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Definable Group Extensions and o-Minimal Group Cohomology via Spectral Sequences

Extensiones de grupo definibles y cohomología de grupos o-minimal vía sucesiones espectrales

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Abstract. We provide the theoretical foundation for the Lyndon-Hochschild-Serre spectral sequence as a tool to study the group cohomology and with this the group extensions in the category of definable groups. We also present various results on definable modules and actions, definable extensions and group cohomology of definable groups. These have applications to the study of non-definably compact groups definable in o-minimal theories (see [1]).

Key words and phrases. o-Minimality, Definable extensions, o-Minimal cohomology, Definable G–module, LHS spectral sequences.

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Resumen. Se presenta el fundamento teórico para las sucesiones espectrales de Lyndon-Hochschild-Serre como una herramienta para estudiar la cohomología de grupos y con ésta las extensiones de grupos en la categoría de los grupos definibles. También se presentan varios resultados en módulos definibles y acciones, extensiones definibles y cohomología de grupos definibles. Estos tienen aplicaciones en el estudio de los grupos definibles no definiblemente compactos en teorías o-minimales (see [1]).

Palabras y frases clave. o-minimalidad, extensiones definibles, cohomología o-minimal, G–módulo definible, sucesión espectral de LHS.

1. Introduction

Although the origins of group cohomology arose in the topology before the twentieth century, it was not until 1943 when Eilenberg and MacLane fixed the current definition of group cohomology [3, 5]. In the context of o-minimal theories, Edmundo adopted group cohomology theory to definable groups in o–minimal structures and proved the existence of a definable section for any definable extension [2].

This paper presents the Lyndon-Hochschild-Serre (briefly LHS) spectral sequences as a tool to study group cohomology. This can sometimes provide an understanding of the cohomology groups which is not evident from the group itself.

These LHS sequences, together with results on definable modules and actions, definable extensions and the group cohomology of definable groups described in this paper, are one of the tools used for the study of some (non-compact) ordered groups definable in an o–minimal theory. Specifically in [1] we provide a complete classification of the decomposable ordered groups of dimension 2 and 3 using the property of super–solvability, together with the results presented in this paper. Throughout, we work in an o–minimal expansion $\mathcal{R}$ of a real closed field in which all o–minimal groups are definably isomorphic. In fact, $\mathcal{R}$ is an exponential real closed field.

The structure of the paper is as follows. In Section 2, we define the concepts of a definable module and definably equivalent actions, and present some elementary but useful results about definable homomorphisms and actions (Proposition 2.3 and Proposition 2.4). Then we use these propositions in the characterization of definable actions on the additive group of $\mathcal{R}$ and we establish that any two non trivial definable actions of $\mathbb{R}^n$ on $(\mathbb{R}, +)$, are definably equivalent (Proposition 2.9). Section 3 introduces spectral sequences (Definition 3.1) and develops some technical facts about the existence of a filtration given by a normal subgroup on the cochain complex for a G–module. This implies the existence of a spectral sequence known as the LHS spectral sequence, providing precise terms for the associated graded group of the n–th cohomology group if one knows the exact terms of the sequence (Fact 2 and Corollary 3.8). Section 4 formulates some useful properties of the cohomology groups considered in the given o–minimal expansion $\mathcal{R}$ and also calculates the 0–th and first cohomology group for $(\mathbb{R}, +)$ considered as definable $(\mathbb{R}^n, +)$–module. Finally, in Section 5, we provide the basic notions of group extension theory in the o–minimal context and the bijection between the group of definable extensions and the second cohomology group.

2. Definable G–modules

Let $\mathcal{R} = (R, <, \cdots)$ be an o–minimal expansion of a real closed field in which all o–minimal groups are definably isomorphic and therefore definable means $\mathcal{R}$–definable. In fact, note that $\mathcal{R}$ is an exponential real closed field. We assume the reader's familiarity with basic o–minimality (see [7]). We start by mentioning some notions and results on definable modules that will be used through the paper.

2.1. Definable G–Modules

Definition 2.1. Let G be a definable group. A *definable G–module* (M, γ) is a G–module such that M is a definable abelian group and the action map $\gamma : G \times M \to M$, $\gamma(x, a) := \gamma(x)(a)$ is definable. In this way we get a homomorphism $\gamma : G \to \operatorname{Aut}(M)$ from G to the group of all definable automorphisms of M. As usual γ is called the action on M, and M is trivial if $\gamma(g)(m) = m$ for all $g \in G$ and $m \in M$.

Definition 2.2. Two actions of G on M γ_1 and γ_2 are *definably equivalent* if there is a definable group automorphism ψ of G such that $\gamma_1 = \gamma_2 \circ \psi$.

We now introduce Proposition 2.3 and Proposition 2.4 and then, we will used them for the characterization of $(R, +)$ as a definable G–module, with G a group homeomorphic to R^n.

The next fact is easy to prove. It is based in the fact that given a definable function of two variables, then if we fix one of the arguments we obtain a definable function in the remaining variable.

Fact 1. *Let (M, γ) be a definable G–module and $m \in M$. Then the function $\psi : G \to M$, $\psi(g) = \gamma(g)(m)$ is definable.*

Proposition 2.3. *Let $(G, \oplus)$ and $(G', \odot)$ be definable groups with $G \subseteq R$, $G' \subseteq R^n$ and G an infinite set. If $\oplus$ and $\odot$ are continuous and $f : (G, \oplus) \to (G', \odot)$ is a definable homomorphism then f is continuous.*

Proof. By the Cell Decomposition Theorem there is a decomposition

$$\big\{(-\infty, a_0), (a_0, a_1), (a_{n-1}, a_n), (a_n, +\infty), \{a_0\}, \dots, \{a_n\}\big\}$$

of R with $a_0 < a_1 < \cdots < a_n$ partitioning G and such that f restricted to each of those subintervals is continuous.

Let $\{b_1, \dots, b_k\} = \{a_0, a_1, \dots, a_n\} \bigcap G$. To show the continuity of f in G is enough to verify the continuity at b_i for each $i \in \{1, \dots, k\}$. We will see that there is $h \in G$ such that for all $i \in \{1, \dots, k\}$, f is continuous at $b_i \oplus h^{-1}$ and at h. Consider the bijection $g_i : G \to G$, $g_i(h) = b_i \oplus h^{-1}$.

Let $X_i := g_i^{-1}\big(\{b_1, \dots, b_k\}\big)$. Then $|X_i| = n$. Let $X = \left(\bigcup_{1 \le i \le k} X_i\right) \bigcup \{b_1, \dots, b_k\}$.

Let $h \in G \setminus X$, so f is continuous at h and, for each $i \in \{1, \dots, k\}$, $b_i \oplus h^{-1} \notin \{b_1, \dots, b_k\}$. Therefore there is $h \in G$ for all $i \in \{1, \dots, k\}$ such that f is continuous at h and at $b_i \oplus h^{-1}$.

Since $\oplus$ and $\odot$ are continuous and $f(b_i) = f\big(b_i \oplus h^{-1}\big) \odot f(h)$ then f is continuous at b_i for each $i \in \{1, \dots, k\}$ and hence the proof of the proposition is complete. ☑

Proposition 2.4. *Let G be a definable group whose product will be denoted multiplicatively. Let f and g be definable homomorphisms from $(R,+)$ to G. If $f(1)=g(1)$ then $f=g$.*

Proof. Let $X=\big\{x\in R \mid f(x)=g(x)\big\}$. Then X is a definable subgroup of $(R,+)$ and moreover, $\{0,1\}\subseteq X$. Since the only definable subgroups of R are $\{0\}$ and R $\big([7, \text{Lemma in I§4}]\big)$ then $X=R$. ☑

Proposition 2.5. *Let $(\mathrm{Aut}(R),\circ)$ be the group of definable automorphisms of $(R,+)$. Then $(\mathrm{Aut}(R),\circ)$ is isomorphic to $\big(R\smallsetminus\{0\},\bullet\big)$.*

Proof. Let $\theta\in(\mathrm{Aut}(R),\circ)$ and let g be the automorphism of R, $g(x)\coloneqq x\theta(1)$. Since $g(1)=\theta(1)$, by Proposition 2.4, $\theta(x)=x\theta(1)$ for all $x\in R$.

Let $\psi:(\mathrm{Aut}(R),\circ)\to(R\smallsetminus\{0\},\bullet)$ given by $\psi(\theta)=\theta(1)$. Is easy to see that ψ is a homomorphism. Furthermore, if $\psi(\theta_1)=\psi(\theta_2)$ then $\theta_1(1)=\theta_2(1)$ and hence by Proposition 2.4, $\theta_1=\theta_2$. Let $k\in R\smallsetminus\{0\}$ and $h(x)\coloneqq xk$. Clearly $h\in\mathrm{Aut}(R)$ and $\psi(h)=k$ then ψ is a surjection. From the above, ψ is an isomorphism and thus $(\mathrm{Aut}(R),\circ)\simeq(R\smallsetminus\{0\},\bullet)$. ☑

Notation. By Proposition 2.5, given a definable G–module $(R,+)$ with action γ, we have that for each $g\in G$ there is k_g in $R\smallsetminus\{0\}$ such that $\gamma(g)(m)=k_gm$ for all $m\in M$. Thus, slightly abusing notation, from now on we will denote the action γ simply by $\gamma(g)=k_g$ meaning that $\gamma(g)(m)=mk_g$.

2.2. Definable Actions of $\mathbb{R}^n$ on $(\mathbb{R},+)$

Even though all the following results work in the above assumed structure $\mathcal{R}$, we will assume that we are working with the actual real numbers and, abusing notation, refer to $\mathbb{R}$ as the additive group. From now on $(\mathbb{R}^n,+)$ will be the usual cartesian group induced by $(\mathbb{R},+)$.

We will determine the possible definable $\mathbb{R}$–modules on $\mathbb{R}$ which are completely determined by their actions.

Proposition 2.6. *Let $(\mathbb{R},\gamma)$ be a definable $\mathbb{R}$–module. Then for each $g\in\mathbb{R}$ there is $k_g>0$ such that $\gamma(g)=k_g$.*

Proof. By Proposition 2.5, for each $g\in\mathbb{R}$, there is some $k_g\in\mathbb{R}\smallsetminus\{0\}$ such that $\gamma(g)(m)=mk_g$ for all $m\in\mathbb{R}$. Consider $\psi:\mathbb{R}\to\mathbb{R}\smallsetminus\{0\}$, $\psi(g)=\gamma(g)(1)=k_g$ which is a definable function by Fact 1; ψ is also a group homomorphism from $(\mathbb{R},+)$ to $\big(\mathbb{R}\smallsetminus\{0\},\bullet\big)$ since $\gamma:(\mathbb{R},+)\to\mathrm{Aut}(M)$ is a group homomorphism and

$$\begin{aligned}\psi(g+h) &= \gamma(g+h)(1)\\ &= \big(\gamma(g)\circ\gamma(h)\big)(1)\\ &= \gamma(g)(k_h)\\ &= k_g k_h\\ &= \gamma(g)(1)\gamma(h)(1)\\ &= \psi(g)\psi(h).\end{aligned}$$

Then, by Proposition 2.3, ψ is continuous. Since $\gamma(0)(1)=1$ and $\mathbb{R}$ is connected, $\psi(\mathbb{R})\subseteq\mathbb{R}^{>0}$, i.e., $\gamma(g)(m)=mk_g$ for all $m\in\mathbb{R}$ and some $k_g\in\mathbb{R}^{>0}$. ☑

Corollary 2.7. *Let $(\mathbb{R},\gamma)$ be a definable $\mathbb{R}$–module. Then, there is $c\in\mathbb{R}$ such that $\gamma(g)=e^{cg}$ for all $g\in\mathbb{R}$.*

***Proof*.** By Proposition 2.6, $\gamma(1)(1)=e^c$ for some $c\in\mathbb{R}$. Consider the definable group homomorphisms $\theta_1,\theta_2:(\mathbb{R},+)\to(\mathbb{R}^{>0},\bullet)$, $\theta_1(g)=e^{cg}$ and $\theta_2(g)=\gamma(g)(1)$. Since $\theta_1(1)=\theta_2(1)$, by Proposition 2.4, $\theta_1=\theta_2$, so $\gamma(g)(m)=e^{cg}m$ for all $g,m\in\mathbb{R}$. ☑

The following is a consequence of the previous result.

Corollary 2.8. *Let $(\mathbb{R},+)$ be a definable $(\mathbb{R}^n,+)$–module with non trivial action γ. Then*

$$\gamma(x_1,\ldots,x_i,\ldots,x_n)=e^{c_1x_1+\cdots+c_nx_n}$$

for all $x_1,\ldots,x_n\in\mathbb{R}$, with $c_i\neq0$ for some $i\in\{1,\ldots,n\}$.

***Proof*.** Let $i\in\{1,\ldots,n\}$. Consider $(\mathbb{R},+)$ as a definable $(\mathbb{R},+)$–module with the action γ_i given by $\gamma_i(x)(m)=\gamma(0,\ldots,x,\ldots,0)(m)$ where $(0,\ldots,x,\ldots,0)$ has x in the i–th component and zeros elsewhere. Then, by Corollary 2.7, $\gamma_i(g)=e^{c_i}$ for some $c_i\in\mathbb{R}$. Hence,

$$\begin{aligned}\gamma(x_1,\ldots,x_i,\ldots,x_n)(m) &= \big(\gamma_1(x_1)\circ\cdots\circ\gamma_i(x_i)\circ\cdots\circ\gamma_n(x_n)\big)(m)\\ &= me^{c_1x_1+\cdots+c_nx_n}\end{aligned}$$

for $c_1,\ldots,c_n\in\mathbb{R}$. Since γ is not trivial $c_i\neq0$ for some $i\in\{1,\ldots,n\}$. ☑

Proposition 2.9. *Consider the non trivial actions $\gamma'(x_1,\ldots,x_n)=e^{c_1x_1+\cdots+c_nx_n}$ and $\gamma(x_1,\ldots,x_i,\ldots,x_n)=e^{x_1}$ for which $\big((\mathbb{R},+),\gamma\big)$ and $\big((\mathbb{R},+),\gamma'\big)$ are definable $\big(\mathbb{R}^n,+\big)$–modules. Then γ and γ' are definably equivalent actions.*

Proof. For $n = 1$, consider the definable automorphism of $(\mathbb{R}, +)$, $\psi(x) = c_1 x$.

Let $n \geq 2$. Since γ' is a non trivial action, there is $i \in \{1, \dots, n\}$ such that $c_i \neq 0$. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be the map given by

$$\psi(x_1, \dots, x_i, \dots, x_n) = (c_1 x_1 + \cdots + c_n x_n, x_2, \dots, x_{i-1}, \underset{i\text{-th}}{x_1}, x_{i+1}, \dots, x_n).$$

It holds that,

(1) ψ is injective: $\psi(x_1, \dots, x_n) = \psi(y_1, \dots, y_n)$ if and only if $x_j = y_j$, $j \in \{1, 2, \dots, n\} \setminus \{i\}$ and $c_1 x_1 + \cdots + c_n x_n = c_1 y_1 + \cdots + c_n y_n$. Then $c_i x_i = c_i y_i$. Since $c_i \neq 0$, $x_i = y_i$.

(2) ψ is clearly a homomorphism and hence an automorphism of $(\mathbb{R}^n, +)$.

(3) $\gamma \circ \psi = \gamma'$.

Then, γ and γ' are definably equivalent. ☑

3. Spectral Sequences and the Cochain Complex for a *G*–Module

We begin this section with the definition of spectral sequence and its convergence and also the introduction of the chain complex of a G–module.

3.1. Spectral Sequences

Definition 3.1. A *spectral sequence* is a collection of abelian groups $E_r^{p,q}$ for all $p, q \geq 0$ and for all $r \geq 1$, equipped with maps

$$d : E_r^{p,q} \to E_r^{p+r, q-r+1}$$

such that

$$d \circ d : E_r^{p,q} \to E_r^{p+2r, q-2r+2}$$

is the zero map and,

$$E_{r+1}^{p,q} \simeq \frac{\ker d : E_r^{p,q} \to E_r^{p+r,q-r+1}}{\operatorname{im} d : E_r^{p-r,q+r-1} \to E_r^{p,q}}$$

for all $p, q \geq 0$ and for all $r \geq 1$. In the case that $q - r + 1 < 0$, $E_r^{p+r,q-r+1} := \{0\}$.

Definition 3.2. Given a spectral sequence $(E_r^{p,q}, d)$ for each $p, q \geq 0$ there is a positive integer r such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$. Simply take $r > \max\{p, q+1\}$. For such r we have that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_{r+k}^{p,q}$, $k \geq 1$. This stable abelian group is denoted by $E_\infty^{p,q}$.

Definition 3.3. A spectral sequence $(E_r^{p,q}, d)$, $p, q \geq 0$ and $r \geq 1$, is said to *converge* to the groups H^n, denoted by $E_r^{p,q} \Rightarrow H^{p+q}$, if for each $n \geq 0$ there is a filtration

$$0 = F^{n+1}H^n \subseteq F^n H^n \subseteq F^{n-1}H^n \subseteq \cdots \subseteq F^0 H^n = H^n$$

such that $E_\infty^{p,n-p} \simeq F^p H^n / F^{p+1} H^n$ for all p, $0 \leq p \leq n+1$.

Thus, in a convergent spectral sequence, the terms $E_\infty^{p,q}$ give information about H^n; precisely, about the associated graded group H^n, we have that $\mathrm{Gr}(H^n) \simeq \bigoplus_{p+q=n} E_\infty^{p,q}$ with $p, q \geq 0$.

3.2. The Cochain Complex for a G–Module

We will now recall some of the main definitions of group cohomology given in [6].

Let G be a group and let (M, γ) be a G–module with action γ. To distinguish the operation, we will denote the operation on G multiplicatively while the operation on M will be denoted additively.

Definition 3.4. For $n \in \mathbb{N}$, we will denote by $C^n(G, M, \gamma)$ (sometimes just $C^n(G, M)$ when the action γ is clear from the context) the (abelian) group given by the set of all the functions from G^n to M with the standard addition. The elements of such a group will be called *n–cochains* and $C^0(G, M) = M$.

Definition 3.5. The *coboundary map* $\delta : C^n(G, M, \gamma) \to C^{n+1}(G, M, \gamma)$ is defined by

$$\delta(f)(g_1, \ldots, g_{n+1}) = \gamma(g_1)\big(f(g_2, \ldots, g_{n+1})\big) + \sum_{i=1}^{n}(-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

It is clear that $\delta(f) \in C^{n+1}(G, M, \gamma)$. Since $\delta \circ \delta = 0$, (C, δ) is a cochain complex, we can define the *cohomology groups* $H^n(G, M) = H^n(G, M, \gamma) = \frac{Z^n(G,M,\gamma)}{B^n(G,M,\gamma)}$, $n \geq 0$, where

$$Z^n(G, M, \gamma) = \ker \delta : C^n(G, M, \gamma) \to C^{n+1}(G, M, \gamma),$$
$$B^n(G, M, \gamma) = \operatorname{im} \delta : C^{n-1}(G, M, \gamma) \to C^n(G, M, \gamma), \quad \text{for } n > 0$$

and

$$B^0(G, M, \gamma) \coloneqq \{0\}.$$

The elements of $Z^n(G,M,\gamma)$ are called *cocycles* and the elements of $B^n(G,M,\gamma)$ *coboundaries*. $H(C) = \underset{n\geq 0}{\oplus} H^n(G,M,\gamma)$ is the *cohomology group of the complex*.

We now introduce a filtration in the cochain complex (C,δ) with the purpose to formulate a description of the cohomology groups through a particular spectral sequence. This idea was formulated by Lyndon, Hochschild and Serre, who from the cohomology of a normal subgroup of G and of its associated quotient group, chose a particular filtration that provides precise terms for the spectral sequence. Let K be a normal subgroup of G. We define the following bounded filtration: $F^pC = C$, $p \leq 0$. For $p > 0$, let $F^pC = \underset{n\geq 0}{\oplus} F_pC \cap C^n(G,M)$ where, for $p \leq n$, $F_pC \cap C^n(G,M)$ is the group of all $f \in C^n(G,M)$ for which $f(g_1,\dots,g_n)$ depends only on $g_1,\dots,g_{n-p}$ and the cosets $g_{n-p+1}K,\dots, g_nK$, while $F_pC \cap C^n(G,M) = \{0\}$, for $p > n$.

This filtration determines a spectral sequence $(E_r^{p,q}, d)$, $p,q \geq 0$, $r \geq 1$, known as the Lyndon-Hochschild-Serre (briefly LHS) spectral sequence, that converges to the cohomology group of the complex. Serre and Hochschild showed in [6] that the family of groups E_1 and E_2 has an explicit and very useful form to calculate cohomology groups (see Fact 2).

***Remark* 3.6.** Let f be a representative of an element in $H^q(K,M)$, let $z_1,\dots,z_i \in K$ and $g \in G$. We will denote by $[g]$ the coset gK in G/K.

Following the previous notation, $H^q(K,M)$ can be considered as a G/K–module with the action $\widetilde{\gamma} : G/K \to \operatorname{Aut}\big(H^q(K,M)\big)$ given by

$$\widetilde{\gamma}\big([g]\big)(f)(z_1,\dots,z_q) = \gamma(g)\big(f(g^{-1}z_1g,\dots,g^{-1}z_qg)\big).$$

Fact 2. *Let G be a group, K a normal subgroup of G and (M,γ) a G–module. Then for the cochain complex $\Big(C = \underset{n\geq 0}{\oplus} C^n(G,M),\delta\Big)$ and the filtration $\{F^pC\}_{p\geq 0}$ defined above,*

$$E_2^{p,q} \simeq H^p\big(G/K, H^q(K,M)\big)$$

and for each $n \geq 0$,

$$\begin{aligned}\operatorname{Gr}\big(H^n(C)\big) &= F^0H^n/F^1H^n \oplus F^1H^n/F^2H^n \oplus \cdots \oplus F^nH^n/\{0\}\\ &\simeq E_\infty^{0,n} \oplus E_\infty^{1,n-1} \oplus \cdots \oplus E_\infty^{n-1,1} \oplus E_\infty^{n,0}.\end{aligned}$$

3.3. O–Minimal Group Cohomology and LHS Spectral Sequences

We will introduce the cohomology groups as in Definition 3.5 only in the o–minimal context. For a definable G–module (M,γ), the n–th group of the definable cochains with the standard addition is

$$C^n(G,M,\gamma) = \big\{f : G^n \to M \mid f \text{ is a definable function}\big\}.$$

The coboundary map δ, as in Definition 3.5, generates the complex of definable cochains $\Big(C = \underset{n\geq 0}{\oplus} C^n(G,M,\gamma),\delta\Big)$ and the cohomology groups $H^n(G,M) = H^n(G,M,\gamma) \coloneqq Z^n(G,M,\gamma)/B^n(G,M,\gamma)$ where $Z^n(G,M,\gamma)$ is the set of definable n–cocycles and $B^n(G,M,\gamma)$ the set of definable n–coboundaries.

***Remark* 3.7.** Observe that $H^0(G,M)$ is a definable subgroup of M. However, in general we do not know if the cohomology group $H^n(G,M)$, $n \geq 1$ is a definable group. With the objective of understanding definable extensions of G by $\mathbb{R}$ compatible with the structure of the definable G–module $\mathbb{R}$ for some definable group G, we can apply the LHS spectral sequences to calculate the second cohomology group $H^2(G,\mathbb{R})$. This involves understanding the groups $H^j\big(G/K, H^i(K,\mathbb{R})\big)$ with $K \trianglelefteq G$, $i,j \in \{0,1,2\}$. Notice that, by definition, $H^i(K,\mathbb{R})$ is a $\mathbb{R}$–vector space and, if it is finite dimensional, we can identify it with $\mathbb{R}^n$ for some $n \in \mathbb{N}$. After this identification, it follows that in such a case, $H^i(K,\mathbb{R})$ is a definable G/K–module.

The following corollary is the consideration of Fact 2 in the cases of the first and second cohomology group.

Corollary 3.8. *Let (M,γ) be a definable G–module and let K be a definable normal subgroup of G. For the cochain complex $\Big(C = \underset{n\geq 0}{\oplus} C^n(G,M,\gamma),\delta\Big)$ there are filtrations of the groups $H^1 = H^1(G,M,\gamma)$ and $H^2 = H^2(G,M,\gamma)$ of the form*

$$\{0\} \trianglelefteq F^1H^1 \trianglelefteq H^1,$$
$$\{0\} \trianglelefteq F^2H^2 \trianglelefteq F^1H^2 \trianglelefteq H^2$$

such that

$$\begin{aligned} F^1H^1 &= H^1\big(G/K, M^K\big),\\ H^1/F^1H^1 &\preceq \big(H^1(K,M)\big)^{G/K},\\ F^2H^2, &= \frac{H^2\big(G/K,M^K\big)}{\text{im }\ d: \big(H^1(K,M)\big)^{G/K} \to H^2\big(G/K,M^K\big)},\\ F^1H^2/F^2H^2 &\preceq H^1\big(G/K, H^1(K,M)\big), \end{aligned}$$

and

$$H^2/F^1H^2 = \big(H^2(K,M)\big)^{G/K}.$$

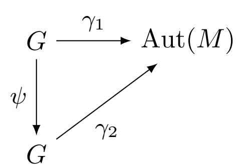
An application of the above result is presented in [1, Section 3] with the calculations of some cohomology groups over an o–minimal expansion of the field of real numbers.

4. More on o–Minimal Cohomology Groups

In this section we formulate some useful properties of the cohomology groups considered in the assumed o–minimal expansion $\mathcal{R}$. The next proposition shows that cohomology groups associated with definably equivalent actions are isomorphic. This result is subsequently applied to Proposition 4.6 and Proposition 5.6.

Proposition 4.1. *Let G be a definable group and M a definable G–module. If γ_1, γ_2 are two definably equivalent actions of G on M then the cohomology groups $H^n(G, M, \gamma_1)$ and $H^n(G, M, \gamma_2)$ are isomorphic.*

Proof. By hypothesis, there is $\psi : G \to G$ a definable automorphism such that the following diagram is commutative:



Let $f \in Z^n(G, M, \gamma_2)$. Define

$$\overline{f}(g_1, \dots, g_n) \coloneqq f\big(\psi(g_1), \dots, \psi(g_n)\big).$$

Note that

$$\begin{aligned}
\delta(\overline{f})(g_1, \dots, g_{n+1}) &= \gamma_1(g_1)\big(\overline{f}(g_2, \dots, g_{n+1})\big) + \\
&\quad \sum_{i=1}^{n} (-1)^i \overline{f}(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + \\
&\quad (-1)^{n+1} \overline{f}(g_1, \dots, g_n) \\
&= \gamma_2 \circ \psi(g_1)\Big(f\big(\psi(g_2), \dots, \psi(g_{n+1})\big)\Big) + \\
&\quad \sum_{i=1}^{n} (-1)^i f\big(\psi(g_1), \dots, \psi(g_i)\psi(g_{i+1}), \dots, \psi(g_{n+1})\big) + \\
&\quad (-1)^{n+1} f\big(\psi(g_1), \dots, \psi(g_n)\big) \\
&= \delta(f)\big(\psi(g_1), \dots, \psi(g_{n+1})\big) \\
&= 0.
\end{aligned}$$

Thus, $\overline{f} \in Z^n(G, M, \gamma_1)$. $[f]$ will denote the equivalence class of f on $H^n(G, M, \gamma_i)$, $i \in \{1, 2\}$.

Let

$$\begin{aligned}\sigma : H^n(G, M, \gamma_2) &\to H^n(G, M, \gamma_1)\\ [f] &\mapsto \big[\,\overline{f}\,\big].\end{aligned}$$

We will see that σ is well defined. Let $h \in [f]$ and $\delta(l) = h - f$. We will show that $\big(\overline{h} - \overline{f}\big) \in B^n(G, M, \gamma_1)$.

$$\begin{aligned}\big(\overline{h} - \overline{f}\big)(g_1, \ldots, g_n) &= \overline{h}(g_1, \ldots, g_n) - \overline{f}(g_1, \ldots, g_n)\\ &= h\big(\psi(g_1), \ldots, \psi(g_n)\big) - f\big(\psi(g_1), \ldots, \psi(g_n)\big)\\ &= (h - f)\big(\psi(g_1), \ldots, \psi(g_n)\big)\\ &= \delta(l)\big(\psi(g_1), \ldots, \psi(g_n)\big)\\ &= \gamma_2 \circ \psi(g_1)\big(l\big(\psi(g_2), \ldots, \psi(g_n)\big)\big) +\\ &\quad \sum_{i=1}^{n-1}(-1)^i\big(\psi(g_1), \ldots, \psi(g_i)\psi(g_{i+1}), \ldots, \psi(g_n)\big) +\\ &\quad (-1)^n l\big(\psi(g_1), \ldots, \psi(g_{n-1})\big)\\ &= \delta\big(\overline{l}\big)(g_1, \ldots, g_n).\end{aligned}$$

σ is also an injective map since, if $\sigma\big([f_1]\big) = \sigma\big([f_2]\big) \Leftrightarrow \big[\,\overline{f_1}\,\big] = \big[\,\overline{f_2}\,\big]$, there is $l' \in C^{n-1}(G, M, \gamma_1)$ such that $\overline{f_1} - \overline{f_2} = \delta(l')$. Define the cochain

$$l(g_1, \ldots, g_{n-1}) \coloneqq l'\big(\psi^{-1}(g_1), \ldots, \psi^{-1}(g_{n-1})\big).$$

$$\begin{aligned}\big(\overline{f_1} - \overline{f_2}\big)(g_1, \ldots, g_n) &= \delta(l')(g_1, \ldots, g_n)\\ &= \gamma_2\big(\psi(g_1)\big)\big(l\big(\psi(g_2), \ldots, \psi(g_n)\big)\big) +\\ &\quad \sum_{i=1}^{n}(-1)^i l\big(\psi(g_1), \ldots, \psi(g_i)\psi(g_{i+1}), \ldots, \psi(g_n)\big) +\\ &\quad (-1)^n l\big(\psi(g_1), \ldots, \psi(g_{n-1})\big)\\ &= \delta(l)\big(\psi(g_1), \ldots, \psi(g_n)\big)\\ &= (f_1 - f_2)\big(\psi(g_1), \ldots, \psi(g_n)\big)\end{aligned}$$

and hence $[f_1] = [f_2]$.

Since $\sigma\big([f_1 + f_2]\big) = \big[\,\overline{f_1 + f_2}\,\big] = \big[\,\overline{f_1} + \overline{f_2}\,\big] = \sigma\big([f_1]\big) + \sigma\big([f_2]\big)$, and for $[g] \in H^n(G, M, \gamma_1)$, $g^\star(g_1, \ldots, g_n) \coloneqq g\big(\psi^{-1}(g_1), \ldots, \psi^{-1}(g_n)\big)$ is such that $\sigma\big([g^\star]\big) = [g]$, then σ is an isomorphism. This shows that $H^n(G, M, \gamma_1) \simeq H^n(G, M, \gamma_2)$. ☑

In order to give a convenient characterization of 0–th and first cohomology group we introduce the next definition.

Definition 4.2. Let $f \in C^1(G, M, \gamma)$. We say that f is a *principal homomorphism* if there is $m \in M$ such that $f(g) = \gamma(g)m - m$ for all $g \in G$. We say that f is a *crossed homomorphism* if $f(gh) = f(g) + \gamma(g)f(h)$ for all $g, h \in G$.

Theorem 4.3.

(1) $H^0(G, M, \gamma) = M^G = \{m \in M \mid (\forall g \in G)(\gamma(g)x = x)\}$.

(2) $H^1(G, M, \gamma)$ *is the set of all crossed homomorphisms from* G *to* M *modulo the principal homomorphisms.*

Proof. By definition, (1) follows directly. For (2), $f \in Z^1(G, M, \gamma)$ if and only if $\delta(f)(g, h) = \gamma(g)f(h) - f(gh) + f(g) = 0$ if and only if f is a crossed homomorphism. An element $f \in B^1(G, M, \gamma)$ if and only if $f(g) = \delta(m)(g) = \gamma(g)m - m$, for some $m \in M$ if and only if f is a principal homomorphism. ☑

The purpose of the next propositions is calculate the 0–th and first cohomology group for $(\mathbb{R}, +)$ a definable $(\mathbb{R}^n, +)$–module considering the trivial and non trivial actions.

Proposition 4.4. *Let* G *be a definable group and let* $(\mathbb{R}, +)$ *be a definable* G*–module with action* γ*.*

(1) If γ *is the trivial action,* $H^0(G, \mathbb{R}, \gamma) = \mathbb{R}$.

(2) If γ *is a non trivial action,* $H^0(G, \mathbb{R}, \gamma) = \{0\}$.

Proof. If γ is a non trivial action there is $x \in G$ such that $\gamma(x) \neq \mathrm{id}_{\mathbb{R}}$ and therefore, there is $a \in \mathbb{R}$, $\gamma(x)(a) \neq a$; so, $a \notin \mathbb{R}^G$. Then, $\mathbb{R}^G \precnsim \mathbb{R}$ and since the only definable subgroups of $(\mathbb{R}, +)$ are either $\mathbb{R}$ or $\{0\}$, $H^0(G, \mathbb{R}) = \{0\}$. The case of the trivial action is obvious. ☑

Proposition 4.5. *Let* $(\mathbb{R}, +)$ *be a definable* $(\mathbb{R}^n, +)$*–module with the trivial action. Then* $H^1(\mathbb{R}^n, \mathbb{R}) \simeq (\mathbb{R}^n, +)$.

Proof. By Theorem 4.3, $H^1(\mathbb{R}^n, \mathbb{R}) = \mathrm{Hom}\big((\mathbb{R}^n, +), (\mathbb{R}, +)\big)$. Let f a homomorphism from $(\mathbb{R}^n, +)$ to $(\mathbb{R}, +)$. For $i \in \{1, \ldots, n\}$, let $R_i \coloneqq \{(0, \ldots, x, \ldots, 0) \mid x \in \mathbb{R}\}$. $f{\downharpoonright}_{R_i}$ is a definable homomorphism from $(R_i, +) \simeq (\mathbb{R}, +)$ to $\mathbb{R}$. By Proposition 2.4 $f(0, \ldots, x, \ldots, 0) = xf(e_i)$, for all $x \in \mathbb{R}$. Thus,

$$f\big((x_1, \ldots, x_i, \ldots, x_n)\big) = x_1 f(e_1) + \cdots + x_i f(e_i) + \cdots + x_n f(e_n)$$

for $(x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n$.

Finally, $\theta : \mathrm{Hom}\big((\mathbb{R}^n, +), (\mathbb{R}, +)\big) \to (\mathbb{R}^n, +)$ given by

$$\theta(f) = \big(f(e_1), \ldots, f(e_i), \ldots, f(e_n)\big)$$

is an isomorphism and the proposition follows. ☑

Proposition 4.6. *Let* $(\mathbb{R},+)$ *be a* $\mathbb{R}^n$*–module with non trivial action. Then* $H^1(\mathbb{R}^n,\mathbb{R})=\{0\}$.

***Proof*.** By Proposition 2.9, any non trivial action of $\mathbb{R}^n$ on $\mathbb{R}$ is definably equivalent to the action $\gamma(x_1,\dots,x_n)(a)=e^{x_1}a$. Also, by Proposition 4.1, it is enough to calculate $H^1(\mathbb{R}^n,\mathbb{R},\gamma)$. By Theorem 4.3, $H^1(\mathbb{R}^n,\mathbb{R},\gamma)$ is the set of all definable crossed homomorphisms in $\mathcal{C}^1(\mathbb{R}^n,\mathbb{R},\gamma)$ modulo the definable principal homomorphisms.

Let $f\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R},\gamma)$ be a definable crossed homomorphism. Then,

$$f\big((x_1,\dots,x_n)+(y_1,\dots,y_n)\big)=f\big((x_1,\dots,x_n)\big)+e^{x_1}f\big((y_1,\dots,y_n)\big).$$

For $x_1,\dots,x_n,y\in\mathbb{R}$ we have

$$\begin{aligned}f\big((x_1,\dots,x_n)\big)&=f\big((x_1,0,\dots,0)\big)+e^{x_1}f\big((0,x_2,\dots,x_n)\big)\\&=f\big((0,x_2,\dots,x_n)\big)+f\big((x_1,0,\dots,0)\big);\end{aligned}$$

then

$$f\big((x_1,\dots,x_n)\big)=f\big((x_1,0,\dots,0)\big),\quad\text{for all}\quad x_1,\dots,x_n\in\mathbb{R}.$$

Now,

$$\begin{aligned}f\big((x_1+y,0,\dots,0)\big)&=f\big((x_1,0,\dots,0)\big)+e^{x_1}f\big((y,0,\dots,0)\big)\\&=f\big((y,0,\dots,0)\big)+e^{y}f\big((x_1,0,\dots,0)\big),\end{aligned}$$

hence

$$f\big((x_1,0,\dots,0)\big)(e^y-1)=f\big((y,0,\dots,0)\big)(e^{x_1}-1).$$

Thus, for $y=1$,

$$f\big((x_1,\dots,x_n)\big)=\left(\frac{e^{x_1}-1}{e-1}\right)f(e_1).$$

Then, f is a definable principal homomorphism. So, $H^1(\mathbb{R}^n,\mathbb{R},\varphi)=\{0\}$. ☑

5. Definable Group Extensions and Second Cohomology Group

We start this section with the basic definitions of group extension theory in the o–minimal context as were established in [2]. From now on G will denote a definable group and M a definable G–module.

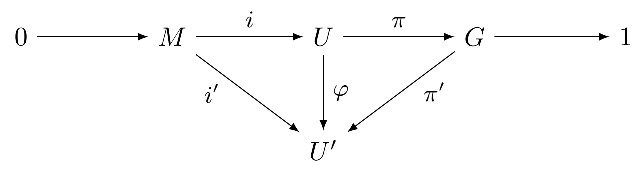
Definition 5.1.

(1) (U,i,π) is a *definable extension* of G by M if we have an exact sequence

$$0\longrightarrow M\xrightarrow{\ i\ }U\xrightarrow{\ \pi\ }G\longrightarrow 1$$

in the category of definable groups with definable homomorphisms. A definable map $s : G \to U$ is said *definable section* of (U, i, π) if for all $g \in G$, $\pi \circ s(g) = g$. And s is said a *definable basic section* if $s(1_G) = 1_U$.

(2) Two definable extensions (U, i, π) and (U', i', π') of G by M are *definably equivalent* if there is a definable homomorphism $\varphi : U \to U'$ such that the following diagram



is commutative.

The next fact assures the existence of a definable section for a definable extension. The proof is basically the same that appears in [2, Theorem 3.10] only that, instead of M, we use the normal subgroup $i(M) = \ker \pi \trianglelefteq U$.

Fact 3. *Let (U, i, π) be a definable extension of G by M. Then there is a definable section $s : G \to U$.*

Corollary 5.2. *Let (U, i, π) be a definable extension of G by M. Then there is a definable section $s : G \to U$ such that $s(1_G) = 1_U$.*

Proof. By Fact 3, there is a definable section $s' : G \to U$ such that $\pi \circ s' = \mathrm{id}_G$. Let $s(g) := \big(s'(1_G)\big)^{-1} s'(g)$ for $g \in G$, then s is a definable map. Finally, since $s'(1_G) \in \ker \pi$, we have that

$$\begin{aligned} \pi \circ s(g) &= \pi\Big(\big(s'(1_G)\big)^{-1} s'(g)\Big) \\ &= \pi\Big(\big(s'(1_G)\big)^{-1}\Big)\pi\big(s'(g)\big) \\ &= g. \end{aligned}$$

☑

***Remark* 5.3.** Any definable extension (U, i, π) of G by M induces an action of G on M, $\gamma_U : G \to \mathrm{Aut}(M)$ given by $\gamma_U(g)(a) := i^{-1}\Big(\big\langle s(g)\big\rangle i(a)\Big)$, where i^{-1} is the inverse homomorphism of $i : M \to i(M) \trianglelefteq U$, s is any definable basic section of (U, i, π) (which exists by Corollary 5.2) and, $\big\langle s(g)\big\rangle$ is the inner automorphism by $s(g)$. The map γ_U is well defined and is effectively an action of G on M (the reader can verify this with simple and direct calculations).

Definition 5.4. A definable extension (U, i, π) of G by M is said *compatible* with the definable G–module (M, γ) if $\gamma_U(g) = \gamma(g)$ for all $g \in G$.

Note that in the set of all definable extensions of G by M, the relation "are definably equivalent extensions" defines an equivalence relation. Simple calculations show that if (U, i, π) is a definable extension of G by M which is compatible with the definable G–module (M, γ), any extension in the class of (U, i, π) will be also compatible with (M, γ). The set of equivalence classes of definable extensions of G by M compatibles with the G–module (M, γ) will be denoted by $\operatorname{Ext}(G, M, \gamma)$. The set $\operatorname{Ext}(G, M, \gamma)$ can be endowed with a product that gives a group structure (see [5, Sections 4 and 5] or [3, Section 4]). The identity element in $\operatorname{Ext}(G, M, \gamma)$ will have by representative the trivial definable extension $(M \rtimes_\gamma G, i_0, \pi_0)$ (the semidirect group between M and G given by the action γ) where $i_0(m) = (m, 1_G)$, $\pi_0(m, g) = g$. The product on $M \rtimes_\gamma G$ is given by the formula

$$(m_1, g_1)(m_2, g_2) = \big(m_1 + \gamma(g_1)m_2, g_1g_2\big).$$

Below we present in Proposition 5.5 and Fact 4 a special kind of extensions, the induced extension by a cocycle and the induced extension by an extension. These let us to establish a bijection between the second cohomology group and the group $\operatorname{Ext}(G, M, \gamma)$.

Proposition 5.5. *Let $f \in Z^2(G, M, \gamma)$ and $U_f \coloneqq M \times G$. We define in U_f a binary operation given by the formula*

$$(m_1, g_1)(m_2, g_2) = \big(m_1 + \gamma(g_1)m_2 + f(g_1, g_2), g_1g_2\big)$$

and, $i_f(m) = (m, 1_G)$, $\pi_f(m, g) = g$. Then (U_f, i_f, π_f) is an element of $\operatorname{Ext}(G, M, \gamma)$. *$(U_f, i_f, \pi_f)$ will be called the induced extension by the cocycle f.*

Proof. The associativity of the product on U_f follows basically by the cocycle condition of f because of

$$\begin{aligned}\big((m_1, g_1)(m_2, g_2)\big)(m_3, g_3) &= \big(m_1 + \gamma(g_1)m_2 + f(g_1, g_2), g_1g_2\big)(m_3, g_3) = \\ &\big(m_1 + \gamma(g_1)m_2 + f(g_1, g_2) + \gamma(g_1g_2)m_3 + f(g_1g_2, g_3), g_1g_2g_3\big)\end{aligned}$$

and

$$\begin{aligned}(m_1, g_1)\big((m_2, g_2)(m_3, g_3)\big) &= (m_1, g_1)\big(m_2 + \gamma(g_2)m_3 + f(g_2, g_3), g_2g_3\big) = \\ &\big(m_1 + \gamma(g_1)m_2 + \gamma(g_1g_2)m_3 + \gamma(g_1)f(g_2, g_3) + f(g_1, g_2g_3), g_1g_2g_3\big).\end{aligned}$$

The equality follows by the fact that

$$f(g_1, g_2) + f(g_1g_2, g_3) - \gamma(g_1)f(g_2, g_3) - f(g_1, g_2g_3) = 0.$$

The identity element is $\big(-f(1_G, 1_G), 1_G\big)$ and the inverse of (m, g) is

$$\Big(-\gamma\big(g^{-1}\big)m - \gamma\big(g^{-1}\big)f\big(g, g^{-1}\big) - \gamma\big(g^{-1}\big)f(1_G, 1_G), g^{-1}\Big).$$

Simple calculations show that i_f, π_f are definable homomorphisms and that $0 \longrightarrow M \xrightarrow{i_f} U_f \xrightarrow{\pi_f} G \longrightarrow 1$ is an exact sequence of definable groups.

Finally, let $s(g) = \big(-f(1_G, 1_G), g\big)$. Then s is a definable section and

$$\begin{aligned}
\gamma_U(g)(m) &= i_f^{-1}\Big(s(g)i_f(m)\big(s(g)\big)^{-1}\Big)\\
&= i_f^{-1}\Big(\big(-f(1_G,1_G),g\big)(m,1_G)\Big(-\gamma\big(g^{-1}\big)f\big(g,g^{-1}\big),g^{-1}\Big)\Big)\\
&= i_f^{-1}\big(-f(1_G,1_G)+\gamma(g)m+f(g,1_G),g\big)\Big(-\gamma\big(g^{-1}\big)f\big(g,g^{-1}\big),g^{-1}\Big)\\
&= i_f^{-1}\big(\gamma(g)m,1_G\big)\\
&= \gamma(g)m.
\end{aligned}$$ ☑

The proof of the following fact is given by Edmundo in [2, Proposition 3.23].

Fact 4. *Let* $(U, i, \pi) \in \mathrm{Ext}(G, M, \gamma)$ *and*

$$f(g,h) = i^{-1}\big(s(g)s(h)s(gh)^{-1}\big)$$

for some definable basic section s *of* (U, i, π)*. We define in* $U_s := M \times G$ *a binary operation given by the formula*

$$(m_1, g_1)(m_2, g_2) = \big(m_1 + \gamma(g_1)m_2 + f(g_1, g_2), g_1g_2\big)$$

and $i_s(m) = (m, 1_G)$*,* $\pi_s(m, g) = g$*. Then,* $f \in Z^2(G, M, \gamma)$*,* (U_s, i_s, π_s) *is an element in* $\mathrm{Ext}(G, M, \gamma)$ *and is definably equivalent to* (U, i, π)*.* (U_s, i_s, π_s) *will be called the induced extension by* (U, i, π) *and by the definable section* s *and* f *will be called the definable cocycle associated to the section* s*.*

As well as in the classical case of group cohomology treated by Eilenberg and MacLane [4, (3.2)], the second cohomology group $H^2(G, M, \gamma)$ is in bijection with the group of definable extensions $\mathrm{Ext}(G, M, \gamma)$. Basically, the bijection assigns to an extension in $\mathrm{Ext}(G, M, \gamma)$ the equivalence class of the definable cocycle associated to any definable section of the given extension.

Fact 5 ([2, Remark 3.29])**.** *There is a bijection between the groups* $\mathrm{Ext}(G, M, \gamma)$ *and* $H^2(G, M, \gamma)$*.*

Actually, the bijection ψ is a group isomorphism with the extension product as defined in [5, Section 4 and Section 5] and the sum of classes in $H^2(G, M, \gamma)$ induced by the usual sum of functions.

Finally, we present the following result showing a relationship between definable extensions given by two definably equivalent actions; its proof is based on the isomorphism between cohomology groups with definably equivalent actions (Proposition 4.1). This fact is useful in the characterization of certain groups which are definable extensions as it was seen in [1].

Proposition 5.6. *Let G be a definable group (M, γ_1) and (M, γ_2) two definable G–modules. If γ_1 and γ_2 are two definably equivalent actions through $\psi : G \to G$ with $\gamma_1 = \gamma_2 \circ \psi$ and if (U, i, π) is an element of $\mathrm{Ext}(G, M, \gamma_2)$ then, there is (U', i', π') in $\mathrm{Ext}(G, M, \gamma_1)$ such that U and U' are definably isomorphic groups.*

Proof. Let s be a definable section of (U, i, π) and f its associated cocycle. By Proposition 4.1 there is an isomorphism σ between $H^2(G, M, \gamma_2)$ and $H^2(G, M, \gamma_1)$ such that $\sigma\big([f]\big) = \big[\, \overline{f} \,\big]$ where $\overline{f}(g_1, g_2) = f\big(\psi(g_1), \psi(g_2)\big)$. Thus, $\overline{f} \in Z^2(G, M, \gamma_1)$ and, by Proposition 5.5, the induced extension by $\overline{f}$ is an element of $\mathrm{Ext}(G, M, \gamma_1)$, we will denote such an extension by (U', i', π').

Consider the definable bijection $\theta : U \to U'$, $\theta\big(i(m)s(g)\big) = \big(m, \psi^{-1}(g)\big)$. It is enough to show that θ is a group homomorphism. ☑

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