

Field of Moduli and Generalized Fermat Curves

Cuerpo de moduli y curvas de Fermat generalizadas

RUBEN A. HIDALGO^{1,a,✉}, SEBASTIÁN REYES-CAROCCA²,
MARÍA ELISA VALDÉS^{3,a}

¹Universidad Técnica Federico Santa María, Valparaíso, Chile

²Universidad Autónoma de Madrid, Madrid, España

³Universidad de Concepción, Concepción, Chile

ABSTRACT. A generalized Fermat curve of type (p, n) is a closed Riemann surface S admitting a group $H \cong \mathbb{Z}_p^n$ of conformal automorphisms with S/H being the Riemann sphere with exactly $n + 1$ cone points, each one of order p . If $(p - 1)(n - 1) \geq 3$, then S is known to be non-hyperelliptic and generically not quasiplatonic. Let us denote by $\text{Aut}_H(S)$ the normalizer of H in $\text{Aut}(S)$. If p is a prime, and either (i) $n = 4$ or (ii) n is even and $\text{Aut}_H(S)/H$ is not a non-trivial cyclic group or (iii) n is odd and $\text{Aut}_H(S)/H$ is not a cyclic group, then we prove that S can be defined over its field of moduli. Moreover, if $n \in \{3, 4\}$, then we also compute the field of moduli of S .

Key words and phrases. Algebraic curves, Riemann surfaces, Field of moduli, Field of definition.

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RESUMEN. Una curva de Fermat generalizada de tipo (p, n) es una superficie de Riemann cerrada S la cual admite un grupo $H \cong \mathbb{Z}_p^n$ de automorfismos conformales de manera que S/H sea de género cero y tenga exactamente $n + 1$ puntos cónicos, cada uno de orden p . Si $(p - 1)(n - 1) \geq 3$, entonces se sabe que S no es hiperelíptica y genéricamente no es casiplatónica. Denotemos por $\text{Aut}_H(S)$ el normalizador de H en $\text{Aut}(S)$. Si p es primo y tenemos que (i) $n = 4$ o bien (ii) n es par y $\text{Aut}_H(S)/H$ no es un grupo cíclico no trivial o bien (iii) n es impar y $\text{Aut}_H(S)/H$ no es un grupo cíclico, entonces verificamos que

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S se puede definir sobre su cuerpo de moduli. Más aún, si $n \in \{3, 4\}$, entonces determinamos tal cuerpo de moduli.

Palabras y frases clave. Curvas algebraicas, superficies de Riemann, cuerpo de moduli, cuerpo de definición.

1. Introduction

A field of definition of an algebraic curve C is a subfield $\mathbb{F} < \mathbb{C}$ for which there is an algebraic curve D , defined over \mathbb{F} , isomorphic to C ; we also say that C is definable over \mathbb{F} . In this paper we will only consider irreducible and non-singular curves.

Let C be an irreducible non-singular complex projective algebraic curve (i.e. a closed Riemann surface), defined by homogeneous complex polynomials P_1, \dots, P_r . If $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, where $\text{Gal}(\mathbb{C}/\mathbb{Q})$ denotes the group of field automorphisms of the complex number field \mathbb{C} , then the polynomials $P_1^\sigma, \dots, P_r^\sigma$ (where P_j^σ is obtained from P_j by replacing its coefficients by their σ -images) define a new irreducible non-singular complex projective algebraic curve, say C^σ . The curves C^σ and C might be or not isomorphic Riemann surfaces. If $G(C)$ is the subgroup of $\text{Gal}(\mathbb{C}/\mathbb{Q})$ whose elements are those σ for which C^σ and C are isomorphic, then its fixed field $\mathcal{M}(C)$ is the field of moduli of C . Results of Koizumi [17] ensure that the field of moduli of C is contained in all fields of definition and that it is the intersection of all fields of definitions. Dèbes-Emsalem [4] (see also Hammer-Herrlich [10]) proved that there is a field of definition being an extension of finite degree of the field of moduli.

Note from the definition that if C and D are isomorphic curves, then $\mathcal{M}(C) = \mathcal{M}(D)$. In particular, it permits to define the field of moduli of a closed Riemann surface S , say $\mathcal{M}(S)$, as the field of moduli of any curve defining it. In this way, $\text{Gal}(\mathbb{C}/\mathbb{Q})$ has a natural action over the moduli space \mathfrak{M}_g of closed Riemann surfaces of genus g , and the stabilizer of such an action at the point $[S] \in \mathfrak{M}_g$ is exactly $G(C)$, where C is a curve representing S .

In the particular case that S is definable over $\overline{\mathbb{Q}}$, the above action also provides an action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at the classes of Belyi curves (i.e. closed Riemann surfaces admitting a holomorphic branched cover over $\widehat{\mathbb{C}}$ with at most three branched values; called a Belyi map). Grothendieck [9] noticed that Belyi pairs are in correspondence with dessins d'enfants (i.e. bipartite maps on closed orientable surfaces). In this way, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on dessins d'enfants and it provides a combinatorial method to understand the structure of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (see, for instance, the survey [24] for more details).

If $S = \widehat{\mathbb{C}}$, then its field of moduli is \mathbb{Q} and clearly it is definable over it. If S has genus one, then there is a holomorphic branched cover of degree two over $\widehat{\mathbb{C}}$ whose branch values are $\infty, 0, 1$ and some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. In this case, it is well known that $\mathcal{M}(S) = \mathbb{Q}(j(\lambda))$, where j denotes the Klein elliptic j -function, and that S can be defined over it.

If S has genus $g \geq 2$, then both, the computation of the field of moduli and the determination of whether it is a field of definition, are difficult problems. Necessary conditions for S to be defined over its field of moduli are provided by Weil's Galois descent theorem [23] (see also Section 3). A direct consequence, of Weil's theorem, is that if $\text{Aut}(S)$ is trivial, then the field of moduli is a field of definition. Another consequence, see [4] and [1], is that if $S/\text{Aut}(S)$ has genus zero and the set of all its cone values of a same order is odd, then S can also be defined over its field of moduli. This fact was also stated by Wolfart [24] in the case when $S/\text{Aut}(S)$ has signature $(0; a, b, c)$ (i.e. when S is a quasiplatonic curve). In general, one may check that if $S/\text{Aut}(S)$ has genus zero, then S can be defined either over its field of moduli or over an extension of degree two of it. Explicit examples of hyperelliptic Riemann surfaces, which cannot be defined over their fields of moduli, were provided separately by Earle [5] and Shimura [22]. Quer-Cardona [20] proved that every closed Riemann surface of genus two, admitting conformal automorphisms besides the hyperelliptic involution and the trivial one, can be defined over their field of moduli. Huggins [16] proved that a hyperelliptic Riemann surface S , with hyperelliptic involution ι , for which $\text{Aut}(S)/\langle \iota \rangle$ is neither trivial nor cyclic, can be defined over its field of moduli. Moreover, in the same paper, for each n so that $\text{Aut}(S)/\langle \iota \rangle \cong \mathbb{Z}_n$ there is constructed an example which cannot be definable over its field of moduli. In [5] Earle stated the existence of non-hyperelliptic Riemann surfaces which cannot be defined over their fields of moduli, but no explicit example was provided. In [11, 12] the first author provided explicit examples of non-hyperelliptic Riemann surfaces which cannot be definable over their fields of moduli. Each of these non-hyperelliptic examples is an example of a generalized Fermat curve of type $(2, 5)$ (see below for the definition) and Earle's examples appear as quotient of them by the action of a freely acting subgroup isomorphic to \mathbb{Z}_2^3 . Other examples of non-hyperelliptic curves not definable over their field of moduli were found by Kontogeorgis in [18] and Artebani-Quispe in [1].

A closed Riemann surface S is called a generalized Fermat curve of type (p, n) , with $p \geq 2$ and $n \geq 1$ integers, if there is some group $H < \text{Aut}(S)$ with $H \cong \mathbb{Z}_p^n$, and S/H an orbifold of signature $(0; p, n+1, p)$, i.e., its underlying Riemann surface structure has genus zero and it has exactly $n + 1$ conical points, each one of order p . In this case, H is called a generalized Fermat group of type (p, n) and (S, H) a generalized Fermat pair of type (p, n) . By the Riemann-Hurwitz formula, in this situation, S has genus $g(p, n) = 1 + p^{n-1}((n-1)(p-1) - 2)/2$. In particular, $g(p, n) < 2$ if and only if $(p, n) \in \{(p, 1), (2, 3), (3, 2), (2, 2)\}$, in which case, it is known that S can be defined over its field of moduli. Hyperbolic generalized Fermat curves (that is, the ones with $g(p, n) > 1$) are necessarily non-hyperelliptic [8]. In Section 2 we recall the description of these Riemann surfaces in terms of Fuchsian groups and algebraic curves. Generalized Fermat curves of type $(p, 2)$ correspond to classical Fermat curves $x^p + y^p + z^p = 0$ and the ones of type (p, n) , with

$n \geq 3$, turn out to be the fiber product of $n - 1$ classical Fermat curves. We are interested in the computation of the field of moduli of generalized Fermat curves and to see if they are fields of definition. In [15] we computed the field of moduli for type $(2, 4)$ and proved that it is in fact a field of definition. In [11] we constructed examples of type $(2, 5)$ which cannot be definable over their fields of moduli. In [21] the second author studied types (p, n) , with $p \geq 3$ being a prime and n being either even or $n \in \{3, 5\}$.

Let us observe that, in the case n even, the orbifold S/H has an odd signature $(0; p, \dots, p)$. Generically $\text{Aut}(S) = H$ and it follows from Dèbes-Emsalem's results in [4] that S can be defined over its field of moduli. But, there are cases with $\text{Aut}(S) \neq H$, so it may be that the number of cone points of $S/\text{Aut}(S)$ is not odd. We instead proceed to see at the quotient $S/\text{Aut}_H(S)$. For it, we make a subtle modification of Dèbes-Emsalem's methods for our situation. For this we use the fact that, if $p \geq 2$ is a prime, then a generalized Fermat group of type (p, n) of S is unique up to conjugation [7]. Our main result is the following.

Theorem 1.1. *Let (S, H) be a generalized Fermat curve of type (p, n) , where p is a prime, and let $\text{Aut}_H(S)$ be the normalizer of H in $\text{Aut}(S)$.*

- (1) *If $n = 4$, then S can be defined over its field of moduli.*
- (2) *If n is even and $\text{Aut}_H(S)/H$ is not a non-trivial cyclic group, then S can be defined over its field of moduli.*
- (3) *If n is odd and $\text{Aut}_H(S)/H$ is not a cyclic group, then S can be defined over its field of moduli.*

In [14] we have generalized the above result to other class of Riemann surfaces.

The uniqueness property, up to conjugation, of a generalized Fermat group of type (p, n) , where p is prime, permits us to compute the field of moduli in the case that $n \in \{3, 4\}$ (see Theorem 2.2).

2. Generalized Fermat Curves: Algebraic Description

Let (S, H) be a given generalized Fermat pair of type (p, n) with $p \geq 2$, $(p - 1)(n - 1) \geq 3$ (this ensures that S has genus greater than one). We fix a regular branched cover $\pi : S \rightarrow S/H = \widehat{\mathbb{C}}$ with H as its deck group. We may assume, without loss of generality, that these branch values are $\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}$, where $\lambda_j \in \mathbb{C} \setminus \{0, 1\}$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. In [3] it was noticed that if $p = 2$ and $n \in \{4, 5\}$, then H is unique inside the full group of automorphisms. In [8] it was also proved that, for p a prime number, the generalized Fermat group is unique up to conjugation in $\text{Aut}(S)$. The uniqueness is also proved for any type $(p, 3)$ in [7].

We denote by $\text{Aut}_H(S)$ the normalizer subgroup of H in $\text{Aut}(S)$ and by $\text{Aut}_{orb}(S/H)$ the group of conformal automorphisms of the orbifold S/H (that

is, the subgroup of the group \mathbb{M} of Möbius transformations permuting the $n + 1$ cone points). If $k \in \text{Aut}_H(S)$, then there is a unique $\theta(k) \in \text{Aut}_{orb}(S/H)$ so that $\pi \circ k = \theta(k) \circ \pi$ and the map $\theta : \text{Aut}_H(S) \rightarrow \text{Aut}_{orb}(S/H)$ defines a homomorphism of groups. In this way, $\text{Aut}_H(S)/H$ can be seen as a subgroup of $\text{Aut}_{orb}(S/H)$.

2.1. Fuchsian Description

As S/H is an orbifold with signature $(0; p, n+1, p)$, it follows from the classical uniformization theorem that there is a Fuchsian group Γ with presentation

$$\Gamma = \langle x_1, \dots, x_{n+1} : x_1^p = \dots = x_{n+1}^p = x_1 x_2 \dots x_n x_{n+1} = 1 \rangle,$$

so that $\mathbb{H}^2/\Gamma = S/H$. It is not difficult to see that $\mathbb{H}^2/\Gamma' = S$ and that $H = \Gamma/\Gamma'$, where Γ' is the derived subgroup of Γ . As Γ' is unique in Γ , it follows that θ is surjective, that is, $\text{Aut}_H(S)/H = \text{Aut}_{orb}(S/H)$. In particular, $\text{Aut}_H(S)$ is obtained by lifting the group $\text{Aut}_{orb}(S/H)$ to S by the branched cover map $\pi : S \rightarrow S/H$ (in [8] is explained how to compute the corresponding liftings).

Note that for $n \geq 4$ the group $\text{Aut}_{orb}(S/H)$ is generically trivial, that is, $\text{Aut}_H(S) = H$. In [13] it was proved that there is a prime p_n (depending on n) so that if $p \geq p_n$ is a prime, then $\text{Aut}_H(S) = \text{Aut}(S)$.

2.2. Algebraic Description

It is well known (see for instance [6]) that, by using a basis of the g -dimensional complex space of holomorphic one-forms $H^{1,0}(S)$, the non-hyperelliptic Riemann surface S may be holomorphically embedded as a smooth complex projective algebraic curve of degree $2g - 2$ in $\mathbb{P}_{\mathbb{C}}^{g-1}$; called a canonical curve. In [8] it was proved that S can also be holomorphically embedded in $\mathbb{P}_{\mathbb{C}}^n$ as the following smooth complex projective algebraic curve

$$C(\lambda_1, \dots, \lambda_{n-2}) : \left\{ \begin{array}{cccccc} x_1^p & + & x_2^p & + & x_3^p & = & 0 \\ \lambda_1 x_1^p & + & x_2^p & + & x_4^p & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{n-2} x_1^p & + & x_2^p & + & x_{n+1}^p & = & 0 \end{array} \right\} \subset \mathbb{P}_{\mathbb{C}}^n,$$

and that, in this algebraic representation, the generalized Fermat group H is generated by the diagonal linear projective transformations a_1, \dots, a_n , where a_j is multiplication by $e^{2\pi i/p}$ to the coordinate x_j , and the map

$$\pi : C(\lambda_1, \dots, \lambda_{n-2}) \rightarrow \widehat{\mathbb{C}}$$

is defined by $\pi([x_1 : \dots : x_{n+1}]) = -(x_2/x_1)^p$ (a Galois cover with H as cover group whose branch values are $\infty, 0, 1, \lambda_1, \dots, \lambda_{n-3}$ and λ_{n-2}).

2.3. Computation of the Field of Moduli: Case p Prime

In this part we assume that p is a prime number. The domain

$$\Omega_n = \{(\lambda_1, \dots, \lambda_{n-2}) \in \mathbb{C}^{n-2} : \lambda_j \notin \{0, 1\} \text{ and } \lambda_i \neq \lambda_j\}$$

parametrizes the collection of generalized Fermat curves of type (p, n) by associating to each tuple $(\lambda_1, \dots, \lambda_{n-2}) \in \Omega_n$ the curve $C(\lambda_1, \dots, \lambda_{n-2})$.

Remark 2.1. The space Ω_n is also a model of the moduli space of ordered $(n+1)$ different points over the Riemann sphere. This is related to the fact that generalized Fermat groups of type (p, n) are all conjugated inside the full group of conformal automorphisms of the corresponding generalized Fermat curve.

Let \mathbb{G}_n be the subgroup of holomorphic automorphisms of Ω_n with the property that if $T \in \mathbb{G}_n$ and $T(\lambda_1, \dots, \lambda_{n-2}) = (\mu_1, \dots, \mu_{n-2})$, then $C(\lambda_1, \dots, \lambda_{n-2}) \cong C(\mu_1, \dots, \mu_{n-2})$. The quotient orbifold Ω_n/\mathbb{G}_n is then a model of the moduli space of generalized Fermat curves of type (p, n) and also a model of the moduli space of $(n+1)$ unordered different points on the Riemann sphere.

If we fix a pair $i \neq j$ and set $\mu_k = \lambda_k$ for $k \notin \{i, j\}$, $\mu_i = \lambda_j$ and $\mu_j = \lambda_i$, then $C(\lambda_1, \dots, \lambda_{n-2}) \cong C(\mu_1, \dots, \mu_{n-2})$; an isomorphism is provided by the permutation of the coordinates x_i and x_j . So the holomorphic automorphism of Ω_n defined by permutation of the coordinates λ_i with λ_j belongs to \mathbb{G}_n .

The uniqueness of H , up to conjugation, asserts that

$$C(\lambda_1, \dots, \lambda_{n-2}) \cong C(\mu_1, \dots, \mu_{n-2})$$

if and only if there is some Möbius transformation M such that

$$\{M(\infty), M(0), M(1), M(\lambda_1), \dots, M(\lambda_{n-2})\} = \{\infty, 0, 1, \mu_1, \dots, \mu_{n-2}\}.$$

To find such a Möbius transformation, we may proceed as follows. For each choice $\{a_1, \dots, a_{n+1}\} = \{\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}\}$, we search for a Möbius transformation M so that $M(a_1) = \infty$, $M(a_2) = 0$ and $M(a_3) = 1$. If $\mu_j = M(a_{3+j})$, for $j = 1, \dots, n-2$, then we obtain $T \in \mathbb{G}_n$ by setting $T(\lambda_1, \dots, \lambda_{n-2}) = (\mu_1, \dots, \mu_{n-2})$. This permits to obtain that $\mathbb{G}_n = \langle A, B \rangle$, where

$$\begin{aligned} A(\lambda_1, \dots, \lambda_{n-2}) &= (1/\lambda_1, \dots, 1/\lambda_{n-2}), \\ B(\lambda_1, \dots, \lambda_{n-2}) &= \\ &(\lambda_{n-2}/(\lambda_{n-2} - 1), \lambda_{n-2}/(\lambda_{n-2} - \lambda_1), \dots, \lambda_{n-2}/(\lambda_{n-2} - \lambda_{n-3})). \end{aligned}$$

The transformation A is obtained by setting $a_1 = 0$, $a_2 = \infty$, $a_3 = 1$, $a_{3+j} = \lambda_j$ and B is obtained by setting $a_1 = \lambda_{n-2}$, $a_2 = \infty$, $a_3 = 0$, $a_4 = 1$

and $a_{4+j} = \lambda_j$. In particular, if $n \geq 4$, then $\mathbb{G}_n \cong \mathfrak{S}_{n+1}$ (the symmetric group on $n + 1$ letters), $\mathbb{G}_3 \cong \mathfrak{S}_3$ and \mathbb{G}_2 is trivial. Note that \mathbb{G}_3 is isomorphic to \mathfrak{S}_4/V_2 , $V_4 \cong \mathbb{Z}_2^2$, which is isomorphic to the symmetric group \mathfrak{S}_3 .

A natural degree $3!$ map $j : \Omega_3 \rightarrow \mathbb{C}$ which is invariant under $\mathbb{G}_3 \cong \mathfrak{S}_3$ is the classical Klein j -function

$$j(\lambda) = \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2},$$

obtained by averaging the map $L(\lambda_1) = \lambda_1^2$ under the group \mathfrak{S}_3 .

In [15] it was provided an explicit degree $5!$ map

$$j_4 : \Omega_4 \rightarrow \mathbb{C}^2 : (z, w) \mapsto (j_1(\lambda_1, \lambda_2), j_2(\lambda_1, \lambda_2))$$

so that $j(T(\lambda_1, \lambda_2)) = j(\lambda_1, \lambda_2)$, for every $T \in \mathbb{G}_4 \cong \mathfrak{S}_5$ and every $(\lambda_1, \lambda_2) \in \Omega_4$ and so that j_1 and j_2 are rational maps defined over \mathbb{Z} . We should observe that in [15] the computation was done for the case $p = 2$, but this also works for any prime p . This, in particular, asserts that the moduli space Ω_4/\mathbb{G}_4 can be holomorphically embedded in \mathbb{C}^2 (similar to the case $n = 3$). This map was obtained by averaging the map $L(\lambda_1, \lambda_2) = (\lambda_1^2, \lambda_2^4)$ under the group \mathbb{G}_4 .

Given $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ and a generalized Fermat curve of type (p, n) , say $C(\lambda_1, \dots, \lambda_{n-2})$, we have the new generalized Fermat curve of type (p, n) , this given by $C(\lambda_1, \dots, \lambda_{n-2})^\sigma = C(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$. In general, $C(\lambda_1, \dots, \lambda_{n-2})$ and $C(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$ are not isomorphic as Riemann surfaces. Let us denote by $K(\lambda_1, \dots, \lambda_{n-2})$ the subgroup consisting of all those $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ for which $C(\lambda_1, \dots, \lambda_{n-2})$ and $C(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$ are isomorphic Riemann surfaces. The fixed field of $K(\lambda_1, \dots, \lambda_{n-2})$ is the field of moduli $\mathcal{M}(C(\lambda_1, \dots, \lambda_{n-2}))$ of $C(\lambda_1, \dots, \lambda_{n-2})$.

Theorem 2.2. *Let $p \geq 3$ be a prime and (S, H) be a generalized Fermat pair of type (p, n) , where $n \in \{3, 4\}$. Let us assume the cone points of S/H to be (i) $\infty, 0, 1, \lambda_1$ for $n = 3$ and (ii) $\infty, 0, 1, \lambda_1, \lambda_2$ for $n = 4$. Then the field of moduli of S is*

- (1) $\mathbb{Q}(j(\lambda_1))$, for $n = 3$.
- (2) $\mathbb{Q}(j_1(\lambda_1, \lambda_2), j_2(\lambda_1, \lambda_2))$, for $n = 4$.

Proof. Let $n \in \{3, 4\}$ and $\sigma \in K(\lambda_1, \dots, \lambda_{n-2})$. As $C(\lambda_1, \dots, \lambda_{n-2}) \cong C(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$, there exists $T \in \mathbb{G}_n$ so that $T(\lambda_1, \dots, \lambda_{n-2}) = (\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$. Now, as the map j has coefficients in \mathbb{Q} , we have that $\sigma(j(\lambda_1, \dots, \lambda_{n-2})) = j(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2})) = j \circ T(\lambda_1, \dots, \lambda_{n-2}) = j(\lambda_1, \dots, \lambda_{n-2})$, from which we obtain that, for each $i = 1, \dots, n - 2$,

$\sigma(j_i(\lambda_1, \dots, \lambda_{n-2})) = j_i(\lambda_1, \dots, \lambda_{n-2})$; that is, $\mathbb{Q}(j_i(\lambda_1, \dots, \lambda_{n-2})) < \mathcal{M}(C(\lambda_1, \dots, \lambda_{n-2}))$; so

$$\mathbb{Q}(j_1(\lambda_1, \dots, \lambda_{n-2}), \dots, j_{n-2}(\lambda_1, \dots, \lambda_{n-2})) < \mathcal{M}(C(\lambda_1, \dots, \lambda_{n-2})).$$

Now, if $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ satisfies that $\sigma(j(\lambda_1, \dots, \lambda_{n-2})) = j(\lambda_1, \dots, \lambda_{n-2})$, then we have that $j(\lambda_1, \dots, \lambda_{n-2}) = \sigma(j(\lambda_1, \dots, \lambda_{n-2})) = j(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$. It follows the existence of some $T \in \mathbb{G}_n$ so that $T(\lambda_1, \dots, \lambda_{n-2}) = (\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$. In particular, both generalized Fermat curves $C(\lambda_1, \dots, \lambda_{n-2})$ and $C(\sigma(\lambda_1), \dots, \sigma(\lambda_{n-2}))$ are isomorphic; so $\sigma \in K(\lambda_1, \dots, \lambda_{n-2})$, in particular

$$\mathcal{M}(C(\lambda_1, \dots, \lambda_{n-2})) < \mathbb{Q}(j_1(\lambda_1, \dots, \lambda_{n-2}), \dots, j_{n-2}(\lambda_1, \dots, \lambda_{n-2})). \quad \square$$

Remark 2.3. The fact that \mathbb{C} is algebraically closed field of characteristic zero provides a correspondence between subfields of \mathbb{C} and subgroups of $\text{Aut}(\mathbb{C}/\mathbb{Q})$ similar to Galois extensions. In fact, assume K is a subfield of \mathbb{C} and let \bar{K} be the algebraic closure of K in \mathbb{C} . If $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ acts as the identity in K , then σ restricts to an automorphism of \bar{K} . Now, as K has characteristic zero, then the extension \bar{K}/K is Galois. This property has been used in the second part of the above proof.

Remark 2.4. In general, for $n \geq 5$ there is a degree $(n+1)!$ map (for a suitable $m \geq n-2$)

$$j : \Omega_n \rightarrow \mathbb{C}^m : (\lambda_1, \dots, \lambda_{n-2}) \mapsto (j_1((\lambda_1, \dots, \lambda_{n-2})), \dots, j_m((\lambda_1, \dots, \lambda_{n-2})))$$

which is a branched cover with \mathbb{G}_n as its deck group. But we do not know an explicit one with the property that j_1, \dots, j_m are rational maps defined over \mathbb{Q} . We believe this should be true for $m = n-2$ by averaging the map $L(\lambda_1, \dots, \lambda_{n-2}) = (\lambda_1^2, \lambda_2^4, \dots, \lambda_{n-2}^{2^{n-2}})$ under the group \mathbb{G}_n . If this is true, then Theorem 2.2 will hold in general.

3. Weil’s Theorem

Necessary conditions for an algebraic variety to be definable over its field of moduli were provided by Weil [23] (we write that statement at the level of curves).

Theorem 3.1 (Weil’s Galois descendent theorem [23]). *Let C be a (projective) algebraic curve defined over a finite Galois extension \mathbb{L} of a field \mathbb{K} . Assume that for every $\sigma \in \text{Aut}(\mathbb{L}/\mathbb{K})$ there is an isomorphism $f_\sigma : C \rightarrow C^\sigma$ defined over \mathbb{L} such that for all $\sigma, \tau \in \text{Aut}(\mathbb{L}/\mathbb{K})$ the compatibility condition $f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau$ holds. Then there exists a (projective) algebraic curve E defined over \mathbb{K} and there exists an isomorphism $Q : C \rightarrow E$, defined over \mathbb{L} , such that, for every $\sigma \in \text{Aut}(\mathbb{L}/\mathbb{K})$, the equality $Q^\sigma \circ f_\sigma = Q$ holds.*

If in Weil's Galois descent theorem the curve C is assumed to be irreducible and non-singular, the model E can also be assumed to have the same properties. In this paper we will be in that situation.

Weil's theorem works for finite Galois extensions in any characteristic, but we are working with the non-Galois extension $\mathbb{Q} < \mathbb{C}$. This will not be a problem as we should observe next. Let us assume the irreducible non-singular complex projective algebraic curve C has a finite group of automorphisms (for instance, this holds if C has genus at least 2). We know [17, 10] that C can be defined over a finite Galois extension of its field of moduli. So, let us assume C is already defined over a finite Galois extension $\mathbb{L} < \mathbb{C}$ of its field of moduli \mathbb{K} . Every $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{K})$ restricts to an element $\sigma \in \text{Aut}(\overline{\mathbb{K}}/\mathbb{K})$ and, by Galois theory, we get by restriction $\sigma \in \text{Aut}(\mathbb{L}/\mathbb{K})$. So, there are only a finite number of possibilities for C^σ . Also, since C has a finite number of automorphisms, it follows that there are only a finite number of possibilities for biholomorphisms $f_\sigma : C \rightarrow C^\sigma$, a priori defined over \mathbb{C} . In this way, if $\eta \in \text{Aut}(\mathbb{C}/\overline{\mathbb{K}})$, then $f_\sigma^\eta : C \rightarrow C^\sigma$, and $G_\sigma = \{\eta \in \text{Aut}(\mathbb{C}/\overline{\mathbb{K}}) : f_\sigma^\eta = f_\sigma\}$ has finite index in $\text{Aut}(\mathbb{C}/\overline{\mathbb{K}})$, so f_σ is defined over $\overline{\mathbb{K}}$. All of the above asserts that we may choose a finite Galois extension of \mathbb{K} with the property that C and all f_σ are defined over it. This observation asserts that Weil's Galois descent theorem can be stated in our case as follows.

Theorem 3.2 (Weil's descent theorem). *Let C be an irreducible non-singular complex projective algebraic curve of genus $g \geq 2$. Assume that for every $\sigma \in \text{Aut}(\mathbb{C}/\mathcal{M}(C))$ there is a biholomorphism $f_\sigma : C \rightarrow C^\sigma$ such that for all $\sigma, \tau \in \text{Aut}(\mathbb{C}/\mathcal{M}(C))$ the compatibility condition $f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau$ holds. Then there exists an irreducible non-singular complex projective algebraic curve E defined over $\mathcal{M}(C)$ and there exists a biholomorphism $Q : C \rightarrow E$ such that, for every $\sigma \in \text{Aut}(\mathbb{C}/\mathcal{M}(C))$, the equality $Q^\sigma \circ f_\sigma = Q$ holds.*

It can be seen that Weil's conditions in Theorem 3.2 hold trivially if $\text{Aut}(C)$ is trivial; so in this case C can always be defined over its field of moduli.

4. A Proof of Theorem 1.1

Before we provide a proof of Theorem 1.1, we state a known result (a consequence of Riemann-Roch's theorem) which guaranties the existence of rational points.

Lemma 4.1 (Lemma 5.1 in [16]). *Let \mathbb{K} be a subfield of \mathbb{C} . Let Z be a genus zero non-singular projective algebraic curve, defined over \mathbb{K} , and assume Z has a divisor D of odd degree and rational over \mathbb{K} . Then Z has infinitely many \mathbb{K} -rational points.*

Proof of Theorem 1.1. Let (C, H) a generalized Fermat pair of type (p, n) . We only need to consider the case $(p-1)(n-1) \geq 3$ since the exceptional cases

are of genus 0 or 1 which are known to be definable over their fields of moduli. As the Riemann orbifold $\mathcal{O} = C/\text{Aut}_H(C)$ has genus zero, we may identify it with the Riemann sphere $\widehat{\mathbb{C}}$. Let $P : C \rightarrow \widehat{\mathbb{C}}$ be a Galois cover with $\text{Aut}_H(C)$ as its group of deck transformations.

Set $K_C = \{\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) : C^\sigma \cong C\}$. If $\sigma \in K_C$, then there is a biholomorphism $f_\sigma : C \rightarrow C^\sigma$. The uniqueness of H , up to conjugation, asserts that we may choose f_σ so that $f_\sigma H f_\sigma^{-1} = H^\sigma$. In particular, $f_\sigma \text{Aut}_H(C) f_\sigma^{-1} = \text{Aut}_{H^\sigma}(C^\sigma)$ and it follows the existence of a Möbius transformation g_σ so that $P^\sigma \circ f_\sigma = g_\sigma \circ P$. It is clear that g_σ is uniquely determined by σ . This follows from the fact that if $\widehat{f}_\sigma : C \rightarrow C^\sigma$ is another isomorphism such that $\widehat{f}_\sigma H \widehat{f}_\sigma^{-1} = H^\sigma$, then $f_\sigma^{-1} \circ \widehat{f}_\sigma \in \text{Aut}_H(C)$.

The uniqueness of the automorphisms g_σ ensures that the collection $\{g_\sigma : \sigma \in K_C\}$ satisfies the conditions of Weil’s Galois descent theorem 3.2. It follows the existence of an irreducible projective algebraic curve Z , defined over $\mathcal{M}(C)$, and a biholomorphism $Q : \widehat{\mathbb{C}} \rightarrow Z$ so that $Q = Q^\sigma \circ g_\sigma$.

Let us now assume we are able to find a set of odd cardinality, say $\{a_1, \dots, a_{2m-1}\} \subset \widehat{\mathbb{C}}$, such that for every $\sigma \in K_C$ it holds that

$$\{g_\sigma(a_1), \dots, g_\sigma(a_{2m-1})\} = \{\sigma(a_1), \dots, \sigma(a_{2m-1})\}. \quad (*)$$

Then the divisor $D = Q(a_1) + \dots + Q(a_{2m-1})$ is a $\mathcal{M}(C)$ -rational divisor. Lemma 4.1 now ensures the existence of a $\mathcal{M}(C)$ -rational point q in Z which is not a branch value of R . Let us fix some point $p \in C$ so that $R(p) = q$. One may check that it is possible, for each $\sigma \in K_C$, to choose $f_\sigma : C \rightarrow C^\sigma$ so that $f_\sigma(p) = \sigma(p)$. In fact, we first note that $R^\sigma(f_\sigma(p)) = R(p) = q$ and that $R^\sigma(\sigma(p)) = \sigma(R(p)) = \sigma(q) = q$. In this way, there is a unique $h_\sigma \in \text{Aut}_{H^\sigma}(C^\sigma)$ so that $h_\sigma(f_\sigma(p)) = \sigma(p)$. Then replacing f_σ by $t_\sigma = h_\sigma \circ f_\sigma$ yields that $t_\sigma(p) = \sigma(p)$ and that still $t_\sigma H t_\sigma^{-1} = H^\sigma$; moreover the t_σ satisfy the conditions of Weil’s descent theorem, so C is definable over its field of moduli.

In order to find a set $\{a_1, \dots, a_{2m-1}\} \subset \widehat{\mathbb{C}}$ satisfying $(*)$ above we proceed as follows. Assume that $C = C(\lambda_1, \dots, \lambda_{n-2})$, for suitable values $\lambda_1, \dots, \lambda_{n-2}$, and let $\pi : C \rightarrow \widehat{\mathbb{C}}$ as defined in Section 2.

The group $\text{Aut}_H(C)/H \cong \text{Aut}_{orb}(\mathcal{O}_H)$ is the finite subgroup of Möbius transformations keeping invariant the set $\{\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}\}$. The finite subgroups of \mathbb{M} (see, for instance, [2, 19]) different from the trivial group are either (i) cyclic groups, (ii) dihedral groups, (iii) the alternating group \mathcal{A}_4 , (iv) the alternating group \mathcal{A}_5 and (v) the symmetric group \mathfrak{S}_4 . Let $N : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a regular branched cover with $\text{Aut}_{orb}(\mathcal{O}_H)$ as its deck group. We may assume, up to composition at the left of N by a suitable Möbius transformation (since the group of Möbius transformations acts triple-transitively), that the branch values of N are inside the set $\{\infty, 0, 1\}$. In this way, $P = N \circ \pi : C \rightarrow \widehat{\mathbb{C}}$ is a regular branched cover with $\text{Aut}_H(C)$ as its deck group.

4.1. Case $n = 4$

As $\text{Aut}_H(C)/H \cong \text{Aut}_{orb}(\mathcal{O}_H)$ keeps invariant the set $\{\infty, 0, 1, \lambda_1, \lambda_2\}$, it is not difficult to see that the only possibilities for $\text{Aut}_{orb}(\mathcal{O}_H)$ are given by either the trivial group, or the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ or the dihedral groups $\mathbb{D}_3, \mathbb{D}_5$. If $\text{Aut}_{orb}(\mathcal{O}_H)$ is neither trivial nor isomorphic to \mathbb{Z}_2 , then C is a quasiplatonic curve; so they are defined over their fields of moduli [24]. If $\text{Aut}_{orb}(\mathcal{O}_H)$ is trivial, then we may proceed as in Section 4.3 below. So, the only case we need to take care of is when $\text{Aut}_{orb}(\mathcal{O}_H) \cong \mathbb{Z}_2$. Under this assumption, the subgroup of \mathbb{M} keeping invariant the set of cone points $\infty, 0, 1, \lambda_1$ and λ_2 is generated by an elliptic transformation $A \in \mathbb{M}$ of order two. We may assume, without loss of generality, that $A(z) = 1/z$ and that $\langle A \rangle$ acts with two orbits of length two and one fix point. So, up to the action of \mathbb{G}_4 on the set Ω_4 of parameters (λ_1, λ_2) , we may assume that $\lambda_2 = \lambda_1^{-1}$ and that $\text{Aut}_{orb}(\mathcal{O}_H) = \langle A(z) = 1/z \rangle$. In this case, a lifting of A is the automorphism of C given as

$$\widehat{A}([x_1 : x_2 : x_3 : x_4 : x_5]) = [x_2 : x_1 : x_3 : \sqrt[p]{\lambda_1}x_5 : \sqrt[p]{\lambda_2}x_4]$$

and $\text{Aut}_H(C) = \langle H, \widehat{A} \rangle \cong \mathbb{Z}_p^4 \rtimes \mathbb{Z}_2$.

Now, we set $\lambda := \lambda_1, K_\lambda := K_C, C_\lambda := C$ and $\mathcal{O}_\lambda := \mathcal{O}_H$ and let $N : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ defined by

$$N(z) = \frac{-\lambda}{(1 + \lambda)^2}(z + 1/z + 2) + 1.$$

The map N defines a degree two branched cover with $\langle A \rangle$ as deck group. It can be checked that $P = N \circ \pi : C_\lambda \rightarrow \widehat{\mathbb{C}}$ defines a regular branched cover with $\text{Aut}_H(C_\lambda)$ as its deck group. Note that

$$\begin{aligned} N(\infty) &= N(0) = \infty, \\ N(\lambda) &= N(\lambda^{-1}) = 0, \\ N(1) &= \rho_\lambda = \left(\frac{\lambda - 1}{\lambda + 1}\right)^2, \\ N(-1) &= 1. \end{aligned}$$

In this way, the cone points of the orbifold $\mathcal{O}_\lambda = C_\lambda / \text{Aut}_H(C_\lambda)$ are given by $\infty, 0$ (both with cone order equal to p), 1 (with cone order equal to 2) and ρ_λ with cone order equal $2p$. In this case, the desired set is $\{a_1 = \rho_\lambda\}$ (if $p \neq 2$, then we may also use $a_1 = 1$).

4.2. Case Aut_H/H is not cyclic

Let us now assume $\text{Aut}_H(C)/H$ is not cyclic. Then this group should be either a dihedral group or isomorphic to either $\mathcal{A}_4, \mathcal{A}_5$ or \mathfrak{S}_4 . Let $N : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a

regular branched cover with $\text{Aut}_H(C)/H$ as its deck group. We may assume that the 3 branch values of N are $\infty, 0$ and 1 . The composition $P = N \circ \pi : C \rightarrow \widehat{\mathbb{C}}$ is a regular branched cover with $\text{Aut}_H(C)$ as its deck group. In the set of branch values of P we have the special subset $\{\infty, 0, 1\}$ which satisfies that, for every $\sigma \in K_C$, it holds that $\{\infty, 0, 1\}$ are the branch values of T^σ . So, in this case, the desired set is $\{a_1 = \infty, a_2 = 0, a_3 = 1\}$.

4.3. Case $\text{Aut}_H(C) = H$

In this case, the desired set is, for n even,

$$\{a_1 = \infty, a_2 = 0, a_3 = 1, a_4 = \lambda_1, \dots, a_{n+1} = \lambda_{n-2}\}. \quad \checkmark$$

5. Explicit Computations: An Example

In the proof of Theorem 1.1 we have seen, as a consequence of Weil’s descent theorem, the existence of a $\mathcal{M}(C)$ -rational point $q \in Z$. In practice, we only have explicitly given the Möbius transformations g_σ . The curve Z and the isomorphism $Q : \widehat{\mathbb{C}} \rightarrow Z$ are in general not known. This is not a problem since the existence of a point $q \in Z$, which is not a branch value of $R : C \rightarrow Z$, is equivalent to find a point $r \in \widehat{\mathbb{C}}$, which is not a branch values of P , satisfying that $\sigma(r) = g_\sigma(r)$, for every $\sigma \in K_C$. In fact $q = Q(r)$ is a $\mathcal{M}(C)$ -rational point since, for $\sigma \in K_C$, $\sigma(Q(r)) = Q^\sigma(\sigma(r)) = Q \circ g_\sigma^{-1}(\sigma(r))$.

5.1. An example

Let us consider the case $n = 4$ and $\text{Aut}_H(S)/H \cong \mathbb{Z}_2$. In this case, $\rho_\lambda = \left(\frac{\lambda-1}{\lambda+1}\right)^2 \neq -1$. In fact, otherwise $\lambda = \pm i$ and, in this case, the order four Möbius transformation $M(z) = (-iz + 1)/(z - i)$ keeps invariant the five cone points (it fixes 1 and permutes cyclically $\infty, -i, 0$ and i). This contradicts our assumption that the orbifold \mathcal{O}_λ has full group of automorphisms the cyclic group of order two generated by $A(z) = 1/z$. Now, for each $\sigma \in K_\lambda$ we have that $P^\sigma = N^\sigma \circ \pi^\sigma = N^\sigma \circ \pi$, where

$$N^\sigma(z) = \frac{-\sigma(\lambda)}{(1 + \sigma(\lambda))^2}(z + 1/z + 2) + 1.$$

5.1.1. Computation of a Point r

Also, as g_σ should preserve the cone points and the cone orders, we should have

$$g_\sigma(\{\infty, 0, 1\}) = \{\infty, 0, 1\} \tag{**}$$

$$g_\sigma(\rho_\lambda) = \rho_{\sigma(\lambda)} = \sigma(\rho_\lambda). \tag{***}$$

In particular, $g_\sigma \in \langle T(z) = 1/z, L(z) = 1/(1-z) \rangle \cong \mathfrak{S}_3$. It follows that $\Phi : K_\lambda \rightarrow \mathbb{M}$, defined as $\Phi(\sigma) = g_\sigma^{-1}$, is a homomorphism of groups. Moreover, $\text{Fix}(\ker(\Phi))$ contains $\mathbb{Q}(\rho_\lambda)$. In particular, we only need to find a point $r \in \mathbb{Q}(\rho_\lambda) \setminus \{0, 1, \rho_\lambda\}$ such that, for every $\sigma \in K_\lambda$ it holds the equality $g_\sigma(r) = \sigma(r)$.

- (1) If $\rho_\lambda = 2$ and $\sigma \in K_\lambda$, then (from $(***)$) we see that $2 = \sigma(2) = g_\sigma(2)$; it follows that $g_\sigma = I$ and that $U = \{I\}$. In this case, we may use any $r \in \mathbb{Q} \setminus \{0, 1, 2\}$.
- (2) If $\rho_\lambda = 1/2$, then we may proceed similarly as above to see that we may use any $r \in \mathbb{Q} \setminus \{0, 1, 1/2\}$.

Let us now assume $\rho_\lambda \notin \{1/2, 2, -1\}$. If we make the choice

$$r = \frac{\rho_\lambda(\rho_\lambda - 2)}{1 - 2\rho_\lambda},$$

then we notice, from $(***)$, that for every $\sigma \in K_\lambda$ it holds that $g_\sigma(r) = \sigma(r)$. As ρ_λ cannot be 0 nor ± 1 , one has that $r \in \mathbb{Q}(\rho_\lambda) \setminus \{0, 1, \rho_\lambda\}$ as required and we will be done in those cases.

5.1.2. Relation of the cases $\rho_\lambda \in \{1/2, 2\}$

If $\rho_\lambda = 2$, then $\{\lambda, 1/\lambda\} = \{-3 - 2\sqrt{2}, -3 + 2\sqrt{2}\}$. If $\rho_\lambda = 1/2$, then $\{\lambda, 1/\lambda\} = \{3 + 2\sqrt{2}, 3 - 2\sqrt{2}\}$.

In any of the above cases, $C_\lambda \cong C_{1/\lambda}$ by the isomorphism

$$f : [x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_1 : x_2 : x_3 : x_5 : x_4]$$

and $C_\lambda \cong C_{-\lambda}$ by the isomorphism

$$g : [x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [\omega x_1 : x_2 : x_3 : x_4 : x_5],$$

where $\omega^p = -1$.

5.1.3. Explicit rational model for $\lambda = \lambda_0 = 3 + 2\sqrt{2}$ and $p = 2$

The curve C_{λ_0} is already defined over the Galois extension $\mathbb{Q}(\sqrt{2})$. Set $\Gamma = \text{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \langle \sigma \rangle \cong \mathbb{Z}_2$, where $\sigma(\sqrt{2}) = -\sqrt{2}$ (we will denote by e the identity element). Since $C_{\lambda_0}^\sigma = C_{1/\lambda_0} \cong C_{\lambda_0}$, and C_{λ_0} , the field of moduli of C_{λ_0} is \mathbb{Q} . Theorem 1.1 asserts that C_{λ_0} can be defined over \mathbb{Q} . As $C_{\lambda_0}^\sigma = C_{1/\lambda_0}$, we have the isomorphism $f_\sigma : C_{\lambda_0} \rightarrow C_{1/\lambda_0}$ where $f_\sigma = f$. It is easy to see that $\{f_e = I, f_\sigma = f\}$ satisfies the conditions of Weil's descent theorem. Let us consider the affine model C of C_{λ_0} given by assuming $x_1 = 1$. The map

$$\begin{aligned} \Phi : C &\rightarrow \mathbb{C}^8 \\ (x_2, x_3, x_4, x_5) &\mapsto (x_2, x_3, x_4, x_5, x_2, x_3, x_5, x_4) = (y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4) \end{aligned}$$

defines a linear isomorphism $\Phi : C \rightarrow \Phi(C)$ (the inverse given by restriction of the projection on the first 4 coordinates). The equations defining $\Phi(C)$ are:

$$\begin{aligned} 1 + y_1^2 + y_2^2 &= 0, & z_1 &= y_1, \\ \lambda_0 + y_1^2 + y_3^2 &= 0, & z_2 &= y_2, \\ 1/\lambda_0 + y_1^2 + y_4^2 &= 0, & y_3 &= z_4, \\ & & y_4 &= z_3. \end{aligned}$$

Set $y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4)$. Let us consider the isomorphism

$$\Theta : \Gamma \rightarrow \Theta(\Gamma) = \widehat{\Gamma} < \text{GL}(\mathbb{C}^8)$$

defined by

$$\Theta(e) = I, \quad \Theta(\sigma)(z, w) = (w, z).$$

Notice that $\Theta(\sigma)(\Phi(C)) \neq \Phi(C)$ (since $C^\sigma \neq C$). A set of generators for the $\widehat{\Gamma}$ -invariant polynomials is given by

$$\begin{aligned} t_1 &= y_1 + z_1, & t_2 &= y_2 + z_2, & t_3 &= y_3 + z_3, & t_4 &= y_4 + z_4, \\ t_5 &= y_1^2 + z_1^2, & t_6 &= y_2^2 + z_2^2, & t_7 &= y_3^2 + z_3^2, & t_8 &= y_4^2 + z_4^2. \end{aligned}$$

Let us consider the map

$$\Psi : \mathbb{C}^8 \rightarrow \mathbb{C}^8 : (y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4) \mapsto (t_1, t_2, t_3, t_4, t_4, t_5, t_6, t_7, t_8).$$

The condition that $\Theta(\sigma)(\Phi(C)) \neq \Phi(C)$ asserts that $\Psi : \Phi(C) \rightarrow \Psi(\Phi(C)) = Z$ is a birational isomorphism, so $R = \Psi \circ \Phi : C \rightarrow Z$ is a birational isomorphism. The inverse $R^{-1} : Z \rightarrow C$ is given by

$$\begin{aligned} x_2 &= t_1/2, & x_3 &= t_2/2, \\ x_4 &= \frac{-6 - 4\sqrt{2} - t_5 + t_4^2 - t_7}{2t_4}, & x_5 &= \frac{6 + 4\sqrt{2} + t_5 + t_4^2 + t_7}{2t_4}. \end{aligned}$$

Equations defining Z (over \mathbb{Q}) are given by

$$\begin{aligned} t_7 - t_8 &= 0, & t_4^4 - 2t_4^2t_8 + 32 &= 0, \\ t_5 + t_8 + 6 &= 0, & t_2^2 - 2t_8 - 8 &= 0, \\ t_3 - t_4 &= 0, & t_1^2 + 2t_8 + 12 &= 0, \\ t_6 - t_8 - 4 &= 0, & & \end{aligned}$$

Notice that, from the above equations, we may write t_8, t_7, t_6, t_5 and t_4 in terms of t_1, t_2 and t_3 . In this way, if we restrict to the above curve the projection on these three coordinates, then we obtain that C is isomorphic to the curve in \mathbb{C}^3 (with coordinates $(u_1 = t_1, u_2 = t_2, u_3 = t_3)$)

$$\widehat{C}_0 = \left\{ \begin{array}{l} u_3^4 - u_2^2 u_3^2 + 8u_3^2 + 32 = 0 \\ u_1^2 + u_2^2 + 4 = 0 \end{array} \right\} \subset \mathbb{C}^3$$

or projectivizing it in $\mathbb{P}_{\mathbb{C}}^3$ (with coordinates $[u_1 : u_2 : u_3 : u_4]$)

$$\widehat{C} = \left\{ \begin{array}{l} u_3^4 - u_2^2 u_3^2 + 8u_3^2 u_4^2 + 32u_4^4 = 0 \\ u_1^2 + u_2^2 + 4u_4^2 = 0 \end{array} \right\} \subset \mathbb{P}_{\mathbb{C}}^3.$$

The curve \widehat{C} is defined over \mathbb{Q} and isomorphic by R to C_{λ_0} as desired.

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DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDAD TÉCNICA FEDERICO SANTA MARÍA
CASILLA 110-V
VALPARAÍSO, CHILE
e-mail: ruben.hidalgo@usm.cl

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD AUTÓNOMA DE MADRID
MADRID, ESPAÑA
e-mail: sebastian.reyes@uam.es

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE CONCEPCIÓN
CONCEPCIÓN, CHILE
e-mail: mariaevaldes@udec.cl

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