Transitivity of the Induced Map $C_n(f)$

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Abstract. A map $f : X \to X$, where $X$ is a continuum, is said to be transitive if for each pair $U$ and $V$ of nonempty open subsets of $X$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. In this paper, we show relationships between transitivity of $f$ and its induced maps $C_n(f)$ and $F_n(f)$, for some $n \in \mathbb{N}$. Also, we present conditions on $X$ such that given a map $f : X \to X$, the induced function $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$.

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1. Introduction

A map $f : X \to X$, where $X$ is a continuum, is said to be transitive if for each pair $U$ and $V$ of nonempty open subsets of $X$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. In [8], Robert Devaney says that a map $f : X \to X$, where $X$ is a metric space, is chaotic on $X$ provided that: i) $f$ has sensitive dependence
on initial conditions, ii) the periodic points of \( f \) are dense in \( X \), and iii) \( f \) is transitive. In [3], it is shown that if the periodic points of \( f \) are dense and \( f \) is transitive, then \( f \) has sensitive dependence on initial conditions; i.e., condition i) is not necessary in Devaney’s definition. Also, it is known that if \( f \) is defined on \([0, 1]\), then \( f \) is chaotic if and only if \( f \) is transitive [4]. Therefore, transitivity is an important property in chaotic dynamical systems.

A continuum is a compact, connected and nonempty metric space. Let \( X \) be a continuum and let \( n \in \mathbb{N} \). The \( n \)-fold hyperspace of \( X \), denoted by \( C_n(X) \), is defined as the set \( C_n(X) = \{ A \subset X : A \text{ is closed, nonempty and has at most } n \text{ components} \} \). The \( n \)-fold symmetric product, denoted by \( F_n(X) \), is defined as \( F_n(X) = \{ A \subset X : A \text{ is nonempty and has at most } n \text{ points} \} \). Given a map \( f : X \to X \) and \( n \in \mathbb{N} \), it is possible to define the induced maps \( C_n(f) : C_n(X) \to C_n(X) \) and \( F_n(f) : F_n(X) \to F_n(X) \). In Section 3 of this paper, after the introduction and preliminaries, we study all possible relationships between the following three statements:

1. \( f \) is transitive.
2. \( C_n(f) \) is transitive, for some \( n \in \mathbb{N} \).
3. \( F_n(f) \) is transitive, for some \( n \in \mathbb{N} \).

In Section 4, we prove that if either \( X \) contains a free arc, \( X \) is a continuum of type \( \lambda \) or \( X \) is a dendrite, then the induced map \( C_n(f) : C_n(X) \to C_n(X) \) is not transitive, for any \( n \in \mathbb{N} \). The transitivity of \( C_1(f) \) was studied by G. Acosta, A. Illanes and H. Mendez in [1].

2. Preliminaries

A continuum is a compact, connected and nonempty metric space. An arc is any space homeomorphic to the closed interval \([0, 1]\). Also, if \( h : [0, 1] \to \alpha \) is a homeomorphism, then \( p = h(0) \) and \( q = h(1) \) are called the end points of the arc \( \alpha \); one says that \( \alpha \) is an arc from \( p \) to \( q \). Given an arc \( \alpha \) with end points \( p \) and \( q \) in a continuum \( X \), we say that \( \alpha \) is a free arc if \( \alpha \setminus \{ p, q \} \) is an open subset of \( X \). A map is assumed to be a continuous function. If \( X \) is a continuum, then given \( A \subset X \), the closure and the interior are denoted by \( \overline{A} \) and \( \text{Int}(A) \), respectively. A dendrite is a locally connected continuum which contains no homeomorphic copy of \( S^1 = \{ z \in \mathbb{C} : ||z|| = 1 \} \). A continuum \( X \) is said to be irreducible provided that there exist \( p, q \in X \) such that no proper subcontinuum of \( X \) contains \( \{ p, q \} \); we say that \( X \) is irreducible between \( p \) and \( q \). A map between continua \( f : X \to Y \) is said to be monotone provided that \( f^{-1}(y) \) is connected for each \( y \in Y \). A continuum \( X \) which is irreducible between \( p \) and \( q \) is said to be of type \( \lambda \) if there is a monotone map \( m : X \to [0, 1] \) such that \( m(p) = 0, m(q) = 1 \) and \( \text{Int}(m^{-1}(t)) = \emptyset \) for each \( t \in [0, 1] \) (see [11] for a complete investigation about continua of type \( \lambda \).
Given a continuum $X$ and a positive integer $n$, the $n$–fold hyperspace of $X$, denoted by $C_n(X)$, is defined as the set $C_n(X) = \{A \subset X : A$ is closed, nonempty and has at most $n$ components$\}$ topologized by the Hausdorff metric $[10]$ Definition 2.1. It is well known that $C_n(X)$ is an arcwise connected continuum $[13]$ Corollary 1.8.12. The $n$–fold symmetric product, denoted by $F_n(X)$, is defined for $F_n(X) = \{A \in C_n(X) : A$ has at most $n$ points$\}$ $[6]$. $F_n(X)$ is endowed with the relative topology as a subspace of $C_n(X)$.

Let $X$ be a continuum and let $D_1, \ldots, D_k$ be nonempty subsets of $X$. We define $\langle D_1, \ldots, D_k \rangle = \{A \in C_n(X) : A \subset \bigcup_{i=1}^{k} D_i$ and $A \cap D_i \neq \emptyset$ for each $i \in \{1, \ldots, k\}\}$. Let $B = \{\langle U_1, \ldots, U_k \rangle : U_i$ is open and $k \in \mathbb{N}\}; B$ is a base for the topology generated by the Hausdorff metric on $C_n(X)$ $[13]$ Theorem 1.8.16].

Let $f : X \to Y$ be a map between continua and let $n \in \mathbb{N}$. Then the function $C_n(f) : C_n(X) \to C_n(Y)$ given by $C_n(f)(A) = f(A)$ for each $A \in C_n(X)$, is called the induced map between the $n$–fold hyperspaces $C_n(X)$ and $C_n(Y)$. The map $F_n(f) : F_n(X) \to F_n(Y)$ given by $F_n(f) = C_n(f)|_{F_n(X)}$ is called the induced map between the $n$–fold symmetric products $F_n(X)$ and $F_n(Y)$. In $[10]$ p. 188, it is shown that $C_n(f)$ is a map. Regarding induced maps, the reader may see $[2] [10] [9] [13]$.

Given a map $f : X \to X$ and $n \in \mathbb{N}$, $f^n$ means the composition $f \circ f \circ \cdots \circ f$, $n$ times. If $n = 0$, $f^0$ is the identity map. Let $x \in X$. The orbit of $x$, denoted by $O(x, f)$, is the set of points $O(x, f) = \{f^n(x) : n \in \mathbb{N} \cup \{0\}\}$. The $\omega$–limit of $x$, denoted by $\omega(x, f)$, is given as the set of accumulation points of the sequence $O(x, f)$. It is easy to see that $\omega(x, f) = \omega(f^k(x), f)$ for each $k \in \mathbb{N}$.

Definition 2.1. Let $X$ be a continuum and let $f : X \to X$ be a map. We say that $f$ is transitive provided that for each pair of nonempty open subsets $U$ and $V$ of $X$, there exists $n \in \mathbb{N}$, such that $f^n(U) \cap V \neq \emptyset$.

Definition 2.2. Let $X$ be a continuum and let $f : X \to X$ be a map. We say that $f$ is exact provided that for each nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$, such that $f^n(U) = X$.

The next claim follows easily from Definitions 2.1 and 2.2.

Claim 2.3. Let $f : X \to X$ be a map. If $f$ is exact then $f$ is transitive.

Theorem 2.4. $[5]$ Proposition 39, p.155] Let $X$ be a continuum and let $f : X \to X$ be a map. Then $f$ is transitive if and only if there exists $x \in X$ such that $\omega(x, f) = X$.

3. On $C_n(f), F_n(f)$ and $f$

Given a continuum $X$ and a map $f : X \to X$, we study the relationships between the following three statements:

(1) $f$ is transitive.
(2) $C_n(f)$ is transitive, for some $n \in \mathbb{N}$.

(3) $F_n(f)$ is transitive, for some $n \in \mathbb{N}$.

**Lemma 3.1.** Let $X$ be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be an exact map. If $B \in C_n(X)$ is such that $\text{Int}(B) \neq \emptyset$, then $\omega(B, C_n(f)) = \{X\}$.

**Proof.** Since $f$ is exact, there exists $k \in \mathbb{N}$ such that $f^k(\text{Int}(B)) = X$. Thus, $f^{mn}(B) = X$ for each $m \geq k$. Therefore, $\omega(B, C_n(f)) = \{X\}$. \hfill $\Box$

Notice that if $f : S^1 \to S^1$ is defined by $f(z) = ze^{2\pi i \theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then $f$ is transitive and the induced map $F_n(f)$ is not transitive, for any $n \in \mathbb{N} \setminus \{1\}$ [Example 3.8]. Therefore, (1) does not imply (3).

**Claim 3.2.** Let $X$ be a continuum and let $n \in \mathbb{N}$. The family $E_n = \{ \langle U_1, \ldots, U_s \rangle \cap F_n(X) : U_i$ is open of $X$ and $s \leq n \}$, is a base for the topology on $F_n(X)$.

**Proof.** Let $\langle V_1, \ldots, V_k \rangle$ be an open subset of $C_n(X)$ such that $\langle V_1, \ldots, V_k \rangle \cap F_n(X) \neq \emptyset$. Let $\{x_1, \ldots, x_s\} \in \langle V_1, \ldots, V_k \rangle \cap F_n(X)$. Note that $s \leq n$. Let $U_i = \bigcap \{V_j : x_i \in V_j, \ j \in \{1, \ldots, k\}\}$, for each $i \in \{1, \ldots, s\}$. It is not difficult to see that $\{x_1, \ldots, x_s\} \in \langle U_1, \ldots, U_s \rangle \cap F_n(X) \subset \langle V_1, \ldots, V_k \rangle \cap F_n(X)$ and the proof is complete. \hfill $\Box$

**Proposition 3.3.** Let $X$ be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If $f$ is exact, then $F_n(f)$ is transitive.

**Proof.** Let $\langle U_1, \ldots, U_l \rangle \cap F_n(X)$ and $\langle V_1, \ldots, V_s \rangle \cap F_n(X)$ be open subsets of $F_n(X)$ such that $l, s \leq n$ (Claim 3.2). Suppose that $l \leq s \leq n$. Since $f$ is exact, there exists $k \in \mathbb{N}$ such that $f^k(U_i) = X$ for each $i \in \{1, \ldots, l\}$. Hence, $f^k(U_i) \cap V_j \neq \emptyset$ for each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, s\}$. Let $x_i \in U_i$ such that $f^k(x_i) \in V_i$, and let $x_j \in U_1$ such that $f^k(x_j) \in V_j$, for each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, s\}$. It is clear that $\{x_1, \ldots, x_s\} \in \langle U_1, \ldots, U_l \rangle \cap F_n(X)$ and $F_n(f)^k(\{x_1, \ldots, x_s\}) \in (V_1, \ldots, V_s) \cap F_n(X)$. Therefore, $F_n(f)^k(\langle U_1, \ldots, U_l \rangle \cap F_n(X)) \cap (\langle V_1, \ldots, V_s \rangle \cap F_n(X)) \neq \emptyset$. Similarly, we conclude the result if we assume that $s \leq l \leq n$. \hfill $\Box$

The following shows that neither (1) nor (3) implies (2).

**Proposition 3.4.** There exists a transitive map $f : X \to X$ such that $F_n(f)$ is transitive, for each $n \in \mathbb{N}$, and $C_n(f)$ is not transitive, for any $n \in \mathbb{N}$.

**Proof.** Let $f : S^1 \to S^1$ be defined by $f(z) = z^2$, for each $z \in S^1$. It is not difficult to see that $f$ is exact. Hence, $F_n(f)$ is transitive, for each $n \in \mathbb{N}$, by Proposition 3.3.
Let \( n \in \mathbb{N} \) and let \( B \in C_n(X) \). We prove that \( \omega(B,C_n(f)) \neq C_n(X) \). Suppose first that \( B \in F_n(X) \). Then \( \omega(B,C_n(f)) = \omega(B,F_n(f)) \subset F_n(X) \). Thus, \( \omega(B,C_n(f)) \neq C_n(X) \). Now, we assume that \( B \in C_n(X) \setminus F_n(X) \). Hence, \( \text{Int}(B) \neq \varnothing \). Therefore, by Lemma \ref{lem:omega} \( \omega(B,C_n(f)) = \{X\} \). The proof now follows from Theorem \ref{thm:transitivity}.

\[ \square \checkmark \]

**Proposition 3.5.** Let \( X \) be a continuum, let \( n \in \mathbb{N} \) and let \( f : X \to X \) be a map. If \( B \in C_n(X) \) (or \( B \in F_n(X) \)) is such that \( \omega(B,C_n(f)) = C_n(X) \) (or \( \omega(B,F_n(f)) = F_n(X) \), respectively) and \( p \in B \), then \( \omega(p,f) = X \).

**Proof.** Let \( U \) be an open subset of \( X \). Since \( \omega(B,C_n(f)) = C_n(X) \), there exists \( k \in \mathbb{N} \) such that \( f^k(B) \in \langle U \rangle \). Thus, \( f^k(B) \subset U \) and \( f^k(p) \in U \). Therefore, \( \omega(p,f) = X \).

\[ \square \checkmark \]

**Theorem 3.6.** Let \( X \) be a continuum and let \( f : X \to X \) be a map. If \( C_n(f) \) or \( F_n(f) \) is transitive, for some \( n \in \mathbb{N} \), then \( f \) is transitive.

**Proof.** It follows from Proposition \ref{prop:omega} and Theorem \ref{thm:transitivity}. Another proof can be found in \cite{revista:35} Theorem 4.

\[ \square \checkmark \]

Theorem \ref{thm:transitivity} completes all possible relationships between (1), (2) and (3).

**Theorem 3.7.** Let \( X \) be a continuum, let \( n \in \mathbb{N} \) and let \( f : X \to X \) be a map. If \( C_n(f) \) is transitive then \( F_n(f) \) is transitive.

**Proof.** Suppose that \( C_n(f) \) is transitive. Then, by Theorem \ref{thm:transitivity} there exists \( B \in C_n(X) \) such that \( \omega(B,C_n(f)) = C_n(X) \).

We prove that \( B \in C_n(X) \setminus C_{n-1}(X) \). Let \( U_1, \ldots, U_n \) be pairwise disjoint, open subsets of \( X \). Since \( \omega(B,C_n(f)) = C_n(X) \), there is a positive integer \( k \) such that \( f^k(B) \in \langle U_1, \ldots, U_n \rangle \). Thus, \( f^k(B) \) has exactly \( n \) components and \( B \in C_n(X) \setminus C_{n-1}(X) \).

Let \( B_1, \ldots, B_n \) be the components of \( B \). Let \( x_i \in B_i \) for each \( i \in \{1, \ldots, n\} \). Let \( A = \{x_1, \ldots, x_n\} \in F_n(X) \). We prove that \( \omega(A,F_n(f)) = F_n(X) \). Let \( V_1, \ldots, V_s \) be pairwise disjoint open subsets of \( X \), \( s \leq n \). Since \( \omega(B,C_n(f)) = C_n(X) \), \( f^k(B) \in \langle V_1, \ldots, V_s \rangle \), for some \( k \in \mathbb{N} \). Observe that each component of \( f^k(B) \) intersects \( f^k(A) \). Thus, \( f^k(A) \in \langle V_1, \ldots, V_s \rangle \). Since \( f^k(A) \in F_n(X) \), \( f^k(A) \in \langle V_1, \ldots, V_s \rangle \cap F_n(X) \) and \( \omega(A,F_n(f)) = F_n(X) \). Therefore, by Theorem \ref{thm:transitivity} \( F_n(f) \) is transitive.

\[ \square \checkmark \]

We finish this section with a question.

**Question 3.8.** Let \( X \) be a continuum and let \( f : X \to X \) be a map. If \( m \neq n \) and \( C_n(f) \) is transitive, then does it follow that \( C_m(f) \) is transitive?
4. Transitivity of $C_n(f)$

In [1, Teorema 7.18], it is shown a map $f : [0,1]^N \to [0,1]^N$ such that the induced map $C_1(f)$ is transitive. It is not known, another map $f : X \to X$, where $X$ is a continuum, such that $C_1(f)$ is transitive. In particular, we do not know if there exists a map $f$ defined on a finite-dimensional continuum such that $C_1(f)$ is transitive [1, Question 7.20]. In this section, with similar arguments to those given in [1], we present some particular cases when the induced map $C_n(f)$ cannot be transitive, for any $n \in \mathbb{N}$.

The proof of the following lemma is the same as the one given in [1, Theorem 4.3] and will be omitted.

**Lemma 4.1.** Let $X$ be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If $B \in C_n(X)$ is such that $\omega(B,C_n(f)) = C_n(X)$, then $\text{Int}(f^k(B)) = \emptyset$, for each $k \in \mathbb{N}$.

**Theorem 4.2.** Let $X$ be a continuum and let $f : X \to X$ be a map. If $X$ contains a free arc, then the induced map $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$.

**Proof.** Let $n \in \mathbb{N}$. Suppose that $C_n(f)$ is transitive. Then, by Theorem 2.4, there exists $B \in C_n(X)$ such that $\omega(B,C_n(f)) = C_n(X)$. Let $\alpha \subset X$ be a free arc with end points $p$ and $q$. Let $U$ and $V$ be nonempty, disjoint open subsets of $\alpha \setminus \{p, q\}$. Let $W_1, \ldots, W_{n-1}$ be nonempty, pairwise disjoint and open subsets of $\alpha \setminus \alpha$. Observe that $\mathcal{U} = \langle \alpha \setminus \{p, q\}, U, V, W_1, \ldots, W_{n-1} \rangle$ is a nonempty open subset of $C_n(X)$. Since $\omega(B,C_n(f)) = C_n(X)$, $f^k(B) \in \mathcal{U}$ for some $k \in \mathbb{N}$. Notice that $W_1, \ldots, W_{n-1}$ and $\alpha \setminus \{p, q\}$ are $n$ nonempty pairwise disjoint open subsets of $X$. Thus, $f^k(B)$ has exactly $n$ components. Let $B_0$ be the component of $f^k(B)$ such that $B_0 \subset \alpha \setminus \{p, q\}$. Since $B_0 \cap U \neq \emptyset$, $B_0 \cap V \neq \emptyset$ and $U \cap V = \emptyset$, we have that $\text{Int}(B_0) \neq \emptyset$. Hence, $\text{Int}(f^k(B)) \neq \emptyset$, contradicting Lemma 4.1. Therefore, $C_n(f)$ is not transitive.

The next lemma is a simple result that follows from the definition of continuum of type $\lambda$.

**Lemma 4.3.** Let $X$ be a continuum of type $\lambda$, where $X$ is irreducible between $p$ and $q$ and let $m : X \to [0, 1]$ be a monotone map such that $m(p) = 0, m(q) = 1$ and $\text{Int}(m^{-1}(t)) = \emptyset$ for each $t \in [0, 1]$. If $K$ is a subcontinuum of $X$ such that $m(K) = [a, b]$, where $a < b$, then $\text{Int}(K) \neq \emptyset$.

**Proof.** Since $m$ is monotone, both $m^{-1}([0, a])$ and $m^{-1}([b, 1])$ are proper subcontinua of $X$. Notice that, $m^{-1}([0, a]) \cap m^{-1}([b, 1]) = \emptyset$, $m^{-1}(a) \cap K \neq \emptyset$ and $m^{-1}(b) \cap K \neq \emptyset$. Thus, $m^{-1}([0, a]) \cup K \cup m^{-1}([b, 1])$ is a continuum such that $\{p, q\} \subset m^{-1}([0, a]) \cup K \cup m^{-1}([b, 1])$. Since $X$ is irreducible between $p$ and
Let and \( \text{Int} (m^{-1}(t)) = \emptyset \) for each \( t \in [0,1] \), and let \( n \in \mathbb{N} \). If \( A \subset C_n(X) \) is such that \( m(A) \) is a nondegenerate component, then \( \text{Int}(A) \neq \emptyset \).

**Corollary 4.4.** Let \( X \) be a continuum of type \( \lambda \), where \( X \) is irreducible between \( p \) and \( q \), let \( m : X \to [0,1] \) be a monotone map such that \( m(p) = 0 \), \( m(q) = 1 \) and \( \text{Int} (m^{-1}(t)) = \emptyset \) for each \( t \in [0,1] \), and let \( n \in \mathbb{N} \). If \( A \subset C_n(X) \) is such that \( m(A) \) is a nondegenerate component, then \( \text{Int}(A) \neq \emptyset \).

**Proof.** Let \( A \subset C_n(X) \) be such that \( m(A) \) has a nondegenerate component. Then \( A \) has a component \( A_0 \) such that \( m(A_0) \) is nondegenerate. Now, the corollary follows from Lemma 4.3.

**Theorem 4.5.** Let \( X \) be a continuum and let \( f : X \to X \) be a map. If \( X \) is a continuum of type \( \lambda \), then \( C_n(f) \) is not transitive, for any \( n \in \mathbb{N} \).

**Proof.** Let \( n \in \mathbb{N} \). Suppose that \( C_n(f) \) is transitive. Then, by Theorem 2.4 there exists \( B \subset C_n(X) \) such that \( \omega(B,C_n(f)) = C_n(X) \). We show that \( \text{Int}(f^k(B)) \neq \emptyset \), for some \( k \in \mathbb{N} \).

Let \( m : X \to [0,1] \) be a monotone map such that \( m(p) = 0 \), \( m(q) = 1 \) and \( \text{Int} (m^{-1}(t)) = \emptyset \) for each \( t \in [0,1] \), where \( X \) is irreducible between \( p \) and \( q \). Let \( 0 = t_0 < t_1 < \cdots < t_{n-1} < a < b < t_n = 1 \). Let \( U_i = m^{-1}((t_{i-1}, t_i)) \), for each \( i \in \{1, \ldots, n\} \). Notice that \( U_1, \ldots, U_n \) are nonempty, pairwise disjoint open subsets of \( X \). Let \( V = m^{-1}((t_{n-1}, a)) \) and \( W = m^{-1}((b, t_n)) \) be disjoint open subsets of \( U_n \). Observe that \( U = \{U_1, \ldots, U_n, V, W\} \) is a nonempty open subset of \( C_n(X) \). Since \( \omega(B,C_n(f)) = C_n(X) \), \( f^k(B) \subset U \) for some \( k \in \mathbb{N} \). Thus, \( f^k(B) \) has exactly \( n \) components. Let \( B_0 \) be the component of \( f^k(B) \) such that \( B_0 \subset U_n \). Since \( B_0 \cap V \neq \emptyset \), \( B_0 \cap W \neq \emptyset \) and \( V \cap W = \emptyset \), we have that \( [a, b] \subset m(B_0) \) is nondegenerate. Therefore, by Corollary 4.4 \( \text{Int}(f^k(B)) \neq \emptyset \), which contradicts Lemma 4.1. Therefore, \( C_n(f) \) is not transitive.

In the remainder of this paper, we will focus on maps which are defined on dendrites. A **cut point** of \( X \) is a point \( p \) such that \( X \setminus \{p\} \) is not connected. We write \( \text{Cut}(X) \) to represent the family of cut points of a dendrite \( X \).

**Proposition 4.6.** Let \( X \) be a dendrite, let \( n \in \mathbb{N} \) and let \( f : X \to X \) be a map. If \( p \in \text{Cut}(X) \) and \( B \subset C_n(X) \) is such that \( \omega(B,C_n(f)) = C_n(X) \), then \( p \in f^k(B) \), for some \( k \in \mathbb{N} \).

**Proof.** Let \( U \) and \( V \) be nonempty, disjoint and open subsets of \( X \) such that \( X \setminus \{p\} = U \cup V \). The sets \( U \cup \{p\} \) and \( V \cup \{p\} \) are subcontinua of \( X \), by [12, Theorem 4, p. 133]. Also, by [12, Theorem 5, p. 173], there exist subcontinua \( M \) and \( N \) of \( X \), such that \( \{p\} \subset M \subset U \) and \( \{p\} \subset N \subset V \). Hence, \( M \cup N \) is a continuum, \( (M \cup N) \cap U \neq \emptyset \), \( (M \cup N) \cap V \neq \emptyset \) and \( M \cup N \neq X \). It is not difficult to show that there are nonempty open subsets \( W_1, \ldots, W_{n-1} \) and \( W \) of \( X \) such that:
(1) $M \cup N \subset W$ and $X \setminus \overline{W} \neq \emptyset$;

(2) $\bigcup_{i=1}^{n} W_i \subset X \setminus W$;

(3) $W_i \cap W_j = \emptyset$ for $i \neq j$.

Let $\mathcal{U} = \langle W, U \cap W, V \cap W, W_1, \ldots, W_{n-1} \rangle$ be an open subset of $C_n(X)$. If $x_i \in W_i$, for each $i \in \{1, \ldots, n-1\}$, then $M \cup N \cup \{x_1, \ldots, x_{n-1}\} \in \mathcal{U}$. Thus, $\mathcal{U} \neq \emptyset$.

Since $\omega(B, C_n(f)) = C_n(X)$, $f^k(B) \in \mathcal{U}$, for some $k \in \mathbb{N}$. Thus, $f^k(B)$ has exactly $n$ components. Let $B_0$ be the component of $f^k(B)$ such that $B_0 \subset W$. Since $B_0 \cap (U \cap W) \neq \emptyset$, $B_0 \cap (V \cap W) \neq \emptyset$, we have that $p \in B_0$. Therefore, $p \in f^k(B)$ and our proof is complete.

If $a, b \in X$ and $X$ is a dendrite, then there exists a unique arc $\alpha$ joining $a$ and $b$ in $X$. We will denote that $\alpha$ by $ab$. The idea of the following proof is similar to [1 Theorem 6.2].

**Theorem 4.7.** Let $X$ be a continuum and let $f : X \to X$ be a map. If $X$ is a dendrite, then the induced map $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$.

**Proof.** Let $n \in \mathbb{N}$. Suppose that $C_n(f)$ is transitive. Then, by Theorem 2.4, there exists $B \in C_n(X)$ such that $\omega(B, C_n(f)) = C_n(X)$. The family $\text{Cut}(X)$ has uncountably infinitely many points, by [14 Theorem 10.8]. Let $p \in \text{Cut}(X)$. Then, there exists $k \in \mathbb{N}$ such that $p \in f^k(B)$, by Proposition 4.6. Thus, since $\omega(f^k(B), C_n(f)) = C_n(X)$, by Proposition 3.5, $\omega(p, f) = X$. Therefore, $f^l(p) \neq f^s(p)$ for each $l \neq s$. Since $p \in \text{Cut}(X)$ and $f(p) \neq p$, we have that there exists a component $W$ of $X \setminus \{p\}$ such that $f(p) \in W$. Observe that $L = W \cup \{p\}$ is a proper subcontinuum of $X$, by [12 Theorem 4, p. 133].

**Claim 4.8.** There exists $q_1 \in pf(p)$, $q_1 \neq p$ such that $q_1 \in \text{Cut}(X)$ and $q_1 \in pf(q_1) \cap f(pq_1)$.

Let $r : X \to pf(p)$ be the first point map defined in [14 Lemma 10.24]; i.e., $r(x) \in pf(p)$ is such that $r(x)$ is a point of any arc in $X$ from $x$ to any point of $pf(p)$. Thus, $(r \circ f|_{pf(p)}) : pf(p) \to pf(p)$ is a map and $C_1 = \{z \in pf(p) : (r \circ f|_{pf(p)})(z) = z\}$ is a nonempty closed subset of $X$. Let $q_1 \in C_1$ be the closest point to $p$ in $pf(p)$. Since $p \neq f(p) = r(f(p))$, $q_1 \neq p$.

Since $r(f(q_1)) = q_1$, it is clear that $q_1 \in pf(q_1) \cap f(q_1)f(p)$. Also, $f(q_1)f(p) \subset f(pq_1)$. Thus, $q_1 \in pf(q_1) \cap f(pq_1)$. We show that $q_1 \in \text{Cut}(X)$. Suppose that $q_1 \neq f(p)$. Hence, $q_1 \in pf(p) \setminus \{p, f(p)\}$ and $q_1 \in \text{Cut}(X)$, by [14 Theorem 10.7]. Similarly, assume that $q_1 = f(p)$. Since $\omega(p, f) = X$, $f(f(p)) \neq f(p)$ and $f(f(p)) \neq pf(p)$. Therefore, $q_1 \in pf(f(p)) \setminus \{p, f(f(p))\}$ and, by [14 Theorem 10.7], $q_1 \in \text{Cut}(X)$.

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Claim 4.9. The arc $pq_1 \subset f^m(B)$, for some $m \in \mathbb{N}$.

Since $p, q_1 \in \text{Cut}(X)$, there are $a, b \in X$ such that $pq_1 \subset ab \setminus \{a, b\}$ Theorem 10.7]. Let $U_1, \ldots, U_n, V_1$ and $V_2$ be nonempty, connected and open subsets of $X$ such that:

1. $U_1, \ldots, U_n$ are pairwise disjoint subsets of $X$.
2. $ab \subset U_1$.
3. $V_1 \cup V_2 \subset U_1 \setminus pq_1$, $a \in V_1$, $b \in V_2$ and $V_1 \cap V_2 = \emptyset$.

Since $X$ is a dendrite, it is not difficult to check that $U_1, \ldots, U_n, V_1$ and $V_2$ do indeed exist. Also, $\mathcal{U} = \{V_1, V_2, U_1, \ldots, U_n\}$ is nonempty open subset of $C_n(X)$. Since $\omega(B, C_n(f)) = C_n(X)$, $f^m(B) \in \mathcal{U}$ for some $m \in \mathbb{N}$. Thus, since $U_1, \ldots, U_n$ are nonempty, pairwise disjoint subsets of $X$, we have that there exists a component $B_0$ of $f^m(B)$ such that $B_0 \subset U_1$. Furthermore, $B_0 \cap V_i \neq \emptyset$ for each $i \in \{1, 2\}$. Therefore, since $X$ is hereditarily unicoherent and (3), $pq_1 \subset B_0$ and $pq_1 \subset f^m(B)$.

Observe that $q_1 \in f(pq_1)$, by Claim 4.8. Hence, $\{q_1, f(q_1)\} \subset f(pq_1) \subset f^{m+1}(B)$, by Claim 4.9. Since $f(pq_1)$ is connected, $q_1$ and $f(q_1)$ belong to the same component of $f^{m+1}(B)$. Therefore, $q_1 f(q_1) \subset f^{m+1}(B)$.

The proof of the following claim is similar to the proof of Claim 4.8, we have to use the cut point $q_1$ instead of $p$.

Claim 4.10. There exists $q_2 \in q_1 f(q_1)$, $q_2 \neq q_1$ such that $q_2 \in \text{Cut}(X)$ and $q_2 \in q_1 f(q_2) \cap f(q_1 q_2)$.

Notice that $q_2 f(q_2) \subset f^{m+2}(B)$, $q_1 f(q_1) \subset L$ and $q_2 f(q_2) \subset L$. Then we can inductively construct a sequence $(q_i)_{i=1}^\infty \subset \text{Cut}(X)$ such that $q_i f(q_i) \subset L \cap f^{m+i}(B)$, for each $i \in \mathbb{N}$. Let $\mathcal{L} = \langle L, X \rangle \subset C_n(X)$. Since $C_n(X) \setminus \mathcal{L} = \langle X \setminus L \rangle$ is open, $\mathcal{L}$ is a proper nonempty, closed subset of $C_n(X)$. Furthermore, $f^{m+i}(B) \in \mathcal{L}$ for each $i \in \mathbb{N}$. Thus, $\omega(f^{m}(B), C_n(f)) \subset \mathcal{L}$. Since $\omega(B, C_n(f)) = \omega(f^{m}(B), C_n(f))$, we contradict the fact that $\omega(B, C_n(f)) = C_n(X)$. Therefore, $C_n(f)$ is not transitive. $$\checkmark$$

References


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TRANSITIVITY OF THE INDUCED MAP $C_N(F)$

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