B)_{G(B)}\right) = \mathrm{GKdim}(B_B).$$

Lemma 13. ([9, Lemma 2.2]) *Let B a $\Bbbk$-algebra with a finite dimensional generating subspace V, σ a $\Bbbk$-automorphism of B and δ a σ-derivation. If $\sigma(V) \subseteq V$, then*

$$\mathrm{GKdim}(B[x;\sigma,\delta]) = \mathrm{GKdim}(B) + 1.$$

Proof. Briefly, the idea presented in [9] is the following. We may assume that $1 \in V$. Since $\bigcup_{k=0}^{\infty} V^k = B$ and $\delta(V)$ is finite dimensional, there exists a positive integer m such that $\delta(V) \subset V^m$. Then, by induction on $n \geq 1$, we have $\delta(V^n) \subset V^{m+n}$ for all n. If $W := \Bbbk x \oplus V$, then W is a finite dimensional generating subspace of $B[x;\sigma,\delta]$. One can show that $W^n \subset \sum_{k=0}^{n} V^{mn}x^k$ for all n. Since the sum $\sum_{k=0}^{n} V^{mn}x^k$ is direct, the definition of Gelfand-Kirillov dimension implies that $\mathrm{GKdim}(B[x;\sigma,\delta]) = \mathrm{GKdim}(B) + 1$. ☑

Next we formulate one of the main results in this section. Consider the automorphism σ_n of R in Proposition 2.

Theorem 14. *Let R be a $\Bbbk$-algebra with a finite dimensional generating subspace V and let A be a bijective skew PBW extension of R given by $A = \sigma(R)\langle x_1,\dots,x_n\rangle$. If $\sigma_n(V) \subseteq V$, then*

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + n.$$

Proof. From Theorem 8 it is clear that A is a $\Bbbk$-algebra with a finite filtration. Let X the $\Bbbk$-linear subspace of A spanned by $1, x_1, \dots, x_n$. Then VX is a finite dimensional generating subspace of $G(A) \cong A^{\sigma}$ and hence Proposition 12 implies $\mathrm{GKdim}(A) = \mathrm{GKdim}(G(A))$. Now, from Theorem 5 and Theorem 9 we have that the ring A^{σ} is isomorphic to the skew polynomial ring of automorphism type $R[x_1;\sigma_1]\cdots[x_n;\sigma_n]$. Note that $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ is a $\Bbbk$-algebra and the automorphism σ_n of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ given by $\sigma_n(r) = c_{n,r}$ and $\sigma_n(x_i) = c_{i,n}x_i$ for $r \in R$, $1 \leq i < n$ is a $\Bbbk$-automorphism. If X' is the $\Bbbk$-linear subspace of A spanned by $1, x_1, \dots, x_{n-1}$, then VX' is a finite dimensional generating subspace of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ and, from the assumption that $\sigma_n(V) \subseteq V$, it follows that $\sigma_n(VX') \subseteq VX'$. Lemma 13 guarantees $\mathrm{GKdim}(A) = \mathrm{GKdim}(G(A)) = \mathrm{GKdim}(R) + n$. ☑

***Remark* 15.** Theorem 14 generalizes [15, Theorem A], which established the result above for classic PBW extensions.

Definition 16 ([11] or [18])**.** For a $\Bbbk$-algebra B, an automorphism σ of B is said to be *locally algebraic* if for any $b \in B$ the set $\{\sigma^m(b) : m \in \mathbb{N}\}$ is contained in a finite dimensional subspace of B.

We remark a useful result about rings with a locally algebraic automorphism
.

Lemma 17. ([11, Proposition 1]) *If σ is a* locally algebraic *automorphism of a $\Bbbk$-algebra B, we have* $\mathrm{GKdim}(B[x;\sigma]) = \mathrm{GKdim}(B)+1 = \mathrm{GKdim}\left(B\left[x^{\pm 1};\sigma\right]\right)$.

Next we formulate another result of this Section. Again, consider the automorphism σ_n of R in Proposition 2.

Theorem 18. *Let R be a $\Bbbk$-algebra with a finite dimensional generating subspace V and let A be a bijective skew PBW extension of R given by $A = \sigma(R)\langle x_1,\dots,x_n\rangle$. If σ_n is locally algebraic, then*

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(R) + n.$$

Proof. From Theorem 14 we know that A is a $\Bbbk$-algebra with a finite filtration and that VX is a finite dimensional generating subspace of $A^{\sigma} \cong G(A)$ which implies $\mathrm{GKdim}(A) = \mathrm{GKdim}(G(A))$. We also know that the ring A^{σ} is isomorphic to the skew polynomial ring of automorphism type $R[x_1;\sigma_1]\cdots[x_n;\sigma_n]$ and that the ring $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ is a $\Bbbk$-algebra and the function σ_n of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$ given by $\sigma_n(r) = c_{n,r}$ and $\sigma_n(x_i) = c_{i,n}x_i$ for $r \in R$, $1 \le i < n$ is a $\Bbbk$-automorphism. Hence, if X' is the $\Bbbk$-linear subspace of A spanned by $1, x_1, \dots, x_{n-1}$, then VX' a finite dimensional generating subspace of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$. It is easy to show that.

$$\sigma_n^m(x_i) = \left[\prod_{t=0}^{m-1} \sigma_n^{m-1-t}(c_{n,i})\right] x_i, \quad 1 \le i < n, m \in \mathbb{N}. \tag{6}$$

By assumption, the automorphism σ_n of R is locally algebraic so (6) implies that σ_n, considered as an automorphism of $R[x_1;\sigma_1]\cdots[x_{n-1};\sigma_{n-1}]$, is locally algebraic. By Lemma 17 we conclude $\mathrm{GKdim}(G(A)) = \mathrm{GKdim}(R) + n$. ☑

***Remark* 19.**

(1) Gelfand-Kirillov dimensions in the literature (cf. [3], [10] and [16, Proposition 8.2.7]) agree with Theorems 14 and 18. For instance, the following Gelfand-Kirillov dimensions are well known:

   (a) $\mathrm{GKdim}\left(R[x]\right) = \mathrm{GKdim}(R) + 1$;
   (b) $\mathrm{GKdim}\left(\Bbbk[x_1,\dots,x_n]\right) = n$;
   (c) $\mathrm{GKdim}\left(\mathcal{O}_q(\Bbbk^2)\right) = 2$;
   (d) $\mathrm{GKdim}\left(A_n(R)\right) = \mathrm{GKdim}(R) + n$,
   (e) $\mathrm{GKdim}\left(\mathcal{U}(\mathfrak{g})\right) = \dim_{\Bbbk}(\mathfrak{g})$,
   (f) $\mathrm{GKdim}(RG) = \mathrm{GKdim}(R)$ for any finite group G;

(g) $\operatorname{GKdim}\big(\mathcal{U}_q(\mathfrak{sl}(2,\Bbbk))\big) = 3$;

(h) $\operatorname{GKdim}\big(R \otimes \mathcal{U}(\mathfrak{g})\big) = \operatorname{GKdim}(R) + \dim(\mathfrak{g})$, for a finite dimensional Lie algebra $\mathfrak{g}$;

(i) $\operatorname{GKdim}\big(R * \mathcal{U}(\mathfrak{g})\big) \geq \operatorname{GKdim}(R) + \dim(\mathfrak{g})$; if R is affine, $\operatorname{GKdim}\big(R * \mathcal{U}(\mathfrak{g})\big) = \operatorname{GKdim}(R) + \dim(\mathfrak{g})$;

(j) $\operatorname{GKdim}\big(\mathcal{O}_q(M_n(\Bbbk))\big) = n^2$ (c.f. [17]);

(k) $\operatorname{GKdim}\big(A_2(J_{a,b})\big) = 4$ (c.f. [4]);

(l) $\operatorname{GKdim}\big(A_n(q, p_{ij})\big) = 2n$ (c.f. [6]);

(m) $\operatorname{GKdim}(\mathcal{A}) = n$, where $\mathcal{A}$ is a diffusion algebra (c.f. [8]).

(2) Theorem 18 generalizes the following result due to Zhang in [18]: Let B a finitely generated $\Bbbk$-algebra which is a commutative domain, σ is a $\Bbbk$-endomorphism of A, and if δ is a σ-derivation of B, then the following statements are equivalent:

(a) $\operatorname{GKdim}\big(B[x;\sigma,\delta]\big) < \operatorname{GKdim}(B) + 2$;

(b) $\operatorname{GKdim}\big(B[x;\sigma,\delta]\big) = \operatorname{GKdim}(B) + 1$;

(c) σ is locally algebraic.

(3) Conditions on automorphism σ_n in Theorem 14 and Theorem 18 are necessary as the next examples show. Let $R = \Bbbk\big[y^{\pm 1}, z^{\pm 1}\big]$. Consider the skew PBW extensions of R given by $B = \Bbbk\big[y^{\pm 1}, z^{\pm 1}\big][x;\sigma]$ and $T = \Bbbk\big[y^{\pm 1}, z^{\pm 1}\big]\big[x^{\pm 1};\sigma\big]$ where $\sigma(y) \coloneqq yz$ and $\sigma(z) \coloneqq z$. Then $\operatorname{GKdim}(R) = 2$ and $\operatorname{GKdim}(B) = \operatorname{GKdim}(T) = 4$. Note that T is the group algebra $\Bbbk G$ where G is the group generated by x, y, z with relations $zy = yz$, $zx = xz$ and $y^{-1}x^{-1}yx = z$. The representation

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad y \mapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad z \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives an isomorphism of G with the group of 3×3 matrices

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

with $a, b, c \in \mathbb{Z}$, i.e., the *discrete Heisenberg group*. See [16], Example 8.2.16 for more details.

Other example is given by Ore extensions $B[x;\sigma,\delta]$ with B a $\Bbbk$-algebra and σ a $\Bbbk$-endomorphism of B. Huh and Kim [9] showed the inequality $\operatorname{GKdim}(B[x;\sigma,\delta]) \geq \operatorname{GKdim}(B) + 1$, where equality holds whenever each finite dimensional subspace of B is contained in a finitely generated subalgebra of B that is stable under both σ and δ. In general, the difference $\mathrm{GKdim}(B[x;\sigma,\delta]) - \mathrm{GKdim}(B)$ may be an arbitrary natural number, it may be infinite. Similarly, if R is a $\Bbbk$-algebra and δ is a $\Bbbk$-derivation, then $\mathrm{GKdim}(R[x;\delta]) \geq \mathrm{GKdim}(R) + 1$ (Corollary, 8.2.11).

(4) In [3, 8, 12, 14] Gelfand-Kirillov dimension is computed for several classes of rings and algebras. We remark that none of these algebras generalize skew PBW extensions and Theorem 14 and Theorem 18 allow to compute the Gelfand-Kirillov dimension for many examples of these algebras. See [13] for relations between all these algebras and skew PBW extensions.

2.1. Gelfand-Kirillov Dimension of Skew Quantum Polynomials

In this Section we calculate the Gelfand-Kirillov dimension for skew quantum polynomials. We recall the following definition presented in [13].

Definition 20. Let R be a ring with a fixed matrix of parameters $\mathbf{q} := [q_{ij}] \in M_n(R)$, $n \geq 2$, such that $q_{ii} = 1 = q_{ij}q_{ji} = q_{ji}q_{ij}$ for every $1 \leq i,j \leq n$, and suppose also that it is given a system $\sigma_1, \dots, \sigma_n$ of automorphisms of R. The ring of *skew quantum polynomials over* R, $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]$, is defined as the ring satisfying the following relations:

(i) $R \subseteq R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]$;

(ii) $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]$ is a free left R-module with basis

$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r \text{ and } \alpha_i \in \mathbb{N} \text{ for } r+1 \leq i \leq n\}; \tag{7}$$

(iii) the variables $x_1, \dots, x_n$ satisfy the defining relations

$$x_i x_i^{-1} = 1 = x_i^{-1} x_i, \qquad 1 \leq i \leq r, \tag{8}$$

$$x_j x_i = \sigma_j(x_i) x_j = q_{ij} x_i x_j, \quad r \in R, \quad 1 \leq i,j \leq n, \tag{9}$$

$$x_j r = \sigma_j(r) x_j j, \quad r \in R, \quad 1 \leq i,j \leq n. \tag{10}$$

***Remark* 21.** $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]$ can be viewed as a localization of a skew PBW extension. In fact, we have the quasi-commutative bijective skew PBW extension

$$A := \sigma(R)\langle x_1, \dots, x_n\rangle, \quad \text{with} \quad x_i r = \sigma_i(r) x_i \quad \text{and}$$
$$x_j x_i = q_{ij} x_i x_j, \quad 1 \leq i,j \leq n.$$

If we set $S := \big\{ r x^{\alpha} : r \in R^*,\ x^{\alpha} \in \mathrm{Mon}\{x_1, \dots, x_r\}\big\}$ then S is a multiplicative subset of A and $S^{-1}A \cong R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]$. In fact, if $f \in A$ and $r x^{\alpha} \in S$ are such that $f r x^{\alpha} = 0$, then $0 = f r x^{\alpha} = f x^{\alpha}\big[(\sigma^{\alpha})^{-1}(r)\big]$, so

$0 = fx^\alpha$ since $(\sigma^\alpha)^{-1}(r) \in R^*$, and hence, $f = 0$. From this we get that $rx^\alpha f = 0$. S satisfies the left (right) Ore condition:

If $f = c_1x^{\beta_1} + \cdots + c_tx^{\beta_t}$, then $grx^\alpha = x^\alpha f$, where $g := d_1x^{\beta_1} + \cdots + d_tx^{\beta_t}$ with $d_i := \sigma^\alpha(c_i)c_{\alpha,\beta_i}c^{-1}_{\beta_i,\alpha}\sigma^{\beta_i}(r^{-1})$, and c_{α,β_i}, $c_{\beta_i,\alpha}$ are the elements of R that we obtain when we apply Theorem 4 to A (for the right Ore condition g is defined in a similar way). This means that $S^{-1}A$ exists (AS^{-1} also exists, and hence, $S^{-1}A \cong AS^{-1}$).

Finally, note that the function

$$h' : A \to R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big], \qquad h'(f) := f$$

is a ring homomorphism and it satisfies $h'(S) \subseteq R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]^*$ $\big($in fact, $[rx^\alpha]^{-1} = (\sigma^\alpha)^{-1}(r^{-1})(x^\alpha)^{-1}\big)$, so h' induces the ring homomorphism

$$h : S^{-1}A \to R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big],$$
$$h\Big(\frac{f}{rx^\alpha}\Big) := h'(rx^\alpha)^{-1}h'(f) = (rx^\alpha)^{-1}f.$$

It is clear that h is injective; moreover, h is surjective since $x_i = h\big(\frac{x_i}{1}\big)$, $1 \leq i \leq n$, $x_j^{-1} = h\big(\frac{1}{x_j}\big)$, $1 \leq j \leq r$, $r = h(r)$, $r \in R$.

***Remark* 22.**

(a) When all automorphisms are trivial, the ring of *quantum polynomials over* R is denoted by $R_{\mathbf{q}}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$.

(b) If $R = \Bbbk$ is a field, then $\Bbbk_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$ is the *algebra of skew quantum polynomials*.

(c) For trivial automorphisms we get the *algebra of quantum polynomials* simply denoted by $\mathcal{O}_{\mathbf{q}}$ (see [1]).

(d) If $r = 0$, the ring $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big] = R_{\mathbf{q},\sigma}[x_1, \ldots, x_n]$ is the n-multiparametric skew quantum space over R.

(e) When $r = n$, the ring of skew quantum polynomials over R coincides with $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\big]$, the *$n$-multiparametric skew quantum torus over R*. In this case, if $n = 1$, $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\big] = R\big[x^{\pm 1}; \sigma\big]$, i.e., this ring coincides with the *skew Laurent polynomial ring over R*. If $r = n$ and automorphisms are trivial, i.e., $\sigma_i = i_R$, $1 \leq i \leq n$, we denoted $R_{\mathbf{q}}\big[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\big]$ and it is called *n-multiparametric quantum torus over R*. For $R = \Bbbk$, $\Bbbk_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\big]$ is simple called *n-multiparametric skew quantum torus*, and the particular case $n = 2$ is called *skew quantum torus*; for trivial automorphisms we have the *n-multiparametric quantum torus* and the *quantum torus* (see [7].) The ring $\Bbbk[x^{\pm 1}; \sigma]$ is the *algebra of skew Laurent polynomials*; if $\sigma = i_R$, then $R\big[x^{\pm 1}; \sigma\big] = R\big[x^{\pm 1}\big]$ is the classical *Laurent polynomial ring over R*, and then $\Bbbk\big[x^{\pm 1}\big]$ is the *algebra of Laurent polynomials*.

(f) Following [7, p. 16], let $\Bbbk$ be a field and $\mathbf{q} = (q_{ij})$ a multiplicatively antisymmetric $n \times n$ matrix over $\Bbbk$. The corresponding *multiparameter quantum torus* is the $\Bbbk$-algebra $\mathcal{O}_{\mathbf{q}}\big((\Bbbk^*)^n\big)$ presented by generators $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ and relations $x_i x_j = q_{ij} x_j x_i$ for all i, j. The single parameter version $\mathcal{O}_q\big((\Bbbk^*)^n\big)$, for $q \in \Bbbk^*$, is the special case when $q_{ij} = q$ for all $i < j$.

From these observations we can see that the ring of skew quantum polynomials over R generalizes all the rings considered by Artamonov in [1].

For the next Lemma consider the automorphism σ_n in Theorem 18.

Lemma 23. *Let R be a $\Bbbk$-algebra with a finite dimensional generating subspace V and suppose that σ_n is locally algebraic. Then,*

$$\operatorname{GKdim}\big(R_{\boldsymbol{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]\big) = \operatorname{GKdim}\big(R_{\boldsymbol{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]\big).$$

Proof. Note that $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big]$ is as a quasi-commutative bijective skew PBW extension of the r-multiparametric skew quantum torus over R. More exactly, $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n\big] \cong \sigma(T)\langle x_{r+1}, \dots, x_n\rangle$, with $T \coloneqq R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]$. Note that for T, σ is defined by the σ_j in (9) with $1 \le j \le r$. Moreover, T is a $\Bbbk$-algebra with a finite dimensional generating subspace $VX^{\pm 1}$, where $X^{\pm 1} \coloneqq \big\{1, x_1^{\pm 1}, \dots, x_r^{\pm 1}\big\}$. Given that σ_n is a locally algebraic automorphism of T , the result follows from Theorem 18. ☑

The following Lemma allows us to compute GKdim of the n-multiparametric skew quantum torus.

Lemma 24. *Let $R_{\boldsymbol{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]$ be the r-multiparametric skew quantum torus. If R is a $\Bbbk$-algebra with a finite dimensional generating subspace V, and the automorphism σ_r of $R_{\boldsymbol{q},\sigma}\big[x_1^{\pm 1}, \dots, x_{r-1}^{\pm 1}\big]$ given by $\sigma_r(a) = c_{ra}$, $\sigma_r(x_i) = c_{ir} x_i$ for $a \in R$ and $1 \le i < r$, is locally algebraic, then*

$$\operatorname{GKdim}\big(R_{\boldsymbol{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]\big) = \operatorname{GKdim}(R) + r.$$

Proof. Follows from Lemmas 17 and 23. ☑

For the next Theorem consider the automorphism σ_n of $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1}\big]$ in Lemma 23.

Theorem 25. *Under the same conditions of Lemma 24, Gelfand-Kirillov dimension for skew quantum polynomials over a finitely generated $\Bbbk$-algebra R with locally algebraic automorphism σ_n , is given by*

$$\operatorname{GKdim}\big(R_{\boldsymbol{q},\sigma}\big[x_1^{\pm 1}, \dots, x_r^{\pm 1} x_{r+1}, \dots, x_n\big]\big) = \operatorname{GKdim}(R) + n.$$

Proof. It is clear that $R_{\mathbf{q},\sigma}\left[x_1^{\pm 1},\dots,x_r^{\pm 1}\right]$ is a $\Bbbk$-algebra. Denote with $X^{\pm 1}$ the $\Bbbk$-linear subspace of $R_{\mathbf{q},\sigma}\left[x_1^{\pm 1},\dots,x_r^{\pm 1}\right]$ spanned by $1, x_1^{\pm 1},\dots,x_r^{\pm 1}$. Then $VX^{\pm 1}$ is a finite dimensional generating subspace of $R_{\mathbf{q},\sigma}\left[x_1^{\pm 1},\dots,x_r^{\pm 1}\right]$. The assertion follows from Theorem 18 and Lemma 24. ☑

3. Examples

In [13] it was proved that all rings and algebras in the following tables are bijective skew PBW extensions. Under conditions of Theorems 14 and 18 we have computed its Gelfand-Kirillov dimension.

Noncommutative ring	**GKdim**
Polynomial ring	$\mathrm{GKdim}(R)+n$
Skew polynomial ring of derivation type	$\mathrm{GKdim}(R)+n$
Universal enveloping algebra of Lie algebras	$\mathrm{GKdim}(R)+n$
Universal enveloping algebra of Kac-Moody Lie algebras	$\mathrm{GKdim}(k)+m+n$
Universal enveloping rings $\mathcal{U}(V,R,\Bbbk)$	$\mathrm{GKdim}(k)+n$
Differential operator rings $V(R,L)$	$\mathrm{GKdim}(k)+n$
Tensor and crossed product	$\mathrm{GKdim}(R)+n$
Twisted or smash product differential operator ring	$\mathrm{GKdim}(R)+n$

Table 1. Gelfand-Kirillov dimension for some PBW extensions.

Noncommutative ring	**GKdim**
Weyl algebra	$2n$
Quantum plane	2
Algebra of q-differential operators	2
Algebra of shift operators	2
Mixed algebra	3
Algebra for multidimensional discrete linear systems	$2n$
Algebra B	3

Table 2. Gelfand-Kirillov dimension for some Ore extensions of derivation type.

Noncommutative ring	GKdim
Algebra of linear partial differential operators	$2n$
Algebra of linear partial shift operators	$2n$
Algebra of linear partial difference operators	$2n$
Algebra of linear partial q-dilation operators	$n+m$
Algebra of linear partial q-differential operators	$n+m$
Operator differential rings	m

TABLE 3. Gelfand-Kirillov dimension for operator algebras.

Noncommutative ring	GKdim
Diffusion algebras	n
Generalized Weyl algebras $R\{\theta, \xi\}$	$\operatorname{GKdim}(R)+2$
Quadratic algebras in 3 variables	3
Clifford algebras	$\operatorname{GKdim}(R)+2n$

TABLE 4. Gelfand-Kirillov dimension for others examples of skew PBW extensions.

Noncommutative ring	GKdim
Additive analogue of the Weyl algebra	$2n$
Multiplicative analogue of the Weyl algebra	n
Quantum algebra $\mathcal{U}'\big(\mathfrak{so}(3,\Bbbk)\big)$	3
Dispin algebra $\mathcal{U}\big(osp(1,2)\big)$	3
Woronowicz algebra $\mathcal{W}_{\nu}\big(\mathfrak{sl}(2,\Bbbk)\big)$	3
Algebra $\mathbf{U}$	$3n$
The Complex algebra $V_q\big(\mathfrak{sl}(3,\mathbb{C})\big)$	10
Manin algebra $\mathcal{O}_q\big(M_2(\Bbbk)\big)$	4
Algebra of quantum matrices $\mathcal{O}_q\big(M_n(\Bbbk)\big)$	n^2
q-Heisenberg algebra $\mathbf{H}_n(q)$	$3n$
Quantum enveloping algebra $\mathcal{U}_q\big(\mathfrak{sl}(2,\Bbbk)\big)$	3
Hayashi's algebra $W_q(J)$	$3n$
Differential operators on a quantum space $D_{\mathbf{q}}\big(S_{\mathbf{q}}\big)$	$2n$
Quantum Weyl algebra $A_2(J_{a,b})$	4
Quantum Weyl algebra $A_2^{\overline{q},\Lambda}$	2
Quantum Weyl algebra of Maltsiniotis $A_n^{\mathbf{q},\lambda}$	n^2
Quantum Weyl algebra of Maltsiniotis $A_n(q,p_{ij})$	n^2
Multiparameter quantized Weyl algebra $A_n^{Q,\Gamma}$	n^2
Quantum Weyl algebra $A_n(\overline{q},\Lambda)$	n^2
Quantum symplectic space $\mathcal{O}_q\big(\mathfrak{sp}\big(\Bbbk^{2n}\big)\big)$	n^2

TABLE 5. Gelfand-Kirillov dimension for some quantum algebras.

Finally, Table 6 contains the Gelfand-Kirillov dimensions for some examples of skew quantum polynomials (see Remark 22.)

Noncommutative ring	**GKdim**
Skew Laurent extension $R[x^{\pm 1};\sigma_1]$	$\mathrm{GKdim}(R)+1$
Skew Laurent polynomials $\Bbbk[x^{\pm 1};\sigma_1]$	1
Classical Laurent polynomial ring $R[x^{\pm 1}]$	$\mathrm{GKdim}(R)+1$
Algebra of Laurent polynomials $\Bbbk[x^{\pm 1}]$	1
n-Multiparametric skew quantum space $R_{\mathbf{q},\sigma}[x_1,\dots,x_n]$	$\mathrm{GKdim}(R)+n$
n-Multiparametric quantum space $R_{\mathbf{q}}[x_1,\dots,x_n]$	$\mathrm{GKdim}(R)+n$
n-Multiparametric skew quantum space $\Bbbk_{\mathbf{q},\sigma}[x_1,\dots,x_n]$	n
n-Multiparametric quantum space $\Bbbk_{\mathbf{q}}[x_1,\dots,x_n]$	n
n-Multiparametric skew quantum torus $R_{\mathbf{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1}]$	$\mathrm{GKdim}(R)+r$
n-Multiparametric quantum torus $R_{\mathbf{q}}[x_1^{\pm 1},\dots,x_r^{\pm 1}]$	$\mathrm{GKdim}(R)+r$
n-Multiparametric skew quantum torus $\Bbbk_{\mathbf{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1}]$	r
Ring of skew quantum polynomials $R_{\mathbf{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$	$\mathrm{GKdim}(R)+n$
Ring of quantum polynomials $R_{\mathbf{q}}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$	$\mathrm{GKdim}(R)+n$
Algebra of skew quantum polynomials $\Bbbk_{\mathbf{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$	n
Algebra of quantum polynomials $\mathcal{O}_{\mathbf{q}}=\Bbbk_{\mathbf{q}}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$	n

TABLE 6. Gelfand-Kirillov dimension of skew quantum polynomials.

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