

Generalized Rigid Modules

Módulos generalizados rígidos

ERDAL GUNER, SAIT HALICIOGLU✉

Ankara University, Ankara, Turkey

ABSTRACT. Let α be an endomorphism of an arbitrary ring R with identity. The aim of this paper is to introduce the notion of an α -rigid module which is an extension of the rigid property in rings and the α -reduced property in modules defined in [8]. The class of α -rigid modules is a new kind of modules which behave like rigid rings. A right R -module M is called α -rigid if $ma\alpha(a) = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$. We investigate some properties of α -rigid modules and among others we also prove that if $M[x; \alpha]$ is a reduced right $R[x; \alpha]$ -module, then M is an α -rigid right R -module. The ring R is α -rigid if and only if every flat right R -module is α -rigid. For a rigid right R -module M , M is α -semicommutative if and only if $M[x; \alpha]_{R[x; \alpha]}$ is semicommutative if and only if $M[[x; \alpha]]_{R[[x; \alpha]]}$ is semicommutative.

Key words and phrases. Reduced modules, Semicommutative modules, Armendariz modules, Rigid modules.

2010 Mathematics Subject Classification. 16U80, 16S36.

RESUMEN. Sea α un endomorfismo de un anillo arbitrario R con identidad. El propósito de este artículo es introducir la noción de un módulo α -rígido el cual es una extensión de la propiedad de rigidez en anillos y la propiedad de α -reducibilidad en módulos definida en [8]. La clase de módulos α -rígidos es una nueva clase de módulos los cuales se comportan como anillos rígidos. Un R -módulo derecho M es llamado α -rígido si $ma\alpha(a) = 0$ implica que $ma = 0$ para cualquier $m \in M$ y $a \in R$. Nosotros investigamos algunas propiedades de módulos α -rígidos y entre otras nosotros también probamos que si $M[x; \alpha]$ es un $R[x; \alpha]$ -módulo derecho reducido, entonces M es un R -módulo derecho α -rígido. El anillo R es α -rígido si y sólo si cada R -módulo bandera derecha es α -rígido. Para un R -módulo derecho rígido M , M es α -semiconmutativo si y sólo si $M[x; \alpha]_{R[x; \alpha]}$ es semiconmutativo si y sólo si $M[[x; \alpha]]_{R[[x; \alpha]]}$ es semiconmutativo.

Palabras y frases clave. Módulos reducidos, módulos semiconmutativos, módulos de Armendariz, módulos rígidos.

1. Introduction

Throughout this paper R denotes a ring with identity, modules are unital right R -modules and all ring homomorphisms are unital (unless explicitly stated otherwise). The letter α denotes an endomorphism of R and it will be fixed in the sequel. Let M be a right R -module. The module M is called α -compatible if for any $m \in M$ and any $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$. An α -compatible module M is called α -reduced if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. The module M is called *reduced* if it is $\mathbf{1}$ -reduced where $\mathbf{1}$ denotes the identity endomorphism of M . Hence a ring R is reduced (i.e., has no nonzero nilpotent elements) if and only if the right R -module R_R is a reduced module. In [9], Rege and Chhawchharia introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^s b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i and j .

We write $R[x]$, $R[[x]]$, $R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R , respectively. Similarly, $R[x; \alpha]$, $R[[x; \alpha]]$, $R[x, x^{-1}; \alpha]$ and $R[[x, x^{-1}; \alpha]]$ denote the skew polynomial ring, the skew power series ring, the skew Laurent polynomial ring and the skew Laurent power series ring over R , respectively. Following Lee and Zhou [8] for a right R -module M , we write

$$\begin{aligned} M[x; \alpha] &= \left\{ \sum_{i=0}^s m_i x^i : s \geq 0, m_i \in M \right\}, \\ M[[x; \alpha]] &= \left\{ \sum_{i=0}^{\infty} m_i x^i : m_i \in M \right\}, \\ M[x, x^{-1}; \alpha] &= \left\{ \sum_{i=-s}^t m_i x^i : s \geq 0, t \geq 0, m_i \in M \right\}, \\ M[[x, x^{-1}; \alpha]] &= \left\{ \sum_{i=-s}^{\infty} m_i x^i : s \geq 0, m_i \in M \right\}. \end{aligned}$$

Each of these is an abelian group under an obvious addition operation. Moreover $M[x; \alpha]$ becomes a right module over $R[x; \alpha]$ under the following scalar product operation:

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i \alpha^i(a_j) \right) x^k,$$

where $m(x) = \sum_{i=0}^s m_i x^i \in M[x; \alpha]$ and $f(x) = \sum_{i=0}^t a_i x^i \in R[x; \alpha]$. Similarly, $M[[x; \alpha]]$ is a module over $R[[x; \alpha]]$. The modules $M[x; \alpha]$ and $M[[x; \alpha]]$

are called, respectively, the *skew polynomial extension* and the *skew power series extension* of M . If $\alpha \in \text{Aut}(R)$, then with a similar scalar product, $M[[x, x^{-1}; \alpha]]$ (resp. $M[x, x^{-1}; \alpha]$) becomes a module over $R[[x, x^{-1}; \alpha]]$ (resp. $R[x, x^{-1}; \alpha]$). The modules $M[x, x^{-1}; \alpha]$ and $M[[x, x^{-1}; \alpha]]$ are called the *skew Laurent polynomial extension* and the *skew Laurent power series extension* of M , respectively.

In [7], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and R is said to be an α -*rigid ring* if the endomorphism α is rigid. Recently, rigid rings are generalized to central rigid rings by Ungor et. al. in [6]. A ring R is called *central rigid* if for any $a, b \in R$, $a^2b = 0$ implies ab is central. Motivated by the results and properties of α -rigid rings that have been studied in Hirano [2], Hong et. al. [3, 4] and Krempa [7], we introduce and study a generalization of α -rigid rings which we call α -rigid modules. We investigate some properties of α -rigid modules, and the relations between α -rigid modules, α -semicommutative modules and extended Armendariz modules. Every α -reduced module is α -rigid but the converse need not be true in module case in general, however it is true for rings, that is, a ring is α -reduced if and only if it is α -rigid (Corollary 2.17). We supply an example to show that there exists an α -rigid module which is not α -reduced (Example 2.18). For a right R -module M , we prove that if $M[x; \alpha]$ is a reduced right $R[x; \alpha]$ -module, then M is an α -rigid right R -module (Theorem 2.19). We also show that if M is an α -semicommutative and rigid module, then M is α -skew Armendariz, in particular, if M is semicommutative and rigid, then M is Armendariz (Theorem 2.28). We combine α -rigid modules with α -semicommutative modules and α -Armendariz modules and prove that (i) If M is an α -Armendariz right R -module, then M is α -semicommutative if and only if $M[x; \alpha]$ is semicommutative as an $R[x; \alpha]$ -module. (ii) An Armendariz right R -module M is α -semicommutative if and only if $M[x]$ is an α -semicommutative right $R[x]$ -module.

2. Generalized Rigid Modules

In this section, we introduce the concept of an α -rigid module which is an extension of the rigid property in rings. We investigate some properties of α -rigid modules and α -semicommutative modules.

We start with the following definition.

Definition 2.1. Let M be a right R -module. We call M an α -*rigid module* if $ma\alpha(a) = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$. The module M is called *rigid* if it is **1**-rigid.

By Definition 2.1, it is clear that a ring R is α -rigid if and only if the right R -module R_R is α -rigid.

We mention some classes of rigid modules.

Examples 2.2.

- (1) Every α -reduced module is α -rigid.
- (2) Let p be a prime number, n a positive integer and \mathbb{Z}_{p^n} the ring of integers modulo p^n . Then $p^{n-1}\mathbb{Z}_{p^n}$ is a rigid module over \mathbb{Z}_{p^n} .
- (3) All rigid modules are reduced over von Neumann regular rings.
- (4) Semiprime rigid modules are reduced.

Lemma 2.3.

- (1) The class of α -rigid modules is closed under submodules, direct products, and therefore under direct sums.
- (2) Let $\{M_i : i \in I\}$ be a class of modules. Then each $M_i (i \in I)$ is α -rigid if and only if $\prod_{i \in I} M_i$ is α -rigid if and only if $\bigoplus_{i \in I} M_i$ is α -rigid.

Proof. Clear from the definitions. ✓

Proposition 2.4. Let M be an α -rigid module and let $a \in R$ satisfying $\alpha(a^2) = 0$. Then $Ma = 0$.

Proof. For any $m \in M$, we have $ma\alpha(a)\alpha(a\alpha(a)) = ma\alpha(a^2)\alpha^2(a) = 0$. By hypothesis, $ma\alpha(a) = 0$ which implies $ma = 0$. ✓

Corollary 2.5. Let R be an α -rigid ring. Then α is a monomorphism and R is a reduced ring.

Note that the class of α -rigid modules is not closed under extensions.

Example 2.6. Let F be a field and R the upper triangular matrix ring $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$

over F , $\alpha : R \rightarrow R$ given by $\alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}$, the right R -module $M = \begin{bmatrix} 0 & F \\ F & F \end{bmatrix}$, the submodule $K = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. It is easy to check that K and M/K are α -rigid modules. The element $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ satisfies $a^2 = 0$, $Ma \neq 0$. By Proposition 2.4, M is not α -rigid.

The ring R is called *semicommutative* if whenever $a, b \in R$ satisfy $ab = 0$, we have $aRb = 0$. In Definition 2.7 we recall an extension of this definition to modules. Agayev and Harmanci investigated α -semicommutative rings and modules in [1] and focused on the semicommutativity of subrings of matrix rings. In this note we continue the study of α -semicommutative modules.

Definition 2.7. Let M be a right R -module. Then M is called α -*semicommutative* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR\alpha(a) = 0$. The module M is called *semicommutative* if it is **1**-semicommutative. The ring R is said to be α -*semicommutative* if the right R -module R_R is α -semicommutative.

We observe the following lemma for future use.

Lemma 2.8. *Let M be an α -semicommutative module. For any $m \in M$ and any $a \in R$, we have*

- (1) $m\alpha^i(a) = 0$ implies $mR\alpha^{i+1}(a) = 0$, for all positive integer i .
- (2) $ma^2 = 0$ implies $ma\alpha(a) = 0$.

Proof. The proof is straightforward. ✓

Next we show that α -semicommutativity of modules is not preserved under extensions.

Example 2.9. Let R , M , K and α be as in Example 2.6. It easy to check that K and M/K are α -semicommutative modules. Let $m = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \in M$ and $r = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $a = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \in R$. Then $ma = 0$. But $mr\alpha(a) \neq 0$. Hence M is not α -semicommutative.

Next we recall a condition which ensures that the α -semicommutativity of the modules N and M/N (for a submodule N of M) implies that of M . A submodule N of a module M is called *prime submodule* of M if whenever $ma \in N$ for $m \in M$ and $a \in R$, then $m \in N$ or $Ma \subseteq N$.

Proposition 2.10. *Let M be a right R -module with a submodule N . Assume that M/N and N are α -semicommutative modules. If N is a prime submodule of M and M/N is torsion-free, then M is α -semicommutative.*

Proof. Let $ma = 0$ for some $m \in M$ and $a \in R$. Then $ma \in N$. If $m \in N$, then $mR\alpha(a) = 0$ since N is α -semicommutative. Assume that $m \notin N$. Hence \overline{m} is nonzero in M/N . By hypothesis, $\overline{m}a = 0$ implies that $\overline{m}R\alpha(a) = 0$. Hence $mR\alpha(a) \subseteq N$. It follows that $M\alpha(a) \subseteq N$ because N is prime. That means $(M/N)\alpha(a) = 0$. Hence $\alpha(a) = 0$ since M/N is torsion-free. This completes the proof. ✓

There are semicommutative modules which are not α -semicommutative.

Example 2.11. Let S be a commutative nonzero ring and $R = S \times S$. Let M be the R -module R_R . Let $\alpha : R \rightarrow R$ be defined by $\alpha(a, b) = (b, a)$. Then α is an automorphism of R . Let $(m_1, m_2) \in M$ and $(a_1, a_2), (b_1, b_2) \in R$ and suppose $(m_1, m_2)(a_1, a_2) = (m_1a_1, m_2a_2) = 0$. Since R is commutative, we have $(m_1, m_2)(b_1, b_2)(a_1, a_2) = (m_1b_1a_1, m_2b_2a_2) = (m_1a_1b_1, m_2a_2b_2) = 0$. Then M is semicommutative. For $(1, 0) \in M$ and $(0, 1) \in R$, we have $(1, 0)(0, 1) = 0$ but $(1, 0)R\alpha(0, 1) \neq 0$. Hence M is not α -semicommutative.

An example of a unital endomorphism α of a nonsemicommutative ring R which makes it α -semicommutative exists in the literature [11, Example 2.4]. Example 2.12 does the same for a nonunital endomorphism α .

Example 2.12. Let F be any field and e_{ij} denote the 3×3 matrix units. Let $R = \begin{bmatrix} F & 0 & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$, let $M = \begin{bmatrix} 0 & 0 & F \\ 0 & F & 0 \\ F & 0 & F \end{bmatrix}$ be a right R -module and let $\alpha : R \rightarrow R$, $\alpha(a_{ij}) = e_{22}a_{22}$, where $(a_{ij}) \in R$. Then α is a well-defined endomorphism of R and which is not unital. Let $m = e_{31} + e_{33} \in M$, $a = e_{13} - e_{33}$ and $r = e_{13}u + e_{33}v \in R$, where $u, v \in F$ with $u + v \neq 0$. Then $ma = 0$ but $mra \neq 0$. Therefore M is not a semicommutative as an R -module.

Now let $m = (m_{ij}) \in M$ and $a = (a_{ij}) \in R$ with $ma = 0$. Then we have

$$m_{13}a_{33} = 0 \quad (1)$$

$$m_{22}a_{22} = 0 \quad (2)$$

$$m_{31}a_{11} = 0 \quad (3)$$

$$m_{31}a_{13} + m_{33}a_{33} = 0 \quad (4)$$

If $n = (n_{ij}) \in R$, then $mn\alpha(a) = e_{22}m_{22}n_{22}a_{22} = 0$ from (2) above. Hence M is α -semicommutative.

However note the following fact.

Proposition 2.13. *Let M be an α -semicommutative and α -rigid module. Then M is semicommutative.*

Proof. Let $m \in M$ and $a \in R$ such that $ma = 0$. Since M is α -semicommutative, $mR\alpha(a) = 0$. For any $r \in R$, we have $mra\alpha(r)\alpha(a) = 0$. Hence $m(ra)\alpha(ra) = 0$. The module M being α -rigid $mra = 0$. It follows that $mRa = 0$. This completes the proof. \square

A *regular element* of a ring R is an element which is not a zero divisor. Let S be a multiplicatively closed subset of R consisting of all central regular elements. We may localize R and M at S and ask when the localization $S^{-1}M$ is α -semicommutative as a right $S^{-1}R$ -module. If $\alpha : R \rightarrow R$ is an endomorphism

of the ring R , then $S^{-1}\alpha : S^{-1}R \rightarrow S^{-1}R$ defined by $S^{-1}\alpha(a/s) = \alpha(a)/s$ is an endomorphism of the ring $S^{-1}R$. Clearly this map extends α and we shall also denote this extended map by α .

Proposition 2.14. *Let S be a multiplicatively closed subset of R consisting of all central regular elements. A right R -module M is α -semicommutative if and only if the right $S^{-1}R$ -module $S^{-1}M$ is α -semicommutative.*

Proof. If the right $S^{-1}R$ -module $S^{-1}M$ is α -semicommutative, then it is clear that M is an α -semicommutative right R -module. Assume that M is α -semicommutative and $(m/s)(a/t) = 0$ in $S^{-1}M$, where $(m/s) \in S^{-1}M$, $a/t \in S^{-1}R$. So $ma = 0$. By assumption, $mR\alpha(a) = 0$. It implies $(m/s)(r/u)(\alpha(a)/t) = 0$ for all $r/u \in S^{-1}R$. Then $(m/s)S^{-1}R(\alpha(a)/t) = 0$ and so $S^{-1}M$ is α -semicommutative. \checkmark

Corollary 2.15. *Let M be a right R -module. The right $R[x]$ -module $M[x]$ is α -semicommutative if and only if the right $R[x, x^{-1}]$ -module $M[x, x^{-1}]$ is α -semicommutative.*

Proof. Let $S = \{1, x, x^2, \dots\}$. Then S is a multiplicatively closed subset consisting of all central regular elements of $R[x]$. Since $S^{-1}M[x] = M[x, x^{-1}]$ and $S^{-1}R[x] = R[x, x^{-1}]$, the result is clear from Proposition 2.14. \checkmark

Theorem 2.16. *Let M be a right R -module. If M is an α -reduced module, then M is an α -rigid module. The converse holds if M is an α -semicommutative module.*

Proof. Let M be a right R -module. By [8, Lemma 1.2(2)(a)], M is an α -reduced module if and only if for any $m \in M$ and any $a \in R$,

- (i) $ma = 0$ implies $mR\alpha(a) = 0$, and
- (ii) $ma\alpha(a) = 0$ implies $ma = 0$.

Therefore M is an α -reduced module if and only if M is α -rigid and α -semicommutative. \checkmark

Corollary 2.17 is just mentioned and proved implicitly in [3]. But we give the proof for the sake of completeness.

Corollary 2.17. *Let α be an endomorphism of a ring R . Then R is α -reduced if and only if R is α -rigid.*

Proof. Let R be an α -reduced ring. By Theorem 2.16, R is α -rigid. Conversely, assume that $ab = 0$, for any $a, b \in R$. By Corollary 2.5, R is reduced and so $aR \cap Rb = 0$. Next we prove that $ab = 0$ if and only if $a\alpha(b) = 0$. If $ab = 0$, then $[a\alpha(b)][\alpha(a\alpha(b))] = a\alpha(b)\alpha(a)\alpha^2(b) = a\alpha(ba)\alpha^2(b) = 0$. Since R is α -rigid, then $a\alpha(b) = 0$. Conversely, suppose $a\alpha(b) = 0$. Multiplying from the right by $\alpha(a)$ and from the left by b , we have $ba\alpha(ba) = 0$. By assumption $ba = 0$, and then $ab = 0$. Therefore R is α -reduced. \square

The next example shows that the converse implication of the first statement in Theorem 2.16 is not true in general, i.e., there exists a rigid module which is not reduced.

Example 2.18. Let $R = F\langle x, y \rangle$ be the ring of polynomials in two non-commuting indeterminates over a field F . Since the right ideal xR is not two-sided, the right R -module $M = R/xR$ is not semicommutative and so it is not reduced. Let $\bar{0} \neq \bar{m} = m + xR \in M$, where $m \in R$. Assume that $\bar{m}f^2 = 0$ for a nonzero $f \in R$. Then $mf^2 \in xR$. We express mf in the form $a + xg + yh$, where $a \in F, g, h \in R$. We have $af + xgf + yhf = mf^2 \in xR$. Hence $hf = 0$ yielding (since $f \neq 0$) $h = 0$. Further, if $a \neq 0$ then $a = mf - xg$ implies both m and f have nonzero constant terms. However, the element mf^2 in xR has no nonzero constant term, so $a = 0$ necessarily. Hence $mf = xg \in xR$ yielding $\bar{m}f = 0$. Thus M is a rigid module.

Theorem 2.19. Let M be a right R -module. If $M[x; \alpha]$ is a reduced $R[x; \alpha]$ -module, then M is an α -rigid R -module.

Proof. If $M[x; \alpha]$ is reduced, then by [8, Theorem 1.6], M is α -reduced. Hence M is α -rigid from Theorem 2.16. \square

Theorem 2.20. A ring R is α -rigid if and only if every flat right R -module is α -rigid.

Proof. Assume that R is α -rigid. Let M be a flat R -module and $m \in M, r \in R$ with $mr\alpha(r) = 0$. We prove $mr = 0$. Let $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ be an exact sequence with F free and $M = F/K$. We write $m = y + K$ for $m \in M$ and $y \in F$. Then $yr\alpha(r) \in K$. By Villamayor Theorem [10, Theorem 3.62] there exists an homomorphism $f : F \rightarrow K$ so that $f(yr\alpha(r)) = yr\alpha(r)$. Write $u = f(y) - y$. Then $u \in F$ and $ur\alpha(r) = 0$. Since F is an α -rigid module as a direct sum copies of the α -rigid ring R , $ur = 0$, it follows that $f(y)r = yr$. Since $f(y)r = yr \in K$, $mr = y + K = 0$ in M . The other implication is trivial since R_R is a flat R -module. \square

Remark 2.21. Let S be a subring of a ring R with $1_R \in S, \alpha \in \text{End}(R)$ such that $\alpha(S) \subseteq S$. Assume that a right S -module M_S is contained in a right R -module L_R . If L is α -semicommutative as an R -module, then M is α -semicommutative as an S -module.

Let M be a right R -module. We now determine the conditions under which the skew (Laurent) polynomial extension and the skew (Laurent) power series extension of the module M are semicommutative.

Lemma 2.22. *Let M be a right R -module. If M is an α -semicommutative and rigid module, $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x; \alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x; \alpha]]$. If $m(x)f(x) = 0$, then $m_i \alpha^i(a_j) = 0$ for all i and j .*

Proof. If M is an α -semicommutative module, then for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR\alpha(a) = 0$. By the definition of a rigid module, for any $m \in M$ and $a \in R$, $ma^2 = 0$ implies $ma = 0$. Therefore, the proof is similar to that of [8, Lemma 1.5]. \square

The proof of Lemma 2.23 is the same as that of (2) \Rightarrow (1) of Proposition 3.1 of [11].

Lemma 2.23. *Let M be a right R -module. If $M[x; \alpha]$ is a semicommutative $R[x; \alpha]$ -module, then M is α -semicommutative and semicommutative.*

Theorem 2.24. *Let M be a rigid right R -module. Then the following are equivalent:*

- (1) M is α -semicommutative.
- (2) $M[x; \alpha]_{R[x; \alpha]}$ is semicommutative.
- (3) $M[[x; \alpha]]_{R[[x; \alpha]]}$ is semicommutative.

If $\alpha \in \text{Aut}(R)$, then the conditions (1)–(3) are equivalent to each of following:

- (4) $M[x, x^{-1}; \alpha]_{R[x, x^{-1}; \alpha]}$ is semicommutative.
- (5) $M[[x, x^{-1}; \alpha]]_{R[[x, x^{-1}; \alpha]]}$ is semicommutative.

Proof. By Remark 2.21, (5) \Rightarrow (3) \Rightarrow (2) and (5) \Rightarrow (4) \Rightarrow (2) are clear.

(1) \Rightarrow (3) Let M be an α -semicommutative module and assume that $m(x)f(x) = 0$, where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x; \alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x; \alpha]]$. We now prove that $m(x)R[[x; \alpha]]f(x) = 0$. So, it suffices to show that $m(x)g(x)f(x) = 0$ for $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x; \alpha]]$ and $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \alpha]]$. By Lemma 2.22, we have $m_i \alpha^i(a_j) = 0$, for all i and j . Since M is α -semicommutative, by Lemma 2.8 $m_i R\alpha^{i+k}(a_j) = 0$, for all i, j, k . Then

$$\begin{aligned}
m(x)g(x)f(x) &= \left(\sum_{i=0}^{\infty} m_i x^i \right) \left(\sum_{j=0}^{\infty} b_j x^j \right) \left(\sum_{k=0}^{\infty} a_k x^k \right) \\
&= \sum_i \sum_j \sum_k (m_i x^i) (b_j x^j) (a_k x^k) \\
&= \sum_i \sum_j \sum_k m_i \alpha^i (b_j) \alpha^{i+j} (a_k) x^{i+j+k} = 0
\end{aligned}$$

therefore $M[[x; \alpha]]_{R[[x; \alpha]]}$ is semicommutative.

(2) \Rightarrow (1) Clear from Lemma 2.23.

(3) \Rightarrow (5) Let $m(x)f(x) = 0$, where $m(x) = \sum_{i=-k}^{\infty} m_i x^i \in$

$M[[x, x^{-1}; \alpha]]_{R[[x; \alpha]]}$ and $f(x) = \sum_{j=-k}^{\infty} a_j x^j \in R[[x, x^{-1}; \alpha]]$. In order to prove $m(x)R[[x, x^{-1}; \alpha]]f(x) = 0$, it suffices to show that $m(x)g(x)f(x) = 0$ for any $g(x) = \sum_{l=-k}^{\infty} b_l x^l \in R[[x, x^{-1}; \alpha]]$. There exists $k > 0$ such that $m(x)x^k \in M[[x; \alpha]]$, $\left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l \right) x^k \right] \in R[[x; \alpha]]$. But

$$[m(x)x^k] \left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l \right) x^k \right] = m(x)f(x)x^{2k} = 0.$$

Since $M[[x; \alpha]]_{R[[x; \alpha]]}$ is semicommutative,

$$[m(x)x^k]R[[x; \alpha]] \left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l \right) x^k \right] = 0.$$

Hence

$$\begin{aligned}
[m(x)x^k] \left[\left(\sum_{j=-k}^{\infty} \alpha^{-k}(b_j)x^j \right) x^k \right] \left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l \right) x^k \right] &= \\
m(x)g(x)f(x)x^{3k} &= 0.
\end{aligned}$$

So $m(x)g(x)f(x) = 0$ and $M[[x, x^{-1}; \alpha]]_{R[[x, x^{-1}; \alpha]]}$ is semicommutative. \square

Corollary 2.25. *Let M be a rigid module. Then M is semicommutative if and only if so is $M[[x, x^{-1}]]$.*

Hong, Kwak and Rizvi [5] gave a generalization of Armendariz rings. Let α be an endomorphism of the ring R . R is called an α -Armendariz ring if for any $f(x) = \sum_{i=0}^n a_i x^i$, $g(x) = \sum_{j=0}^s b_j x^j \in R[x; \alpha]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j . Lee and Zhou extended the concept of α -Armendariz ring to modules in [8] so that Armendariz rings are generalized to modules. An α -compatible module M is called α -Armendariz if for any $m(x) = \sum_{i=0}^n m_i x^i \in M[x; \alpha]$ and $f(x) = \sum_{j=0}^s a_j x^j \in R[x; \alpha]$, $m(x)f(x) = 0$ implies $m_i \alpha^i(a_j) = 0$ for all i and j . The module M is called Armendariz if it is $\mathbf{1}$ -Armendariz. Let α be an endomorphism of R . By definition $\alpha(f(x)) = \sum_{j=0}^s \alpha(a_j) x^j$ for $f(x) = \sum_{j=0}^s a_j x^j \in R[x]$, α extends also an endomorphism of $R[x]$.

All reduced rings are Armendariz. There are non-reduced Armendariz rings. For a positive integer n , let $\mathbb{Z}/n\mathbb{Z}$ denotes ring of integers modulo n . Obviously, for each positive integer n , the ring $\mathbb{Z}/n\mathbb{Z}$ is not reduced but Armendariz.

In the sequel we consider the relationship between the class of α -semicommutative modules and the class of Armendariz modules.

Theorem 2.26. *Let M be a right R -module. Then the followings hold.*

- (1) *If M is an α -Armendariz R -module, then M is α -semicommutative if and only if $M[x; \alpha]$ is semicommutative as an $R[x; \alpha]$ -module.*
- (2) *If M is an Armendariz R -module, then M is α -semicommutative if and only if $M[x]$ is α -semicommutative as an $R[x]$ -module.*

Proof.

- (1) Let M be an α -Armendariz R -module. Assume that $M[x; \alpha]$ is a semicommutative $R[x; \alpha]$ -module. By Lemma 2.23, M is an α -semicommutative module. Conversely, suppose that M is α -semicommutative. Let $m(x)f(x) = 0$, where $m(x) \in M[x]$, $f(x) \in R[x]$. Since M is α -Armendariz and α -semicommutative, we have $m_i a_j = 0$ and so $m_i R \alpha^n(a_j) = 0$, for $n = 1, 2, 3, \dots$. For any positive integer t , $m_0 \alpha^t(a_0) = 0$, $m_1 \alpha(a) \alpha^{t+1}(a_1) = 0, \dots, m_i \alpha(a) \alpha^{t+i}(a_i) = 0, \dots$ from which we have

$$\begin{aligned}
 m(x) a x^t f(x) &= (m_0 + m_1 x + \dots + m_n x^n) a x^t (a_0 + a_1 x + \dots + a_m x^m) \\
 &= (m_0 a x^t + m_1 \alpha(a) x^{t+1} + \dots + m_n \alpha^n(a) x^{t+n}) (a_0 + \dots + a_m x^m) \\
 &= m_0 a \alpha^t(a_0) x^t + m_0 a \alpha^t(a_1) x^{t+1} + \dots + m_1 \alpha(a) \alpha^{t+1}(a_0) x^{t+1} + \\
 &\quad m_1 \alpha(a) \alpha^{t+1}(a_1) x^{t+2} + \dots \\
 &= 0.
 \end{aligned}$$

It follows that $m(x)R[x;\alpha]f(x) = 0$ and so $M[x;\alpha]$ is semicommutative.

- (2) Let M be an Armendariz module. Suppose that $M[x]$ is α -semicommutative and $ma = 0$ where $m \in M$, $a \in R$. We have $mR[x]\alpha(a) = 0$ and so $mR\alpha(a) = 0$. Hence M is α -semicommutative. Conversely, assume that M is α -semicommutative. Let $m(x)f(x) = 0$ where $m(x) \in M[x]$, $f(x) \in R[x]$. We have $m_i a_j = 0$ since M is Armendariz. By assumption, $m_i R\alpha(a_j) = 0$. This implies that $m_i R x^t \alpha(a_j) = 0$, in particular $m_i R[x]\alpha(a_j) = 0$. Hence $m(x)R[x]\alpha(f(x)) = 0$. Therefore $M[x]$ is α -semicommutative. \square

Corollary 2.27. [11, Corollary 3.2] *Let M be an Armendariz right R -module. Then M is a semicommutative right R -module if and only if $M[x]$ is a semicommutative right $R[x]$ -module.*

According to Hong, Kim and Kwak [4], R is called α -skew Armendariz if $p(x)q(x) = 0$ where $p(x) = \sum_{i=0}^m a_i x^i$ and $q(x) = \sum_{j=0}^n b_j x^j \in R[x;\alpha]$ implies $a_i \alpha^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. In [11], a module M is said to be α -skew Armendariz for any $m(x) = \sum_{i=0}^n m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^s a_j x^j \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_i \alpha^i(a_j) = 0$ for all i and j . Hence M is an Armendariz module if and only if it is $\mathbf{1}$ -Armendariz if and only if it is $\mathbf{1}$ -skew Armendariz. A ring R is skew-Armendariz if and only if the right R -module R_R is a skew-Armendariz module.

Theorem 2.28. *Let M be an α -semicommutative and rigid module. Then M is α -skew Armendariz. In particular, if M is semicommutative and rigid, then M is Armendariz.*

Proof. Let M be an α -semicommutative and rigid module. By Lemma 2.22, if $m(x)f(x) = 0$, then $m_i \alpha^i(a_j) = 0$ for all i and j , where $m(x) = \sum_{i=0}^n m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^s a_j x^j \in R[x;\alpha]$. Hence M is α -skew Armendariz. In particular, if $\alpha = \mathbf{1}$, then M is Armendariz. \square

Acknowledgement. The authors would like to thank the referees for valuable suggestions.

References

- [1] N. Agayev and A. Harmanci, *On Semicommutative Modules and Rings*, Kyungpook Math. J. **47** (2007), 21–30.
- [2] Y. Hirano, *On the Uniqueness of Rings of Coefficients in Skew Polynomial Rings*, Publ. Math. Debrecen **54** (1999), 489–495.
- [3] C. Y. Hong, N. K. Kim, and T. K. Kwak, *Ore Extensions of Baer and p.p.-Rings*, J. Pure and Appl. Algebra **151** (2000), no. 3, 215–226.
- [4] ———, *On Skew Armendariz Rings*, Comm. Algebra **31** (2003), 103–122.
- [5] C. Y. Hong, T. K. Kwak, and S. T. Rizvi, *Extensions of Generalized Armendariz Rings*, Algebra Colloq. **13** (2006), no. 2, 253–266.
- [6] H. Kose, B. Ungor, and S. Halicioglu, *A Generalization of Reduced Rings*, Hacet. J. Math. Stat. **41** (2012), no. 5, 689–696.
- [7] J. Krempa, *Some Examples of Reduced Rings*, Algebra Colloq. **3** (1996), no. 4, 289–300.
- [8] T. K. Lee and Y. Zhou, *Reduced Modules, Rings, Modules, Algebras, and Abelian Groups* (Dekker, New York), vol. 236, Lecture Notes in Pure and Appl. Math., 2004, pp. 365–377.
- [9] M. B. Rege and S. Chhawchharia, *Armendariz Rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 14–17.
- [10] J. J. Rotman, *An Introduction to Homological Algebra*, second ed., Springer, 2009.
- [11] C. Zhang and J. Chen, *α -skew Armendariz Modules and α -Semicommutative Modules*, Taiwanese J. Math. **12** (2008), no. 2, 473–486.

(Recibido en diciembre de 2013. Aceptado en marzo de 2014)

DEPARTMENT OF MATHEMATICS
 ANKARA UNIVERSITY
 06100 TANDOĞAN
 ANKARA, TURKEY
e-mail: `guner@science.ankara.edu.tr`
e-mail: `halici@ankara.edu.tr`

Esta página aparece intencionalmente en blanco