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On the Classification of 3–Bridge Links

Sobre la clasificación de los enlaces de 3 puentes

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Abstract. Using a new way to represent links, that we call a *butterfly representation*, we assign to each 3-bridge link diagram a sequence of six integers, collected as a triple $(p/n, q/m, s/l)$, such that $p \geq q \geq s \geq 2$, $0 < n \leq p$, $0 < m \leq q$ and $0 < l \leq s$. For each 3-bridge link there exists an infinite number of 3-bridge diagrams, so we define an order in the set $(p/n, q/m, s/l)$ and assign to each 3-bridge link L the minimum among all the triples that correspond to a 3-butterfly of L, and call it *the butterfly presentation* of L. This presentation extends, in a natural way, the well known Schubert classification of 2-bridge links.

We obtain necessary and sufficient conditions for a triple $(p/n, q/m, s/l)$ to correspond to a 3-butterfly and so, to a 3-bridge link diagram. Given a triple $(p/n, q/m, s/l)$ we give an algorithm to draw a canonical 3-bridge diagram of the associated link. We present formulas for a 3-butterfly of the mirror image of a link, for the connected sum of two rational knots and for some important families of 3-bridge links. We present the open question: When do the triples $(p/n, q/m, s/l)$ and $(p'/n', q'/m', s'/l')$ represent the same 3-bridge link?

Key words and phrases. Links, 3-bridge links, Bridge presentation, Link diagram, 3-butterfly, Butterfly presentation.

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Resumen. Usando una nueva forma de representar enlaces, que se denomina *representación en mariposa*, se asocia a cada diagrama de 3 puentes de un enlace una sucesión de seis enteros, organizados como una tripla $(p/n, q/m, s/l)$,

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tal que $p \geq q \geq s \geq 2$, $0 < n \leq p$, $0 < m \leq q$ y $0 < l \leq s$. Para cada enlace de 3 puentes existe un número infinito de diagramas de 3 puentes, por lo que se define un orden en el conjunto de triplas de la forma $(p/n, q/m, s/l)$ y se asigna a cada enlace de 3 puentes L el mínimo entre todas las triplas que corresponden a una 3-mariposa de L, y que se llama la *presentación en mariposa* de L. Esta presentación extiende, en una forma natural, la bien conocida clasificación de Schubert de los enlaces de 2 puentes.

Se obtienen condiciones necesarias y suficientes para que una tripla de la forma $(p/n, q/m, s/l)$ corresponda a una 3-mariposa y por tanto, a un diagrama de 3 puentes de un enlace. Dada una tripla $(p/n, q/m, s/l)$ se da un algoritmo para dibujar, en forma canónica, un diagrama de 3 puentes del enlace de 3 puentes asociado. Se presentan fórmulas para la 3-mariposa de la imagen espejos de un enlace de 3 puentes, para la suma conexa de dos nudos racionales y de algunas familias importantes de enlaces de 3 puentes. Queda la pregunta abierta: ¿Cuándo dos triplas $(p/n, q/m, s/l)$ y $(p'/n', q'/m', s'/l')$ representan el mismo enlace de 3 puentes?.

Palabras y frases clave. Enlaces, enlaces de 3 puentes, presentación en puentes, diagrama de enlace, 3-mariposa, presentación en mariposa.

1. Introduction

Our goal in this paper is to study 3-bridge link using a new presentation of links, called *butterfly*, that we introduced in [6], [5], [4], [7] and [8]. Up to now, the classification of 3-bridge links has not been realized, and it presents difficulties that contrast with the case of 2-bridge links, that were completely classified by Schubert, [19].

We will associate to each 3-bridge link diagram a set of 6 integers, extending the classification of 2-bridge links given by Schubert, [19]. As each 3-bridge link admits different 3-bridge diagrams, (in fact, infinitely many), the situation is more difficult and we can not expect the same type of classification that was obtained for 2-bridge links.

In Section 2 we present a brief review of the concept of m-butterfly and some results from [7] and [8]. The main result in [7], that the butterfly number coincides with the bridge number, is the starting point for our work: *to study the 3-butterflies with the objective of classifying the 3-bridge links.* For our purpose, we need to consider only reduced 3-bridge link diagrams, so we will impose restrictions on the type of butterfly we will consider.

In Section 3 we assign a set of positive integers $\{p, n, q, m, s, l\}$ to each 3-butterfly, that we write as a triple $(p/n, q/m, s/l)$ for geometrical reasons, to be explained in that section. We obtain necessary and sufficient conditions for a triple $(p/n, q/m, s/l)$ to correspond to a 3-butterfly and so to a 3-bridge link. This is done by defining a permutation μ associated to each triple $(p/n, q/m, s/l)$ and studying its cyclic decomposition. This cyclic decomposition answers the question of the number of components of the link diagram associated to the 3-butterfly $(p/n, q/m, s/l)$ and provides an algorithm to draw a canonical link diagram associated to it. We define the concept of butterfly equivalence and find some conditions for two 3-butterflies to be equivalent. It is an open problem to find necessary and sufficient conditions for the equivalence of two 3-butterflies and to find a complete set of "*moves*" to transform a 3-butterfly into an equivalent one. All the constructions and algorithms are easily implemented, and we have done so using the software Mathematica. A variation of the permutation μ allows us to find the combinatorial knot (or Gauss code) associated to a link diagram given by $(p/n, q/m, s/l)$. Then it is possible to compute invariants of the link, such as the link group or an invariant polynomial. See [24], [17] and [23] for details on combinatorial knots.

In Section 4 we define *the butterfly representation of a link* L. We do so by ordering the set of all 3-butterflies and taking the minimum, in that order, of all 3-butterflies that represent the link L.

In Section 5 we give a 3-butterfly representation of two basic link constructions: the mirror image of a 3-bridge link and the connected sum of two rational knots. Also, we give a 3-butterfly representation of some families of 3-bridge links: the pretzel $P(a, b, c)$, the torus link $T(p, 3)$ and the 3-bridge knots up to 9 crossings in Rolfsen's table [18]. We end this section by giving a 3-butterfly interpretation of the constructions given in [2], [12] and [13].

In the last section we present some concluding remarks regarding the open problems that arise and the applications of our construction. The basic information on links and knots can be found in [1], [3] or [14].

2. About n-Butterflies

Intuitively, an m-butterfly is a 3-ball $\mathbf{B}^3$ with $m > 0$ polygonal faces on its boundary $S^2 = \partial\mathbf{B}^3$, such that each face C is subdivided by an arc t_C in two subfaces (that have the same number of vertices) that are identified by a "reflection" along this arc t_C. Thurston's construction of the borromean rings, [21] and [22], is a nice example of this construction, that we generalize for all links in [7]. In this example we notice that the cube is actually a closed 3-cell $\mathbf{B}^3$, with twelve faces on its boundary that are identified by reflections along some axes. Moreover, pasting the faces of the cube we obtain S^3 and the set of axes become the borromean rings. In terms of butterflies, that we are going to define in Section 2.1, the cube edges and vertices form the graph and the axes form the trunk of a 6-butterfly, (see Figure 1a).

In this section we give the definitions and basic aspects of the constructions given in [7], and in the rest of the paper we will consider the case $m = 3$. For more details and proofs see [7].

2.1. Definitions and Constructions

Let R be a connected graph embedded in $S^2 = \partial\mathbf{B}^3$, where $\mathbf{B}^3$ is a closed 3-cell, so that $S^2 - R$ is a disjoint union of open 2-cells. We denote each open 2-cell generically by C.

For any $n \in \mathbb{N}$, let P_{2n} be the regular polygon that is the closed convex hull of the $2n^{th}$ roots of unity. We define a parametrization of C to be a function f from P_{2n} to the closure $\overline{C}$ of C, with the following properties:

a) The restriction of f to interior P_{2n} is a homeomorphism from interior P_{2n} to C.

b) The restriction of f to an edge of P_{2n} is a piecewise linear homeomorphism from that edge to an edge in the graph R.

c) f as a map from the edges of ∂P_{2n} to the edges of ∂C is at most 2 to 1.

The existence of a parametrization of C places restrictions on C and on R. We will assume that R is such that each C has a parametrization $f : P_{2n} \to \overline{C}$, for some n, and we fix a parametrization f_C for each C.

Complex conjugation, $z \to \overline{z}$, restricted to P_{2n} or to the boundary of P_{2n} defines an involution and an equivalence relation on the edges and vertices of P_{2n}, and this in turn, induces an equivalence relation on the edges and vertices of $\overline{C}$, and on the points of C as well. That is to say, for v and w points of $\overline{C}$, $v \sim w$ if $f_C^{-1}(v) = f_C^{-1}(w)$ or $f_C^{-1}(v) = \overline{f_C^{-1}(w)}$, where $\overline{f_C^{-1}(v)}$ is defined as the set $\left\{\overline{z}/z \in f_C^{-1}(v)\right\}$.

The equivalence relation on each $\overline{C}$ induces an equivalence relation on $S^2 = \partial\mathbf{B}^3$. That is $x \simeq y$ if and only if there exists a finite sequence $x = x_1, \dots, x_l = y$ with $x_i \sim x_{i+1}$ for $i = 1, \dots, l-1$.

Each P_{2n} contains the line segment $[-1, 1]$, which is the fixed point set of complex conjugation restricted to P_{2n}. The image of this line segment $f_C([-1, 1])$ is called the *trunk* t. A pair (C, t) will be called a *butterfly with trunk* t. The *wings* W and W' are just $f_C(P_{2n} \cap$ *upper half plane*$)$ and $f_C(P_{2n} \cap$ *lower half plane*$)$ and $W \cap W' = t$. Each time that we consider a trunk t we are implicitly considering the equivalence relation described above. We denote by T the collection of all trunks t (over all C).

Let us denote by $M(R, T)$ the space $\mathbf{B}^3/\simeq$ with the topology of the identification map $p : \mathbf{B}^3 \to M(R, T)$, where $\simeq$ is the minimal equivalence relation generated by equivalence relation $\simeq$ defined on S^2.

Equivalence classes of points of C contain two points except for those points in $f([-1, 1])$ where there is only one point. Note that if x is a vertex of R, its complete class under the equivalence relation $\simeq$ is composed entirely of vertices. We classify the vertices as follows: A member of $R \cap T$ will be called an *A-vertex*.

A member of $p^{-1}\big(p(v)\big)$, $v \in R \cap T$, which is not an *A-vertex* will be called an *E-vertex*. A vertex of R which is neither an A-vertex nor an *E-vertex* will be called a *B-vertex* iff $p^{-1}\big(p(v)\big)$ contains at least one non-bivalent vertex of R.

Definition 1. Let R and T be as above. For $m \geq 1$, an *m-butterfly* (R, T) is a 3-ball $\mathbf{B}^3$ with m butterflies (C_i, t_i), $i = 1, \ldots, m$, on its boundary $S^2 = \partial\mathbf{B}^3$, such that (i) the A- and E-vertices are bivalent in R, and (ii) T has m components.

In [7] we proved the following results:

Theorem 2. *[7, Theorem 1, pag. 5] For any m-butterfly (R, T), the space $M(R, T)$ is homeomorphic to S^3 and $p(T)$ is a knot or a link, where $p : \mathbf{B}^3 \to M(R, T)$ is the identification map.*

Theorem 3. *[7, Theorem 3, pag. 11] Every knot or link can be represented by an m-butterfly diagram, for some $m > 0$.*

Example 4. In Figure 1 we show examples of butterflies and its associated links. The example in Figure 1a is the classical example of Thurston and the one in Figure 1b is a 2-butterfly that represents the rational link p/q, for the case 5/3. Notice that the 2-butterfly illustrates the process of pasting the northern and southern hemispheres of S^2 to themselves by reflections through half meridians separated apart $2\pi q/p$. See [11] and [10] for details.

a b

FIGURE 1. Examples of butterflies.

Definition 5. The minimum m among all possible m-butterfly diagrams of a given link L is called the *butterfly number* of L and it is denoted by $m(L)$.

Theorem 6. *[7, Theorem 5, pag17.] For any link L, $m(L) = b(L)$, where $b(L)$ is the bridge number of L.*

In the examples in Figure 2a, (and in the rest of the paper) we consider B as the 3-ball lying beneath the plane $\mathbb{R}^2 + \infty$, so that R is a planar graph. The trunk T is drawn with bold lines and we do not draw the B-vertices. When we perform the identification given by the map p, we say that we "*close the butterfly*" and the link $p(T)$ is formed by the trunk, that becomes the bridges, and the under arcs, (drawn by dashed lines), are described by the orbits of

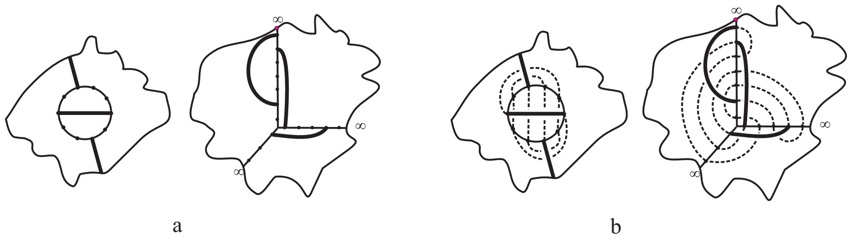


FIGURE 2. Examples of a 2-butterfly and a 3-butterfly. In b we close the butterflies in a.

the A-vertices. Figure 2b shows the links constructed by the 3-butterflies in Figure 2a.

In [7, Sections 4 and 5.] we define algorithms that permit the construction of the diagram of a link associated to an m-butterfly and the construction of the butterfly associated to a link diagram. In Section 4 we will explain the first one for 3-butterflies.

2.2. Simplifying Butterflies

Our definition of butterfly is given in such a way as to allow us to represent every m-bridge diagram of a link, even if it has kinks or it admits type II Reidemeister moves. For the purpose of this paper, we need to work only with reduced 3-bridge diagrams, so we need only a more restricted type of butterflies, in which the graph R has no monovalent vertices.

Let us see that in fact we need to consider only this type of m-butterflies. A link diagram can be reduced by a type I Reidemeister move if and only if when we produce the butterfly, following the algorithm given in [7], we have the situation described in Figure 3a.

Figure 3b shows the situation when the link diagram admits type II Reidemeister moves, and Figure 3c refers to the case when it admits type I and II Reidemeister moves. As we want to study reduced 3-bridge diagrams, we work only with 3-butterflies whose graph has no monovalent vertices. From now on, the term 3-butterfly refers only to this type of 3-butterfly.

3. 3-Butterflies

Given that to every 3-bridge diagram of a link we can associate a 3-butterfly and, conversely, each 3-butterfly produces a 3-bridge diagram, it is natural to study 3-bridge links by using 3-butterflies. In this section we assign to each 3-butterfly a set of six positive integers $\{p, n, q, m, s, l\}$, that satisfy some conditions. For geometrical reasons, to be explained in Remark 9, we will use the

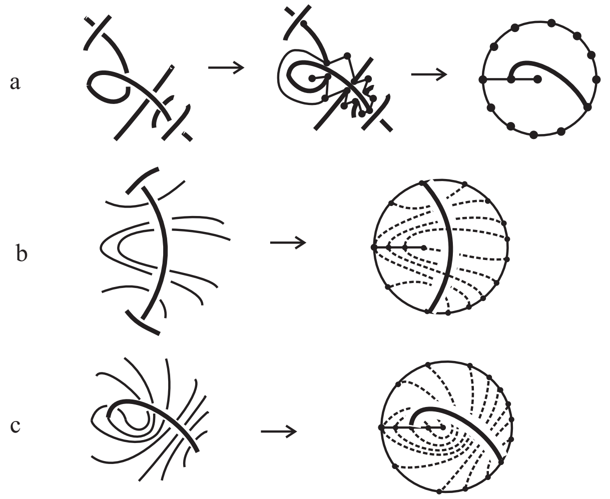


FIGURE 3. Examples of butterflies with a monovalent vertex.

notation $(p/n, q/m, s/l)$ to refer to this set, but they are not to be considered rational numbers. In this way we extend, in a "*natural way*", Schubert classification of rational links.

Definition 1 and the above considerations, lead to the following definition:

Definition 7. A 3-*butterfly* is a 3-ball $\mathbf{B}^3$ with a graph R on its boundary $\partial\mathbf{B}^3 = S^2$, such that $S^2 \setminus R$ is a union of 3 polygonal regions and (i) the A- and E-vertices are bivalent in R, (ii) T has 3 components and (iii) R has no monovalent vertices.

In the general definition of an m-butterfly given in [7], we impose the additional condition (iv) *that the graph R has only A-vertices*, *E-vertices and B-vertices.* This condition is not essential and we can drop it. We say that a 3-butterfly is *reduced* if it satisfies Condition (iv), but in our construction we allow this more general type of butterfly. Notice that Condition (ii) is equivalent to *(ii') There are exactly* 6 *A-vertices.*

3.1. Description of a 3-Butterfly by $(p/n, q/m, s/l)$

Let R be the graph of a 3-butterfly. It separates S^2 in 3 butterflies, called P, Q and S, each one of them parameterized by a polygon with an even number of vertices. We will identify each butterfly with the polygon and so we will talk about the vertices and edges of the butterfly. In each butterfly we have a *trunk*, that divides the butterfly in two wings, and a reflection along the trunk, that identifies the two wings. There can be only two basic forms for the graph R, as shown in Figure 4, that are determined by the way the butterflies P, Q and S intersect.

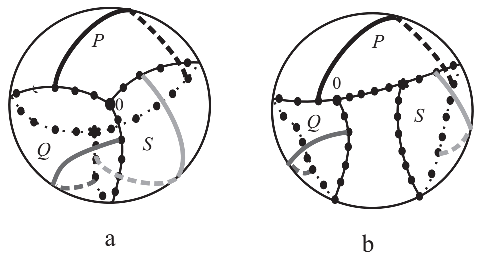


FIGURE 4. The two types of 3-butterflies without monovalent vertices.

Type I: The graph is a theta graph, and the three butterflies intersect in two vertices, denoted 0 and $*$, see Figure 4a. This is the only type of 3-butterflies that we need to consider, as we will see later.

Type II: Two of the polygons do not intersect, see Figure 4b. Again, there are only two intersection points for the three butterflies.

In [15, pag. 480], Negami classifies the 3-bridge diagrams in two types, shown in Figure 5, where each circle stands for a neighborhood of a bridge and each edge represents a collection of parallel subarcs.

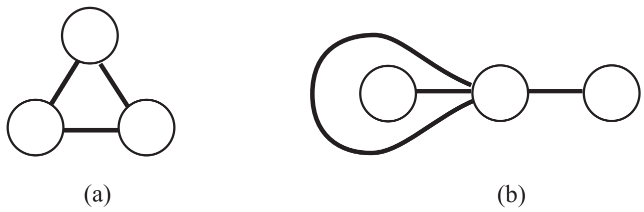


FIGURE 5. Negami [15] classification of 3-bridge diagrams.

He proved that if a link L admits a type (b) 3-bridge diagram D, it can be transformed in a type (a) 3-bridge diagram D' by a finite number of *wave moves*, see [15, pag. 1], and that the crossing number of D' is less than the crossing number of D. Now, when we use the algorithm to produce a 3-bridge diagram associated to a type I (resp. type II) 3-butterfly, we obtain a type (a) (resp. type (b)) 3-bridge diagram, therefore, as we are interested in classifying 3-bridge diagrams, in this paper we need only to consider type I 3-butterflies. It is important to note, that type (b) diagrams can not be ignored in some other problems concerning the study of 3-bridge links. In a future paper we will address the complete description of both types of 3-butterflies and the interpretation of the wave moves in terms of butterflies.

For now on, we will assume that all 3-butterflies are of type I. If we consider plane diagrams of the 3-butterflies, taking the point at ∞ as a point on the interior of the butterfly P, we have diagrams as shown in Figure 6a. If we take ∞ as a point of the graph R, we get the diagram shown in Figure 6b.

Now, if we take a point outside the graph R as the point at infinity, and we reshape the plane and stretch out the vertices, so that the butterflies become disks, we get diagrams as shown in Figure 6c. We will use any of the three types of diagrams. The last diagram is useful to understand the reason we want to interpret the set $\{p, n, q, m, s, l\}$ as a triple $(p/n, q/m, s/l)$.

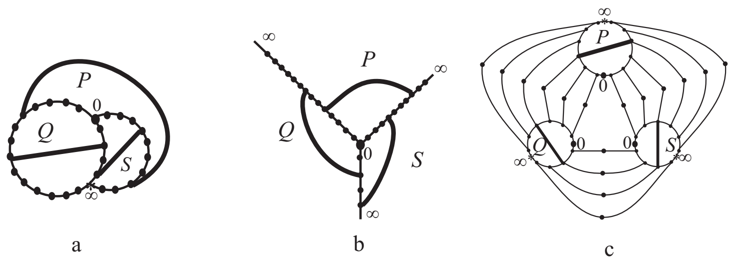


FIGURE 6. Plane diagrams for 3-butterflies.

Let us consider the 3-butterfly given by Figure 6c. All B-vertices are either in the orbit of 0 or $*$, and the link is determined by the orbits of the A-vertices. So we do not draw the B-vertices except for 0 or $*$. We call the set of vertices that are not B-vertices *admissible vertices.* So, if the 3-butterfly is reduced, an admissible vertex is an A-vertex or a E-vertex.

The integers p, q and s are defined as follows:

$$2p = |P| = \text{ Number of admissible vertices of the butterfly } P.$$
$$2q = |Q| = \text{ Number of admissible vertices of the butterfly } Q.$$
$$2s = |S| = \text{ Number of admissible vertices of the butterfly } S.$$

In order to obtain a canonical way to describe a 3-butterfly, we will always assume that

$$p \geq q \geq s \geq 2, \tag{1}$$

and by rotating the plane (and interchanging the points 0 and ∞, if necessary), we can always obtain a 3-butterfly diagram with P at the top, Q to the left and S to the right, and we read it in the counter-clock wise direction, $P\ Q\ S$. The condition $s \geq 2$ is to ensure that each bridge has at least one crossing.

***Remark* 8.** If we are not interested in a canonical way to describe a 3-butterfly, we can omit condition (1). In fact, in Lemma 13 we give equivalences between 3-butterflies that do not satisfy this condition. We can omit the condition $s \geq 2$ if we want to allow bridges without crossings.

We need to know the number of admissible vertices in the intersection of the butterflies. Let us denote by $|P \cap Q|$ the number of admissible vertices between P and Q and define

$$t = |P \cap Q|, \qquad v = |Q \cap S|, \qquad \text{and} \qquad w = |P \cap S|.$$

As each butterfly intersects the other two, t, v and w are positive integers. Counting the admissible vertices of P, Q and S we get, respectively,

$$2p = t + w, \quad 2q = t + v, \quad 2s = v + w.$$

So,

$$t = p + q - s, \qquad v = q + s - p, \qquad w = p + s - q, \tag{2}$$

therefore, the information on the number of admissible vertices of the butterflies is enough to determine the relative position of them. As $v \geq 1$, then

$$p + 1 \leq s + q. \tag{3}$$

Thus, we only need to know the integers p, q and s (or t, v and w), satisfying (1) and (3) to describe the graph R completely.

Now let us describe the trunk. We orient clock wise each disk as shown in Figure 7.

From the point 0, following the orientation, we count the number of admissible vertices between 0 and the A-vertex in which the trunk begins, this is the *rotation index* of the trunk. The end of the trunk is on an opposite vertex.

We call n, m and l the rotation indices of P, Q and S, respectively. Clearly

$$1 \leq n \leq p, \qquad 1 \leq m \leq q, \qquad 1 \leq l \leq s. \tag{4}$$

***Remark* 9.** Notice that $\frac{n}{p}\pi$ corresponds to the angle between the trunk of P and the radius that ends on the first admissible vertex to the right of the point 0. (See Figure 7b.) For butterfly Q the angle is $\frac{m}{q}\pi$ and for S it is $\frac{l}{s}\pi$. This is analogous to the description of a rational link as a 2-butterfly and the reason we want to use the fraction notation to describe a 3-butterfly.

Condition (ii) in Definition 7 of a 3-butterfly means that the three trunks have no common vertices, so the 3-butterfly have 6 different A-vertices. This imposes additional conditions on the set of integers $\{p, n, q, m, s, l\}$. In Figure 8 we show all possible 3-poligonalizations that fail condition (ii) and consequently, do not represent 3-butterflies.

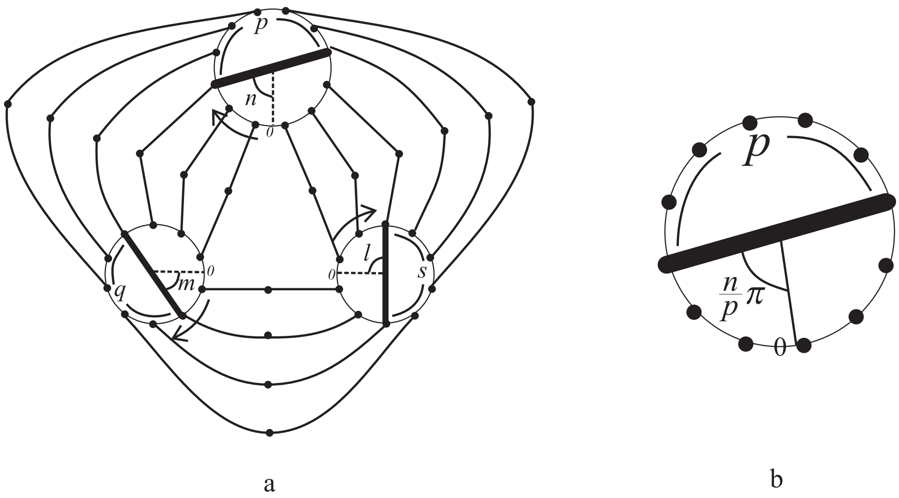


Figure 7. Description of a 3-butterfly. The integers n, m and l describe the trunk position. Figure b shows the angle $\dfrac{n}{p}\pi$.

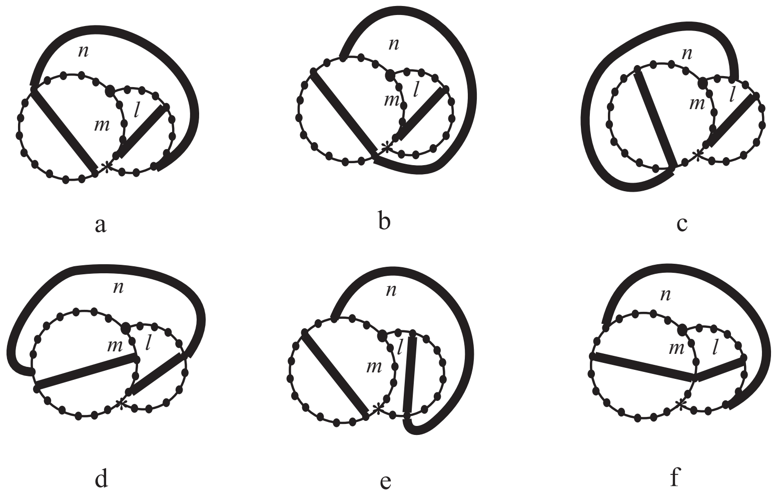


Figure 8. All possible graphs that do not represent 3-butterflies. In each case, two of the trunks have a common vertex.

Each diagram of Figure 8 imposes restrictions on the set $\{p, n, q, m, s, l\}$ that we summarize in the following table

Diagram	Condition
a	$n + m = q + 1$
b	$n + m = 2q - p + 1 \quad \text{and} \quad m > q + s - p$
c	$n + m = 2q + 1 \quad \text{and} \quad m > q + s - p$
d	$n + l = p + 1$
e	$n + l = p - s + 1 \quad \text{and} \quad l < p - s$
f	$m + l = s + 1 \quad \text{and} \quad m \leq q + s - p$

(5)

We have proved the following theorem:

Theorem 10. *Every 3-butterfly defines a unique set of integers $\{p, m, q, n, s, l\}$ such that*

$$\begin{aligned}
&p \geq q \geq s \geq 2, \\
&1 \leq n \leq p, \quad 1 \leq m \leq q, \quad 1 \leq l \leq s, \\
&n + m \neq q + 1, \quad n + l \neq p + 1, \\
&n + m \neq 2q + 1 \quad \text{and} \quad n + m \neq 2q - p + 1 \quad \text{if} \quad m > q + s - p, \\
&n + l \neq p - s + 1 \quad \text{if} \quad l < p - s, \\
&m + l \neq s + 1 \quad \text{if} \quad m \leq q + s - p.
\end{aligned} \tag{6}$$

Conversely, any set of integers $\{p, m, q, n, s, l\}$ that satisfies conditions (6) *defines a* 3*-butterfly. We write this set as the triple $(p/n, q/m, s/l)$ and we represent it by the diagrams shown in Figure 9.*

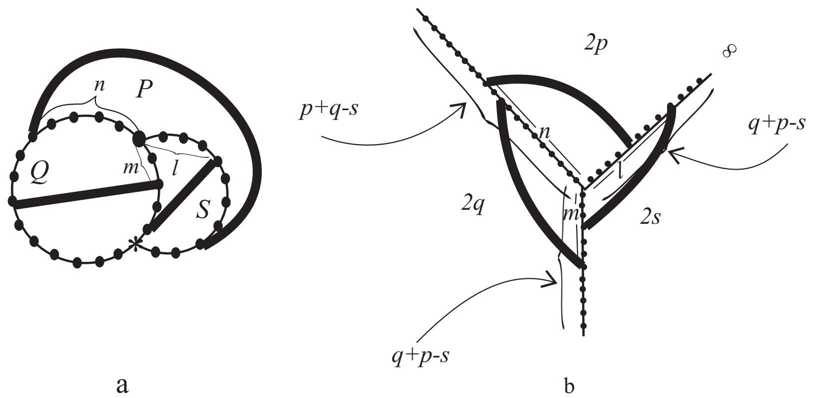


Figure 9. Diagrams for the 3-butterfly $(p/n, q/m, s/l)$.

Proof. It is just a consequence of the previous construction. Note that in each step of the construction we obtain unique numbers, so each 3-butterfly is completely determined by the triple $(p/n, q/m, s/l)$. ☑

The integers p, n, q, m, s, l are independent, in the sense that we can not deduce one of them from the other five. Table 1 shows examples of different 3-butterflies that coincide in five of the numbers. We prove the no equivalence of the associated links by using the Jones polynomial.

3-butterfly		**Jones Polynomial**
(6/2,6/2,5/1)		$q - q^2 + 2q^3 - q^4 + q^5 - q^6$
(6/2,6/2,5/3)	$l \neq l'$	$\frac{1 - 2q + 2q^2 - 4q^3 + 3q^4 - 3q^5 + 2q^6 - q^7}{q^{11/2}}$
(6/2,6/3,5/1)	$m \neq m'$	$\frac{-1 + q - q^2 - q^4}{q^{11/2}}$
(6/1,6/2,5/1)	$n \neq n'$	$\frac{-1 + q - 2q^2 + q^3 - q^4 + q^5 - q^6}{q^{5/2}}$
(8/2,6/2,5/1)	$p \neq p'$	$\frac{-1 - q^2 - q^4 + q^5}{q^{7/2}}$
(6/2,5/2,5/1)	$q \neq q'$	1
(6/2,6/2,4/1)	$s \neq s'$	$\frac{-1 - 2q^2 + q^3 - q^4 + q^5}{q^{3/2}}$

TABLE 1. Each 3-butterfly differs in just one number from the 3-butterfly in the first row.

3.2. 3-Butterfly $(p/n, q/m, s/l)$ Representation of a 3-Bridge Link Diagram

Given a 3-butterfly we produce a 3-bridge diagram of a link, by "*closing the butterfly wings*". For instance, the 3-butterfly $(5/2, 5/2, 5/2)$ gives rise to the Borromean rings. (See Figure 10a.)

Definition 11. We say that a 3-bridge link diagram D has a $(p/n, q/m, s/l)$ butterfly representation if the 3-butterfly $(p/n, q/m, s/l)$ produces the diagram D.

Recall that each 3-bridge link can be represented by an infinite number of 3-bridge diagrams, so it is represented by an infinite number of 3-butterflies $(p/n, q/m, s/l)$. For instance, the knot 8_{20} in Rolfsen's Table can be represented by the 3-butterflies $(6/6, 5/2, 4/4)$ and $(6/2, 6/4, 4/3)$, see Figure 10b and Figure 10c.

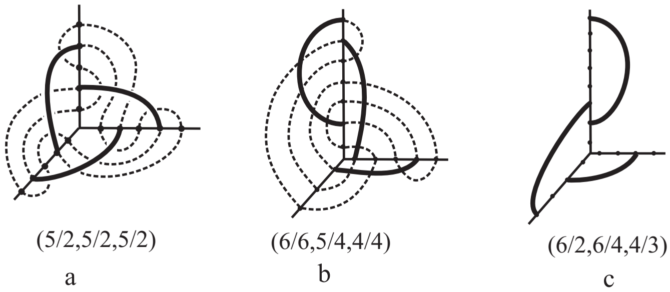


FIGURE 10. Butterfly diagrams for the Borromean rings and the knot 8_{20} in Rolfsen's table.

***Remark* 12.** Note that our convention of 3-bridge diagram forces each bridge to have at least one under crossing, but some authors allow trivial bridges, for instance [16]. In that setting, it could be possible to take $s = 1$ and to describe the rational link p/q as a 3-bridge link with 3-butterfly representation given by $(p/q, p/1, 1/1)$. So, it is clear why we say that our construction is a "natural" extension of Schubert notation.

We say that two 3-butterflies are *equivalent* if they produce the same 3-bridge link. It is an open problem to find a complete set of *moves* that allows us to transform a 3-butterfly into an equivalent one. We have, however, some cases in which we are able to establish the equivalence.

In order to get a normal form for a 3-butterfly we require $p \geq q \geq s \geq 2$, but it is possible to have a 3-butterfly that does not satisfy this condition. The following lemma allows us to write any 3-butterfly in a canonical form.

Lemma 13.

i) The 3*-butterfly* $(p/n, q/m, s/l)$ *is equivalent to the* 3*-butterflies*

$$(q/m, s/l, p/n) \quad \text{and} \quad (s/l, p/n, q/m).$$

ii) If $s > q$, *the* 3*-butterfly* $(p/n, q/m, s/l)$ *is equivalent to* $(p/n', s/l', q/m')$, *where*

$$\begin{aligned} n' &= (n + s - q) \bmod p, \\ m' &= (m + p - s) \bmod q, \qquad \text{and} \\ l' &= (l + q - p) \bmod s. \end{aligned} \tag{7}$$

Proof. i) It is just a rotation of S^2, as the diagrams in Figure 11 show.

ii) For *ii)* we change the viewpoint. In Figure 12a we have the 3-butterfly $(p/n, q/m, s/l)$, with 0 as the initial point from which we count the numbers

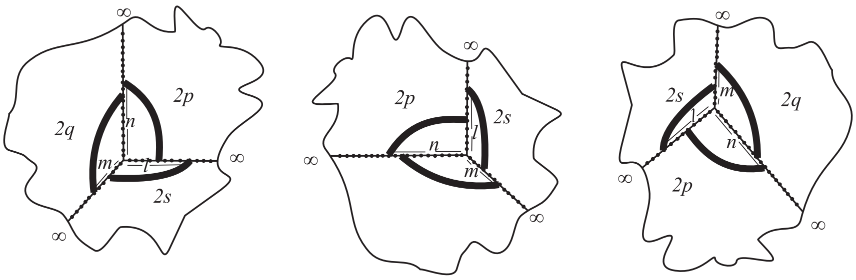


Figure 11. Rotations of a 3-butterfly.

n, m and l, that describe the trunk. In Figure 12b we rotate S^2 and have the same graph to describe the 3-butterfly, but with $*$ as the initial point, and the integers n', m' and l' describe the trunk position with respect to $*$. To find the relations in (7) it is enough to consider the plane diagram for the 3-butterflies given in Figure 12c and Figure 12d.

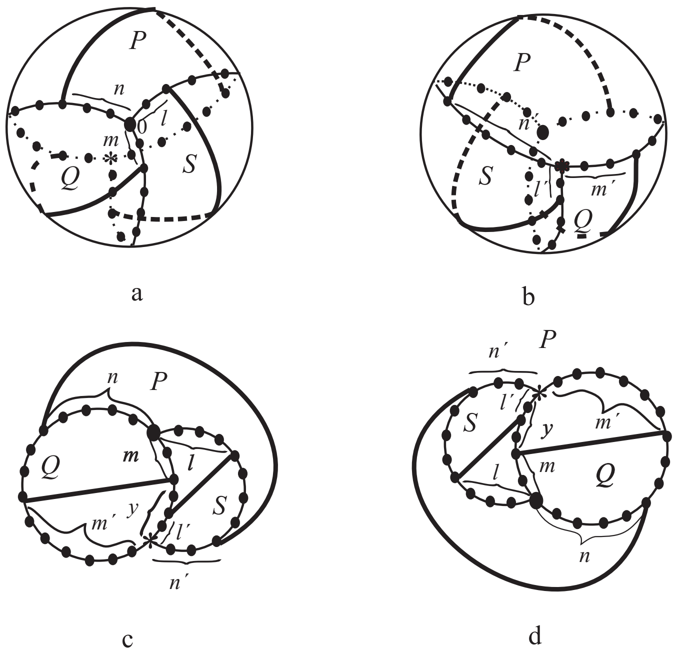
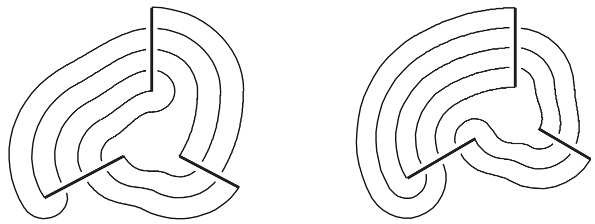


Figure 12. Change of viewpoint to describe a 3-butterfly.

☑

Example 14. We need to be careful with the relative order of the numbers in $(p/n, q/m, s/l)$. Lemma 13 allows us to perform cyclic rotations, but if we change the order, it is possible that we get non-equivalent 3-butterflies. The 3-butterflies $(4/2, 4/1, 3/2)$ and $(4/1, 4/2, 3/2)$ represent the links shown in Figure 13. The first one is a knot and the second is a 2 component link, so the butterflies are non equivalent.



(4/2, 4/1, 3/2) (4/1, 4/2, 3/2)

FIGURE 13. Different links obtained by a change in the order of a 3-butterfly.

In the sequel, when we say the $(p/n, q/m, s/l)$ 3-butterfly we mean that the triple $(p/n, q/m, s/l)$ satisfies the conditions of Theorem 10.

3.3. 3-Butterfly Classification

Now we want to address the questions of the number of components of the 3-bridge link associated to a 3-butterfly and if a 3-butterfly is reduced or not. Recall that a 3-butterfly is reduced if its graph has only A-vertices, E-vertices and B-vertices. We associate to each 3-butterfly $(p/n, q/m, s/l)$ a permutation that allows us to answer both questions. When a 3-butterfly is not reduced, we give a method to find an equivalent reduced 3-butterfly.

Given a 3-butterfly $(p/n, q/m, s/l)$ we label the admissible vertices of the butterfly P with the set $A = \{a_0, \dots, a_i, \dots, a_{2p-1}\}$, $i \in \mathbb{Z}_{2p}$, the admissible vertices of the butterfly Q with the set $B = \{b_0, \dots, b_j, \dots, b_{2q-1}\}$, $j \in \mathbb{Z}_{2q}$ and the admissible vertices of the butterfly S with the set $C = \{c_0, \dots, c_l, \dots, c_{2s-1}\}$, $l \in \mathbb{Z}_{2s}$. For butterfly P (resp. Q and S) the ends of the trunk are labeled by a_0 and a_p (resp. b_0, b_q and c_0, c_s), and we label the vertices in the reverse order of the butterfly orientation, see Figure 7.

The subscripts of set A (resp. B and C) are taken $\bmod (2p)$ $\big($resp. $\bmod (2q)$ and $\bmod (2s)\big)$ but, for simplicity, we use regular notation for the operation and do not write the mod notation.

To the 3-butterfly $(p/n, q/m, s/l)$ we associate two permutations of the set $A \cup B \cup C$, denoted ϕ and γ. The permutation ϕ is determined by the fact that each admissible vertex of the graph R belongs to two butterflies, and therefore, gets two labels. The permutation γ is associated to the identification of the butterflies wings.

Lemma 15. *Given a 3-butterfly* $(p/n, q/m, s/l)$ *satisfying the conditions given in* (6) *and the sets* $A = \{a_0, \ldots, a_{2p-1}\}$, $B = \{b_0, \ldots, b_{2q-1}\}$ *and* $C = \{c_0, \ldots, c_{2s-1}\}$, *the map* $\phi : A \cup B \cup C \to A \cup B \cup C$ *defined by*

$$\begin{aligned} a_{n-i} &\leftrightarrow b_{m+i-1}, \quad &\text{if} \quad 1 \le i \le t, \\ a_{n+j} &\leftrightarrow c_{l-j-1}, \quad &\text{if} \quad 0 \le j \le w-1, \\ b_{m-h} &\leftrightarrow c_{l+h-1}, \quad &\text{if} \quad 1 \le h \le v, \end{aligned} \tag{8}$$

is an order 2 permutation, where $t = p+q-s$, $v = q+s-p$ *and* $w = p+s-q$.

Proof. As ϕ is defined as a product of transpositions, we need to prove that all elements of $A \cup B \cup C$ are considered and that the transpositions are disjoint.

As $t = p+q-s$, $v = q+s-p$ and $w = p+s-q$ are the number of admissible vertices in $P \cap Q$, $Q \cap S$ and $P \cap S$, respectively, and given that $2 \le s \le q \le p$, then $1 \le t \le 2q \le 2p$, $0 \le v < 2s \le 2q$ and $1 \le w \le 2s \le 2q$. The elements of A appear in the transpositions

$$(a_{n-i}, b_{m+i-1}), (a_{n+j}, c_{l-j-1}), \qquad \text{for} \qquad 1 \le i \le t, \qquad 0 \le j \le w-1.$$

Clearly, for $1 \le i,\ l \le t < 2p$, $a_{n-i} \ne a_{n-l}$ if $i \ne l$ and for $0 \le j,\ l \le w-1 \le 2s-1 < 2p$, $a_{n+j} \ne a_{n+l}$ if $j \ne l$. Now, if $a_{n-i} = a_{n+j}$ for i, j, $1 \le i \le t$, $0 \le j \le w-1$ then $n - i \equiv n + j \bmod 2p$, so $i + j \equiv 0 \bmod 2p$, but this is not possible because $1 \le i + j \le t + w - 1 = 2p - 1$. Therefore all the elements in A belong to a transposition. The elements in B are in

$$(a_{n-i}, b_{m+i-1}), (b_{m-h}, c_{l+h-1}), \qquad \text{for} \qquad 1 \le i \le t, \qquad 1 \le h \le v,$$

and if $b_{m+i-1} = b_{m-h}$ for i, h, $1 \le i \le t$, $1 \le h \le v$, then $i + h - 1 \equiv 0 \bmod 2q$, but $2 \le i + h - 1 \le t + v - 1 \le 2q - 1$. So, all the elements in B belong to a transposition, and the same is true for the elements in C. Then ϕ is the product of disjoint bicycles, so its order is 2. ☑

Note that ϕ does not have fixed points, and among the transpositions in ϕ there is no transposition in the set

$$\begin{aligned} \mathcal{F} = \big\{ &(a_0, b_0), (a_0, b_q), (a_0, c_0), (a_0, c_s), (b_0, a_p), (b_0, c_0), \\ &(b_0, c_s), (c_0, a_p), (c_0, b_q), (a_p, b_q), (a_p, c_s), (b_q, c_s) \big\} \end{aligned} \tag{9}$$

since the trunks do not have common vertices. Let us call $\mathcal{F}$ the set of forbidden transpositions.

The construction of ϕ is well defined for any poligonalization of S^2 with 3 polygons, even if they do not satisfy the conditions of Theorem 10. In fact, in terms of the permutation ϕ, we can rewrite Theorem 10 as follows.

Theorem 16. *A triple $(p/n, q/m, s/l)$, with $p \geq q \geq s \geq 2$, $1 \leq n \leq p$, $1 \leq m \leq q$, $1 \leq l \leq s$, describes a 3-butterfly if and only if the associated permutation ϕ (8) does not have any of the transpositions in the forbidden set $\mathcal{F}$ given by (9).*

The following lemma describes the permutation ϕ explicitly. This is useful when we want to use ϕ, since there are some special considerations depending on the relative position of a_0, a_p, b_0, b_q, c_0 and c_s. We omit the tedious details of the proofs.

Lemma 17. *For a 3-butterfly $(p/n, q/m, s/l)$, the associated permutation ϕ is given by:*

$$\phi(a_j) = \begin{cases} b_{m+n-j-1}, & 0 \leq j < n; \\ c_{l+n-j-1}, & n \leq j < n+w; \\ b_{m+n-j-1+2p}, & w+n \leq j < 2p. \end{cases}$$

$$\phi(c_i) = \begin{cases} a_{l+n-i-1}, & 0 \leq i < l; \\ b_{m+l-i-1}, & l \leq i < v+l; \\ a_{l+n-i-1+2s}, & v+l \leq i < 2s. \end{cases} \qquad \text{If} \quad m \leq v$$

$$\phi(b_k) = \begin{cases} c_{l+m-k-1}, & 0 \leq k < m; \\ a_{m+n-k-1}, & m \leq k < t+m; \\ c_{l+m-k-1+2q}, & m+t \leq k < 2q. \end{cases} \qquad \text{If} \quad m > v$$

$$\phi(b_k) = \begin{cases} a_{m+n-k-1+2q}, & 0 \leq k < m-v; \\ c_{l+m-k-1}, & m-v \leq k < m; \\ a_{m+n-k-1}, & m \leq k < 2q. \end{cases}$$

The following permutation γ comes from the wing identification by the reflection along the trunk. Its properties are clear.

Lemma 18. *The function defined in the set $A \cup B \cup C$ by*

$$\begin{aligned} \gamma(a_i) &= a_{2p-i}, \quad 0 \leq i < 2p, \\ \gamma(b_j) &= b_{2q-j}, \quad 0 \leq j < 2q, \\ \gamma(c_h) &= b_{2s-h}, \quad 0 \leq h < 2s, \end{aligned} \tag{10}$$

is an order 2 permutation. The fixed point set is $\{a_0, a_p, b_0, b_q, c_0, c_s\}$.

Now we study $\mu = \phi\gamma$. The cyclic decomposition of μ allows us to determine if a 3-butterfly is reduced or not and to know the number of components of the associated link diagram.

Theorem 19 (**Classification**). *Let $(p/n, q/m, s/l)$ be a 3-butterfly, ϕ and γ its associated permutations given in Lemma 15 and Lemma 18 and let $\mu = \phi\gamma$.*

The 3*-butterfly* $(p/n, q/m, s/l)$ *is reduced if and only if* μ *is the product of three disjoint cycles. Besides, if* $\mathcal{O}(x)$ *is the orbit of* x*, the* 3*-bridge link diagram* L *represented by the reduced* 3*-buttefly satisfies:*

i) L *is a knot if and only if* $a_p \notin \mathcal{O}(a_0)$*,* $b_q \notin \mathcal{O}(b_0)$ *and* $c_s \notin \mathcal{O}(c_0)$*.*

ii) L *is a two component link if and only if one, and only one, of the following conditions is true:* $a_p \in \mathcal{O}(a_0)$*,* $b_q \in \mathcal{O}(b_0)$ *or* $c_s \in \mathcal{O}(c_0)$*.*

iii) L *is a three component link if and only if* $a_p \in \mathcal{O}(a_0)$*,* $b_q \in \mathcal{O}(b_0)$ *and* $c_s \in \mathcal{O}(c_0)$*.*

Proof. The essential point in the proof is to notice that the orbits of the vertices $a_0, a_p, b_0, b_q, c_0, c_s$ describe the three arcs that go under the bridges of the diagram. As we have only three bridges, if there are more than three cycles, then there are vertices that are not admissible and the 3-butterfly is not reduced. It is not possible that the number of cycles is one or two, because in that case the link diagram has only one or two bridges, but we are assuming that the triple $(p/n, q/m, s/l)$ is a 3-butterfly, so it satisfies the conditions of Theorem 10. Analyzing the orbits we get conditions i), ii) and iii). ☑

The previous theorem answers the question of whether a 3-butterfly is reduced or not, but not in a combinatorial way. It would be interesting to find combinatorial conditions on the set $\{p, n, q, m, s, l\}$ that allow us to know, a priori, if the 3-butterfly $(p/n, q/m, s/l)$ is reduced or not.

A variation of the permutation μ is very useful for the actual construction of a combinatorial knot (or Gauss code) for the link, (see [26] or [24] for more information on combinatorial knots) and so we are able to compute link invariants, such as the link group, the Seifert matrix, the Alexander, Jones and HOMFLY polynomials. In this paper we do not use this approach, but we describe an algorithm to draw a link diagram for any 3-butterfly $(p/n, q/m, s/l)$ in the next section.

If the 3-butterfly is not reduced, we drop the vertices corresponding to the orbits of not admissible vertices and recompute the values of the new butterfly. The following example shows some families of not reduced butterflies.

Example 20.

1) For $n \geq 2$, the 3-butterfly $\big((2n+1)/n, (2n+1)/1, (2n+1)/1\big)$ is not reduced and yields the same link diagram of the reduced 3-butterfly $(3/1, 3/1, 3/1)$.

2) For $n \geq 2$, the 3-butterfly $\big(4n/n, 4(n-1)/1, 4(n-1)/1\big)$ is not reduced and yields the same link diagram of the reduced 3-butterfly $\big(4/1, 4/1, 4/1\big)$.

3) For $p > 8$, the 3-butterfly $\big(p/5, (p-3)/(p-6), (p-6)/(p-6)\big)$ is not reduced and yields the same link diagram of the reduced 3-butterfly

$$\left(\frac{(p-3)}{5}, \frac{(p-4)}{(p-7)}, \frac{(p-8)}{(p-8)}\right).$$

3.4. An Algorithm to Draw a Canonical 3-Bridge Link Diagram

There exists a canonical way to draw a rational link, see [9, pag. 21]. In an analogous way, we describe a canonical method to draw a 3-bridge link diagram represented by a 3-butterfly $(p/n, q/m, s/l)$. This is a symmetric diagram and in the process the role of the integers $\{p, n, q, m, s, l\}$ becomes very clear.

We draw the three bridges as three segments, each one of them corresponding to the trunk of one of the butterflies. The initial point of each segment is on a circle, whose radius depends on the value of $n + m + l$. The trunk of butterfly P is drawn by a vertical segment and we call it bridge a. The trunk of butterfly Q (resp. S) is a segment forming a $120°$ angle (resp. $240°$ angle) with the bridge a and it is called bridge b (resp. bridge c).

We divide bridge a in p segments, and we fix two points in each division, one to the left and one to the right, except in the extremes, where there is only one point. We have then $2p$ points and we label them with the elements of the set $A = \{a_0, a_1, \ldots, a_{2p-1}\}$, in a counter-clock wise sense, so the extreme bridges are labeled a_0 and a_p. For bridge b we repeat the process, but we divide the bridge in q segments and label the points with $B = \{b_0, \ldots, b_{2q-1}\}$. For bridge c the number of segments is s and the labels are $C = \{c_0, \ldots, c_{2s-1}\}$. As before, the subscripts of A (resp. B and C) are taken $\bmod(2p)$, $\big($resp. $\bmod(2q)$ and $\bmod(2s)\big)$.

To draw the link diagram we need to join, with appropriate arcs, the points a_i, b_j and c_k , $i \in \mathbb{Z}_{2p}$, $j \in \mathbb{Z}_{2q}$, and $k \in \mathbb{Z}_{2s}$, according to the rules given by permutations ϕ and γ given in (8) and (10).

The number of arcs between the bridges corresponds to the number of vertices in the intersection of the corresponding butterflies, so by (2) we have:

$$\begin{aligned} t = p + q - s &: \text{arcs between the } a \text{ and } b \text{ bridges,} \\ v = q + s - p &: \text{arcs between the } b \text{ and } c \text{ bridges,} \\ w = p + s - q &: \text{arcs between the } c \text{ and } a \text{ bridges.} \end{aligned}$$

It is enough to know how to construct the first arc between each pair of bridges, and the rest of the arcs are "*parallel*" arcs to them. By the description of the 3-butterfly, we find that the first arc between the a and b bridges connects a_{n-1} with b_m, the first arc between the bridges b and c connects b_{m-1} with c_l and the first one between bridges c and a connects c_{l-1} with a_n, see Figure 14.

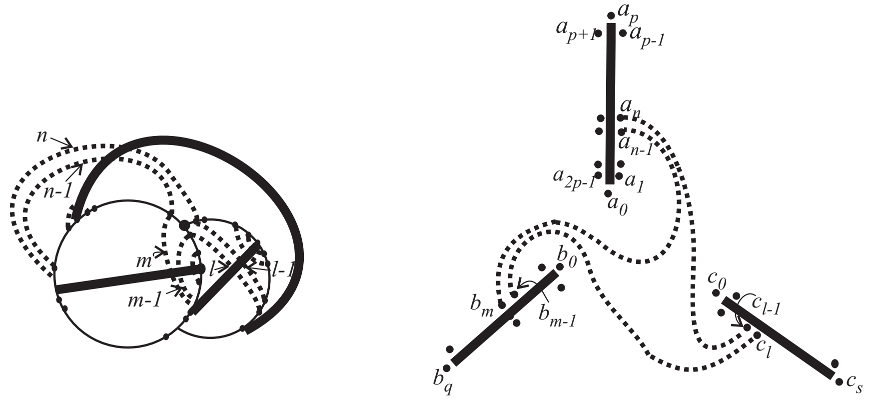


FIGURE 14. First step in the algorithm to draw a 3-bridge diagram associated with the 3-butterfly $(p/n, q/m, s/l)$.

Therefore the t arcs between the a and b bridges, see Figure 15a, are

$$a_{n-1}b_m,\ a_{n-2}b_{m+1},\ a_{n-3}b_{m+2}\ ,\dots,\ a_{n-j}b_{m+j-1}\ ,\dots,\ a_{n-t}b_{m+t-1}, \tag{11}$$

the v arcs between the b and c bridges are

$$b_{m-1}c_l,\ b_{m-2}c_{l+1},\ b_{m-3}c_{l+2}\ ,\dots,\ b_{m-j}c_{l+j-1}\ ,\dots,\ b_{m-v}c_{l+v-1}, \tag{12}$$

and the w arcs between the c and b bridges are

$$c_{l-1}a_n,\ c_{l-2}a_{n+1},\ c_{l-3}a_{n+2}\ ,\dots,\ c_{l-j}a_{n+j-1}\ ,\dots,\ c_{l-w}a_{n+w-1}, \tag{13}$$

Figure 15b shows the complete 3-diagram. The restrictions on $(p/n, q/m, s/l)$ given in (6) assure that the bridges do not connect one with another and in fact we get a 3-bridge diagram.

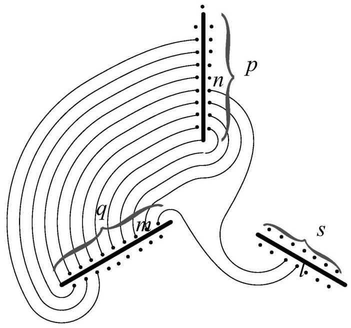
If the butterfly $(p/n, q/m, s/l)$ is reduced, we use all the arcs in the diagram, but if there are vertices that are not E-, A- or B-vertices, they yield arcs that are obsolete and we do not use them in the 3-bridge diagram, but we need them in the construction. So, we just drop the obsolete arcs and vertices, and we find a new 3-bridge diagram that corresponds to a new reduced 3-butterfly.

As a result of the previous construction we obtain the following lemma.

Lemma 21. *If the* 3*-butterfly* $(p/n, q/m, s/l)$ *is reduced, the* 3*-bridges diagram has* $p+q+s-3$ *crossings.*

4. 3-Butterfly Representation of a 3-Bridge Link

A natural way to establish a classification for the 3-bridge links is to *order* the set of 3-butterflies that represent them and take the minimum with respect



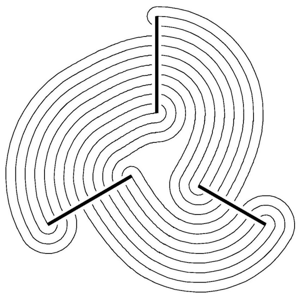


FIGURE 15. Canonical 3-bridge diagram represented by the 3-butterfly $(p/n, q/m, s/l)$.

to this order. We want to minimize the crossing number of the diagram, but this consideration is not enough, because there are different 3-bridges diagrams of the same link, with minimum crossing numbers. For instance the 8_{18} knot is represented by the 3-butterflies $(10/4, 10/2, 8/4)$ and $(10/9, 10/7, 8/5)$, the corresponding 3-bridge diagrams have 25 crossings, (see Figure 16), that is the minimum crossing number for a 3-bridge diagram for this knot. In fact, in [10] Montesinos shows examples of minimal non equivalent 3-plats presentations of the same link, i.e. it is not possible to change one into the other preserving the 3-bridge representation. This example shows the difficulties in the classification of 3-bridge links. So we need to consider other aspects in the classification. In this section we address the problem arising from the lack of uniqueness in the presentation of a 3-bridge link as a 3-butterfly. By a typical procedure in knot theory, we establish a *concept of minimality* among all 3-butterfly representations of a link and define that minimum as the *appropriate* 3-butterfly representation of that link.

Now we define an order in the set $\mathcal{T}$ of all 3-butterflies $(p/n, q/m, s/l)$. As we are interested in the simplicity of the diagram, the first consideration is to minimize the crossing number. By Lemma 21, we know that the 3-bridge diagram represented by the reduced 3-butterfly $(p/n, q/m, s/l)$ has crossing number equal to $p + q + s - 3$. Then we order the set $\mathcal{T}$ by the sum $p + q + s$, then we take a lexicographic order in the triples (p, q, s) and then a lexicographic order in the triples (n, m, l).

Definition 22. In the set $\{(x, y, z) \in \mathbb{N}^3\}$ we define the lexicographic order as

$$(x, y, z) < (x', y', z') \qquad \text{if and only if}$$

$$
\begin{array}{rlllll}
\text{i)} & x+y+z < x'+y'+z' & \text{or} & & & \\
\text{ii)} & x+y+z = x'+y'+z' & \text{and} & x < x' & \text{or} & \\
\text{iii)} & x+y+z = x'+y'+z', & x = x' & \text{and} & y < y'. &
\end{array}
$$

Using this order, we define an order in the set of all 3-butterflies.

Definition 23. Given two 3-butterflies $(p/n, q/m, s/l)$ and $(p'/n', q'/m', s'/l')$, we say that

$$(p/n, q/m, s/l) < \left(p'/n', q'/m', s'/l'\right)$$

if and only if

$$
\begin{array}{rlll}
\text{i)} & (p,q,s) < (p',q',s') & \text{or} & \\
\text{ii)} & (p,q,s) = (p',q',s') & \text{and} & (n,m,l) < (n',m',l')
\end{array}
$$

Using this order, we are able to define the 3-butterfly representation of a link.

Definition 24. Given a 3-bridge link L, we define the 3-butterfly representation of L as the minimum of the set of all the 3-butterflies $(p/n, q/m, s/l)$ such that $(p/n, q/m, s/l)$ is a 3-butterfly representation of some 3-bridge diagram of L.

Thus, the 3-butterfly representation of a link is an invariant of the 3-bridge link.

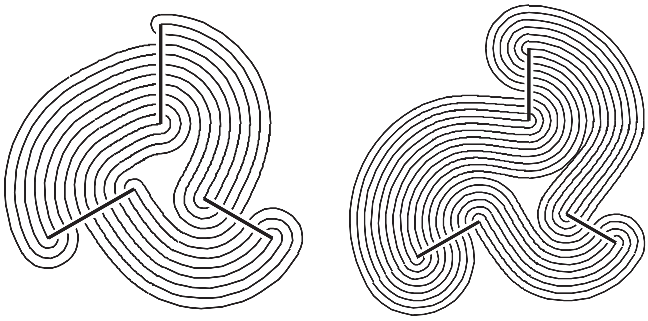
Example 25. For the 3-butterflies of the knot 8_{18} given in Figure 16 we have that $(10/4, 10/2, 8/4) < (10/9, 10/7, 8/5)$, and it is possible to prove that $(10/4, 10/2, 8/4)$ is in fact the 3-butterfly representation of 8_{18}.

We choose this order to minimize the crossing number and to have as much symmetry as possible. Figure 16 shows two 3-bridge diagrams for the 8_{18} knot, corresponding to the 3-butterflies $M_1 = (10/4, 10/2, 8/4)$ and $M_2 = (10/9, 10/7, 8/5)$. Both have the same crossing number and the same value for (p, q, s). Note the symmetry for the diagram in Figure 16a, that corresponds to M_1 and $M_1 < M_2$.

With this 3-butterfly representation for any 3-bridge link we are able to construct a table of all 3-bridge links using the procedure:

1) We order all reduced 3-butterflies.
2) For each 3-butterfly we find its associated 3-bridge link L. Using link invariants, we check if L is in the table or not. If it is not, we add L to the table and continue with the next 3-butterfly.

In [25] we have a list with the first thousand 3-bridge knots, in this order.



a b

FIGURE 16. Two 3-bridge diagrams for the knot 8_{18}.

5. Examples of 3-Bridge Link Families

In this section we show the usefulness of our construction by giving the 3-butterfly presentations of some important families of 3-bridge links: the mirror image, the connected sum of two rational knots, the torus link of type $T(3,p)$, the pretzel links of type $P(a,b,c)$ and the 3-bridge knots, up to 9 crossings in Rolfsen's table, see [18]. We give a short idea of the proofs, by using diagrams, and we omit the details.

The 3-butterfly representation distinguishes a 3-bridge link from its mirror image.

Theorem 26 (Mirror Image)**.** *If a 3-bridge link L has a 3-butterfly representation given by $(p/n, q/m, s/l)$, then its mirror image L^* has a 3-butterfly representation given by $(p/n', q/m', s/l')$ with*

$$\begin{aligned} n' &= ((q - s - n) \bmod p) + 1, \\ m' &= ((s - p - m) \bmod q) + 1, \\ l' &= ((p - q - l) \bmod s) + 1. \end{aligned}$$

Proof. It is enough to change the way we "*close*" the 3-butterfly. For the construction of L we suppose that the graph R is in the plane $z = 0$ and the 3-ball B is the semi-space under the plane, see Section 2.1. Now, if we consider the same graph R and the ball the semi-space over the plane, we construct the mirror image L^*. Therefore, we change the way of looking at the 3-butterfly as the diagram in Figure 17 shows. We now read the values of n', m' and l'. ☑

We have the following corollary.

Corollary 27. *If a link L has a 3-butterfly representation given by $(p/n, q/m, s/l)$ and is such that*

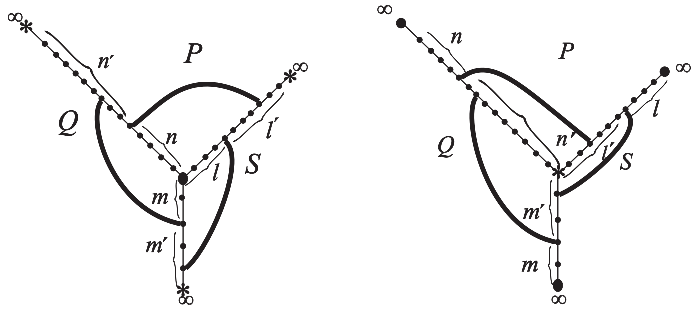


FIGURE 17. A 3-butterfly and its mirror image.

$$n - 1 \equiv (q - s - n) \bmod p,$$
$$m - 1 \equiv (s - p - m) \bmod q, \qquad \textit{and}$$
$$l - 1 \equiv (p - q - l) \bmod s,$$

then $L = L^*$.

This is not an equivalence. There are amphicheiral 3-bridge links with an infinite number of 3-bridge diagrams that are not its mirror image. For instance, the knot 10_{91} is amphicheiral and has $(13/6, 12/4, 10/3)$ as the 3-bridge presentation. The following question arises: Is it true that *if a 3-bridge link is amphicheiral then it admits a 3-butterfly representation satisfying conditions* (27)*?*

Theorem 28 (Connected Sum)**.** *If* $K = p_1/q_1$ *and* $L = p_2/q_2$ *are rational knots, then its connected sum* $K\#L$ *has a* 3*-butterfly representation given by*

$$\big((p_1 + p_2 - 1)/q_1, p_1/(q_1 + 1), p_2/q_2\big) \tag{14}$$

Proof. It is easy to check that the triplet given in (14) satisfies conditions (6), thus it is a 3-butterfly. We draw the associated 3-bridge and we see clearly that it is the connected sum $K\#L$, (see Figure 18). Note that the connected sum was made by joining two of the bridges. ☑

Theorem 29 (Torus links $T(3, p)$)**.** *The* 3*-bridge link given by* $(p/1, p/1, p/1)$ *is the toroidal link* $T(3, -p)$ *and the* 3*-bridge link* $(p/p, p/p, p/p)$ *is its mirror image* $T(3, p)$*.*

Proof. Just draw the torus link in the canonical way describe in Seccion 3.4. Figure 19 shows an example. ☑

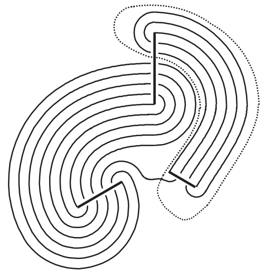


FIGURE 18. 3-bridge diagram of the connected sum of two rational links.

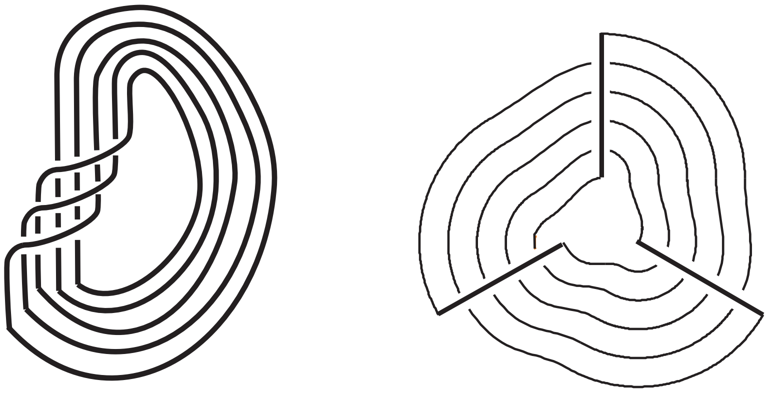


FIGURE 19. The toroidal knot $T(3,-5)$.

Theorem 30 (Pretzel Links)**.** *For integers $0 < a \leq b \leq c$, the pretzel links have the 3-butterfly representation given by:*

1) $P(a,b,c)$ has 3-butterfly representation $\left(\dfrac{c+b}{c}, \dfrac{c+a}{a}, \dfrac{b+a}{b}\right)$.

2) $P(-a,-b,-c)$ has 3-butterfly representation $\left(\dfrac{c+b}{c+1}, \dfrac{c+a}{a+1}, \dfrac{b+a}{b+1}\right)$.

3) $P(a,b,-c)$ has 3-butterfly representation $\left(\dfrac{c+b-1}{c}, \dfrac{c+a-1}{a+1}, \dfrac{b+a}{b-1}\right)$.

4) $P(a,-b,c)$ has 3-butterfly representation $\left(\dfrac{c+b-1}{c+1}, \dfrac{c+a}{a-1}, \dfrac{b+a-1}{b}\right)$.

5) $P(-a,b,c)$ has 3-butterfly representation $\left(\dfrac{c+b}{c-1}, \dfrac{c+a-1}{a}, \dfrac{b+a-1}{b+1}\right)$.

6) $P(-a,-b,c)$ has 3-butterfly representation $\left(\dfrac{c+b-1}{c-1}, \dfrac{c+a-1}{a}, \dfrac{b+a}{b+2}\right)$.

7) $P(-a,b,-c)$ has 3-butterfly representation $\left(\dfrac{c+b-1}{c}, \dfrac{c+a-1}{a+2}, \dfrac{b+a}{b-1}\right)$.

8) $P(a,-b,-c)$ has 3-butterfly representation $\left(\dfrac{c+b}{c+2}, \dfrac{c+a-1}{a-1}, \dfrac{b+a}{b}\right)$.

Proof. We sketch the proof of Case 1 by the sequence of diagrams of Figure 20, Figure 21 and Figure 22.

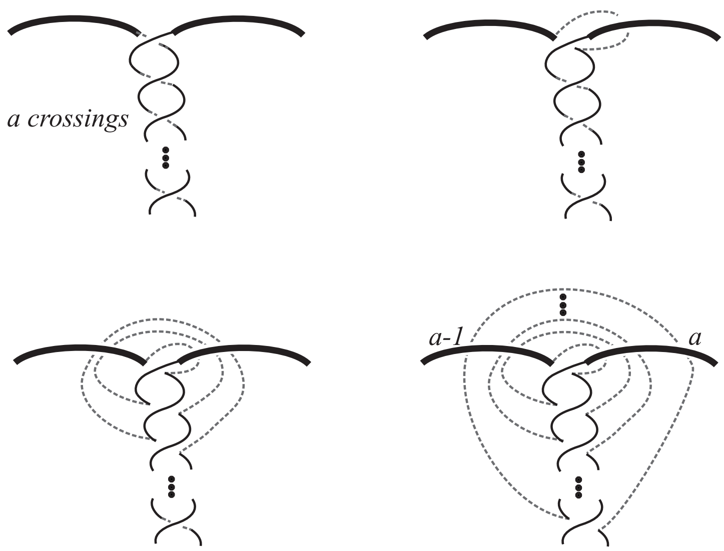


Figure 20. Change of crossing in a tangle.

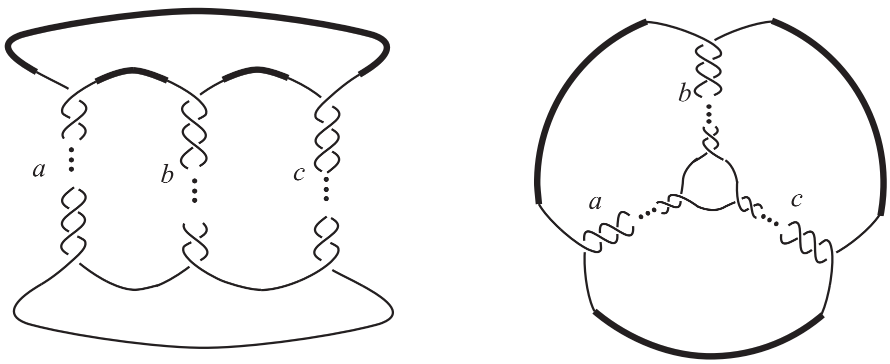


Figure 21. Two ways to draw the pretzel link $P(a,b,c)$.

For the Cases 3, 4 and 5, we may need to simplify the diagram by performing some Reidemeister moves of type II in order to get an appropriate 3 bridge link

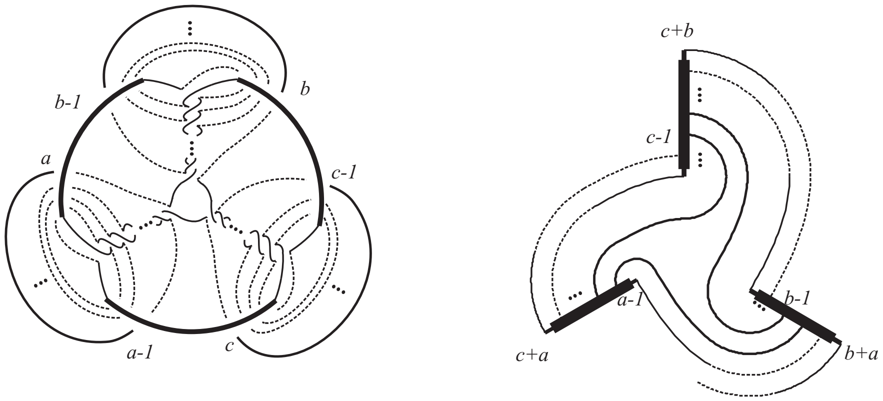


Figure 22. Diagrams of the pretzel link $P(a, b, c)$ for the case $0 \leq a \leq b \leq c$.

diagram. For the other cases we use the formula for the mirror image of the appropriate pretzel. ☑

Table 2 gives the 3-butterfly representations of the 3-bridge knots, up to 9 crossings, in Rolsfsen's table, see [18]. To compile the table, we studied all 3-butterflies up to 30 crossings. We used HOMFLY and Jones polynomials for a first classification and then we turned to *Knotscape [20]* for further refinements. In [25] we present a table with the 3-butterfly representation of all knots, up to 10 crossings, in Rolfsen's table.

$8_5 = (6/3, 5/2, 5/3)$	$8_{10} = (7/3, 7/3, 6/3)$	$8_{15} = (9/5, 9/2, 6/3)$
$8_{16} = (8/3, 8/2, 7/3)$	$8_{17} = (9/3, 8/3, 8/3)$	$8_{18} = (10/4, 10/2, 8/4)$
$8_{19} = (4/4, 4/4, 4/4)$	$8_{20} = (5/2, 5/3, 4/3)$	$8_{21} = (6/6, 6/5, 5/4)$
$9_{16} = (11/1, 11/7, 6/3)$	$9_{22} = (11/4, 9/3, 8/5)$	$9_{24} = (13/1, 13/8, 6/3)$
$9_{25} = (11/5, 9/2, 8/5)$	$9_{28} = (15/2, 15/8, 6/3)$	$9_{29} = (11/8, 11/7, 9/6)$
$9_{30} = (13/11, 11/5, 8/6)$	$9_{32} = (12/10, 11/7, 10/8)$	
$9_{33} = (13/5, 12/2, 10/5)$	$9_{34} = (14/11, 12/8, 12/9)$	
$9_{35} = (6/3, 6/3, 6/3)$	$9_{36} = (9/4, 8/3, 7/4)$	$9_{37} = (12/7, 9/5, 9/2)$
$9_{38} = (13/8, 10/6, 10/2)$	$9_{39} = (12/7, 12/2, 7/3)$	$9_{40} = (12/9, 12/9, 12/9)$
$9_{41} = (9/4, 9/4, 9/4)$	$9_{42} = (6/1, 6/1, 6/2)$	$9_{43} = (7/2, 7/5, 6/4)$
$9_{44} = (8/7, 7/6, 6/6)$	$9_{45} = (8/7, 8/7, 7/6)$	$9_{46} = (6/2, 5/3, 5/4)$
$9_{47} = (8/3, 8/3, 8/3)$	$9_{48} = (9/5, 9/2, 6/1)$	$9_{49} = (7/5, 7/5, 7/5)$

Table 2. 3-butterfly representation of the 3-bridge knots with less than 10 crossings.

We end this section with a brief note on the constructions given in [2], [12] and [13]. In [2] the authors associate to every 3-bridge an special type of 3-bridge link diagram that can be characterized by six integers. They denote this

3-bridge link diagram by $L(p, q, k, h, t, s)$ and the construction is given in an algorithmic way, designed for the computer. The initial point is a diagram of the p/q rational link and they introduce a method to modify the diagram in a way that depends on the parameters k, h, t and s. They do not give conditions on the set of integers to ensure that they represent a 3-bridge link diagram and the 3-bridge diagram they obtain usually has kinks. We can interpret their construction by using 3-butterflies (the details will be presented elsewhere) and we obtain that a 3-butterfly representation of the link $L(p, q, k, h, t, s)$ is given by

$$\left(\frac{p}{(k+q) \bmod p}, \frac{p-t}{p-t-k+1}, \frac{h-t}{s}\right).$$

In [12] and [13], the author obtains a family of 3-bridge diagrams, depending on a set of three numbers. Again, the author starts with a diagram of the p/q rational link and modifies it, depending on a parameter r. The obtained 3-bridge diagram is denoted $K(p, q; r)$. The author studies some interesting properties of the links constructed in this way. In [2], the authors interpret this $K(p, q; r)$ en terms of their notation, and so we get a 3-butterfly representation of the link $K(p, q; r)$. It is an interesting project to extend some of the results given in [2], [12] and [13] to other families of 3-bridge links.

6. Concluding Remarks

There are constructions related to 3-bridge links that can be formulated in terms of the 3-butterfly representation introduced here. Some of them appear here and some others will be left for another paper.

A presentation of the group of a 3-bridge link L can be easily obtained from a 3-butterfly representation of L. It is interesting to find properties of this group depending on the 3-butterfly representation.

Another important topic is to study the two-fold branched covering of a 3-bridge link in terms of a 3-butterfly representation.

In this paper we do not consider orientation. If we want to study oriented 3-bridge links, we need to extend the set of integers that describe the 3-butterfly and allow that n, m and l, (see Section 2.1) be integers. It is interesting to investigate the changes in the restrictions and constructions.

A question remains unanswered: Is it possible to state a set of "butterfly moves" to change a 3-butterfly into an equivalent one?

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