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Sectional-Anosov Flows in Higher Dimensions

Flujos seccionales Anosov en dimensiones superiores

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Abstract. A *sectional-Anosov flow* on a manifold is a C^1 vector field inwardly transverse to the boundary for which the maximal invariant is sectional hyperbolic [10]. We prove that every attractor of every vector field C^1 close to a transitive sectional-Anosov flow with singularities on a compact manifold has a singularity. This extends the three-dimensional result obtained in [9].

Key words and phrases. Transitive, Maximal invariant, Sectional-Anosov flow.

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Resumen. Un *flujo seccional-Anosov* sobre una variedad es un C^1 campo vectorial transversal a la frontera apuntando hacia el interior, para el cual su conjunto maximal invariante es un conjunto seccional hiperbólico [10]. Probamos que todo atractor de todo campo vectorial C^1 próximo a un flujo seccional-Anosov transitivo con singularidades sobre una variedad compacta tiene una singularidad. Este resultado extiende el resultado tres-dimensional obtenido en [9].

Palabras y frases clave. Transitivo, maximal invariante, flujo seccional-Anosov.

1. Introduction

The *sectional-Anosov flows* were introduced in [10] as a generalization of the *Anosov flows*. These also includes the *saddle-type hyperbolic attracting sets* and the *geometric and multidimensional Lorenz attractors* [1, 5, 7]. Some properties of these flows have been shown in the literature [2, 4]. For instance, [9] proved that every attractor of every vector field C^1 close to a transitive sectional-Anosov flow with singularities on a compact 3-manifold has a singularity. Moreover, [3] generalized this result from transitive to nonwandering

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ones. In this paper we further extend [9] but now to higher dimensions. More precisely, we prove that every attractor of every vector field C^1 close to a transitive sectional-Anosov flow with singularities of a compact manifold has a singularity. Let us state our result in a precise way.

Consider a compact Riemannian manifold M of dimension $n \geq 3$ (a *compact n-manifold* for short). We denote by ∂M the boundary of M. Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denote by X_t the flow of X, $t \in \mathbb{R}$.

The ω-limit set of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to \infty$.

Given $\Lambda \in M$ compact, we say that Λ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We also say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; *singular* if it contains a singularity and *attracting* if $\Lambda = \cap_{t>0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. An attractor is *nontrivial* if it is not a closed orbit.

Note that the set formed by a single singularity is a transitive set and in this case such transitive set is trivial.

The *maximal invariant* set of X is defined by $M(X) = \bigcap_{t\geq 0} X_t(M)$.

We denote by $m(L)$ the minimum norm of a linear operator L, i.e., $m(L) = \inf_{v\neq 0} \frac{\|Lv\|}{\|v\|}$.

Definition 1.1. A compact invariant set Λ is *partially hyperbolic* if there is a continuous invariant splitting

$$T_\Lambda M = E^s \oplus E^c$$

such that the following properties hold for some positive constants C, λ:

(1) E^s is *contracting*, i.e.,

$$\left\| DX_t(x) \mid_{E^s_x} \right\| \leq Ce^{-\lambda t},$$

for all $x \in \Lambda$ and $t > 0$.

(2) E^s *dominates* E^c, i.e.,

$$\frac{\left\| DX_t(x) \mid_{E^s_x} \right\|}{m\big(DX_t(x) \mid_{E^c_x}\big)} \leq Ce^{-\lambda t},$$

for all $x \in \Lambda$ and $t > 0$.

We say that the central subbundle E^c_x of Λ as above is *sectionally expanding* if for all two-dimensional subspace L_x of E^c_x

$$\dim\left(E^c_x\right) \geq 2 \quad \textit{and} \quad \left|J(DX_t(x)\,|_{L_x})\right| \geq C^{-1}e^{\lambda t}, \qquad \forall x \in \Lambda \quad \textit{and} \quad t > 0,$$

where $J(\bullet)$ is the jacobian.

Definition 1.2. A *sectional hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is sectionally-expanding.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

Definition 1.3. We say that X is a *sectional-Anosov flow* if $M(X)$ is a sectional hyperbolic set.

Our result is the following.

Theorem 1.4. *Let X be a transitive sectional-Anosov flow with singularities of a compact n-manifold. Then, every attractor of every vector field C^1 close to X has a singularity.*

The proof of this theorem follows closely that of [9]. More precisely, we assume by contradiction that there exists a sequence X^n of vector fields C^1 close to X each one exhibiting a non-singular attractor A^n. We then prove that A^n accumulates on a singularity of X and, consequently, for n large, we will prove that the corresponding attractor A^n does contain a singularity. This give us the desired contradiction. To prove such assertions we will extend some tools in [9] including the definitions of both Lorenz-like singularity and singular cross-section.

2. Lorenz-Like Singularities and Singular Cross-Sections in Higher Dimension

Let M be a compact Riemannian n-manifold, $n \geq 3$. Fix $X \in \mathcal{X}^1(M)$ inwardly transverse to ∂M. Denote by X_t the flow of X, $t \in \mathbb{R}$, and by $M(X)$ the maximal invariant of X.

Definition 2.1. A compact invariant set Λ of X is *hyperbolic* if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that

- E^X is the vector field direction over Λ.
- E^s is *contracting*, i.e., $\left\|DX_t(x)\,|_{E^s_x}\right\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- E^u is *expanding*, i.e., $\left\|DX_{-t}(x)\,|_{E^u_x}\right\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

A closed orbit or attractor is hyperbolic if it does as a compact invariant set.

It follows from the stable manifold theory [8] that if p belongs to a hyperbolic set Λ, then the sets

$$W^{ss}_X(p) = \big\{x : d\big(X_t(x), X_t(p)\big) \to 0, t \to \infty\big\} \quad \text{and}$$
$$W^{uu}_X(p) = \big\{x : d\big(X_t(x), X_t(p)\big) \to 0, t \to -\infty\big\}$$

are C^1 immersed submanifolds of M which are tangent at p to the subspaces E^s_p and E^u_p of T_pM respectively. Similarly, the sets

$$W^s_X(p) = \bigcup_{t\in\mathbb{R}} W^{ss}_X\big(X_t(p)\big) \quad \text{and}$$
$$W^u_X(p) = \bigcup_{t\in\mathbb{R}} W^{uu}_X\big(X_t(p)\big)$$

are also C^1 immersed submanifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ at p respectively. Moreover, for every $\epsilon > 0$ we have that

$$W^{ss}_X(p,\epsilon) = \big\{x : d\big(X_t(x), X_t(p)\big) \le \epsilon, \forall t \ge 0\big\} \quad \text{and}$$
$$W^{uu}_X(p,\epsilon) = \big\{x : d\big(X_t(x), X_t(p)\big) \le \epsilon, \forall t \le 0\big\}$$

are closed neighborhoods of p in $W^{ss}_X(p)$ and $W^{uu}_X(p)$ respectively.

There is also a stable manifold theorem in the case when X is sectional-Anosov. Indeed, denoting by $T_{M(X)}M = E^s_{M(X)} \oplus E^c_{M(X)}$ the corresponding sectional-hyperbolic splitting over $M(X)$ we have from [8] that the contracting subbundle $E^s_{M(X)}$ can be extended to a contracting subbundle E^s_U in M, where U is a neighborhood of $M(X)$. Moreover, such an extension is tangent to a continuous foliation denoted by W^{ss} (or W^{ss}_X to indicate dependence on X). By adding the flow direction to W^{ss} we obtain a continuous foliation W^s (or W^s_X) now tangent to $E^s_M \oplus E^X_M$. Unlike the Anosov case W^s may have singularities, all of which being the leaves $W^{ss}(\sigma)$ passing through the singularities σ of X. Note that W^s is transverse to ∂M because it contains the flow direction (which is transverse to ∂M by definition).

It turns out that every singularity σ of a sectional-Anosov flow X satisfies $W^{ss}_X(\sigma) \subset W^s_X(\sigma)$. Furthermore, there are two possibilities for such a singularity, namely, either $\dim\big(W^{ss}_X(\sigma)\big) = \dim\big(W^s_X(\sigma)\big)$ $\big($and so $W^{ss}_X(\sigma) = W^s_X(\sigma)\big)$ or $\dim\big(W^s_X(\sigma)\big) = \dim\big(W^{ss}_X(\sigma)\big) + 1$. In the later case we call it Lorenz-like according to the following definition.

Definition 2.2. We say that a singularity σ of a sectional-Anosov flow X is *Lorenz-like* if $\dim\big(W^s(\sigma)\big) = \dim\big(W^{ss}(\sigma)\big) + 1$.

Hereafter, we will denote $\dim\big(W^{ss}_X(\sigma)\big) = s$, $\dim\big(W^{u}_X(\sigma)\big) = u$ and therefore $\dim\big(W^{s}_X(\sigma)\big) = s+1$ by definition. Moreover $W^{ss}_X(\sigma)$ separates $W^{s}_{loc}(\sigma)$ in two connected components denoted by $W^{s,t}_{loc}(\sigma)$ and $W^{s,b}_{loc}(\sigma)$ respectively.

Next we define singular cross-section in the higher dimensional context. First, we will denote a cross-section by Σ and its boundary by $\partial\Sigma$. Also, the hypercube $I^k = \big[-1,1\big]^k$ will be submanifold of dimension k, with $k \in \mathbb{N}$.

Thus, we begin by considering $B^u[0,1] \approx I^u$ and $B^{ss}[0,1] \approx I^s$ where

$B^{ss}[0,1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^{ss}(\sigma))} = \mathbb{R}^s$ and

$B^{u}[0,1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^{u}(\sigma))} = \mathbb{R}^{n-s-1} = \mathbb{R}^u$.

Definition 2.3. A *singular cross-section* of a Lorenz-like singularity σ consists of a pair of submanifolds Σ^t, Σ^b, where Σ^t, Σ^b are cross-sections and

Σ^t is transversal to $W^{s,t}_{loc}(\sigma)$.

Σ^b is transversal to $W^{s,b}_{loc}(\sigma)$.

Note that every singular cross-section contains a pair singular submanifolds l^t, l^b defined as the intersection of the local stable manifold of σ with Σ^t, Σ^b respectively and additionally $\dim(l^*) = \dim\big(W^{ss}(\sigma)\big)$ $(* = t, b)$.

Thus, a singular cross-section Σ^* will be a *hypercube of dimension* $(n-1)$, i.e., diffeomorphic to $B^u[0,1] \times B^{ss}[0,1]$. Let $f : B^u[0,1] \times B^{ss}[0,1] \longrightarrow \Sigma^*$ be the diffeomorphism, such that

$$f\big(\{0\} \times B^{ss}[0,1]\big) = l^*$$

and $\{0\} = 0 \in \mathbb{R}^u$. Define

$$\partial\Sigma^* = \partial^h\Sigma^* \cup \partial^v\Sigma^*$$

by

$$\partial^h\Sigma^* = \big\{\text{union of the boundary submanifolds which are transverse to } l^*\big\}$$

and

$$\partial^v\Sigma^* = \big\{\text{union of the boundary submanifolds which are parallel to } l^*\big\}.$$

From this decomposition we obtain that

$$\partial^h\Sigma^* = \Big(I^u \times \big[\cup_{j=0}^{s-1}\big(I^j \times \{-1\} \times I^{s-j-1}\big)\big]\Big) \bigcup \Big(I^u \times \big[\cup_{j=0}^{s-1}\big(I^j \times \{1\} \times I^{s-j-1}\big)\big]\Big)$$

and

$$\partial^v\Sigma^* =$$
$$\Big(\big[\cup_{j=0}^{u-1}\big(I^j\times\{-1\}\times I^{u-j-1}\big)\big]\times I^s\Big)\bigcup\Big(\big[\cup_{j=0}^{u-1}\big(I^j\times\{1\}\times I^{u-j-1}\big)\big]\times I^s\Big),$$

where $I^0 \times I = I$.

Hereafter we denote $\Sigma^* = B^u[0,1] \times B^{ss}[0,1]$.

3. Sectional Hyperbolic Sets in Higher Dimension

In this section we prove a preliminary result for transitive sectional-Anosov flows. To begin with, we present the following useful property.

Lemma 3.1. *Let X be a sectional-Anosov flow, X a C^1 vector field in M. If Y is C^1 close to X, then every nonempty, compact, non singular, invariant set H of Y is hyperbolic* saddle-type *(i.e. $E^s \neq 0$ and $E^u \neq 0$).*

Proof. See [11]. The proof in [11] is made in dimension three, but the same proof yields the same conclusion in any dimension. ☑

Lemma 3.2. *Let X be a transitive sectional-Anosov flow C^1 in M. If $O \subset M(X)$ is a periodic orbit of X, then O is a hyperbolic saddle-type periodic orbit. In addition, if $p \in O$ then the set*

$$\left\{q \in W_X^{uu}(p) : M(X) = \omega_X(q)\right\}$$

is dense in $W_X^{uu}(p)$.

Proof. By Lemma 3.1, we have that O is hyperbolic and saddle-type. Let W be an open set in $W_X^{uu}(p)$. This set W exists since the point p belongs to the periodic orbit O which is hyperbolic. Define

$$B = \bigcup_{0\le t\le 1} X_t(W).$$

This set has dimension at least two, and so,

$$B' = \bigcup_{x\in B} W_X^{ss}(x)$$

contains an open set V with $B \cap V \neq \varnothing$.

Since $M(X)$ is the maximal invariant of X, $B \subset W_X^u(p)$ and $p \in O$, we obtain $B \cap V \subset M(X)$. Let $q \in M(X)$ such that $M(X) = \omega_X(q)$. Then, the forward orbit of q intersects V and so it intersects B' too. It follows from the definition of B' that the positive orbit of q is asymptotic to the forward orbit of some $q' \in B$. In particular, $M(X) = \omega_X(q) = \omega_X(q')$. This proves that $\left\{q \in W_X^{uu}(p) : M(X) = \omega_X(q)\right\}$ is dense in $M(X)$ as desired. ☑

The following theorem appears in [4].

Theorem 3.3. *Let X be a transitive sectional-Anosov flow C^1 for M. Then, every $\sigma \in Sing(X) \cap M(X)$ is Lorenz-like and satisfies*

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

Proof. We begin by proving two claims.

Claim 3.4. If $x \in \big(M(X) \setminus Sing(X)\big)$, then $X(x) \notin E^s_x$.

Proof. Suppose by contradiction that there is $x_0 \in \big(M(X) \setminus Sing(X)\big)$ such that $X(x_0) \in E^s_{x_0}$. Then, $X(x) \in E^s_x$ for every x in the orbit of x_0 since $E^s_{M(X)}$ is invariant. So, by continuity,

$$X(x) \in E^s_x \text{ for every } x \in \alpha(x_0). \tag{1}$$

It follows that $\omega(x)$ is a singularity for all $x \in \alpha(x_0)$. In particular, $\alpha(x_0)$ contains a singularity σ which is necessary hyperbolic of saddle-type.

Now we have two cases: $\alpha(x_0) = \{\sigma\}$ or not.

If $\alpha(x_0) = \{\sigma\}$ then $x_0 \in W^u(\sigma)$. For all $t \in \mathbb{R}$ define the unitary vector

$$v^t = \frac{DX_t(x_0)\big(X(x_0)\big)}{\|DX_t(x_0)\big(X(x_0)\big)\|}.$$

It follows that

$$v^t \in T_{X_t(x_0)}W^u(\sigma) \cap E^s_{X_t(x_0)}, \qquad \forall t \in \mathbb{R}.$$

Take a sequence $t_n \to \infty$ such that the sequence v^{-tn} converges to v^∞. Clearly v^∞ is an unitary vector and, since $X_{-t_n}(x_0) \to \sigma$ and E^s is continuous we obtain

$$v^\infty \in T_\sigma W^u(\sigma) \cap E^s_\sigma.$$

Therefore v^∞ is an unitary vector which is simultaneously expanded and contracted by $DX_t(\sigma)$, a contradiction. This contradiction shows the result in the first case.

For the second case, we assume $\alpha(x_0) \neq \{\sigma\}$. Then, $\big(W^u(\sigma) \setminus \{\sigma\}\big) \cap \alpha(x_0) \neq \emptyset$. Pick $x_1 \in \big(W^u(\sigma) \setminus \{\sigma\}\big) \cap \alpha(x_0)$. It follows from (1) that $X(x_1) \in E^s_{x_1}$ and then we get a contradiction as in the first case replacing x_0 by x_1. This contradiction proves the claim. ☑

Claim 3.5. If $\sigma \in Sing(X)$, then $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$.

Proof. Take $x \in W^{ss}(\sigma) \setminus \{\sigma\}$. Then, $E^s_x = T_x W^{ss}(\sigma)$. Moreover, since $W^{ss}(\sigma)$ is an invariant, we obtain $X(x) \in T_x W^{ss}(\sigma)$. We conclude that $X(x) \in E^s_x$ for all $x \in W^{ss}(\sigma)$ and now Claim 3.4 applies. ☑

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This theorem implies the following two useful properties.

Proposition 3.6. *Let X be a transitive sectional-Anosov flow C^1 of M. Let σ be a singularity of X in $M(X)$ (so σ is Lorenz-like by Theorem 3.3). Then, there is a singular-cross section Σ^t, Σ^b of σ in M such that.*

$$\big(M(Y)\big) \cap \big(\partial^h \Sigma^t \cup \partial^h \Sigma^b\big) = \varnothing,$$

for every C^r vector field Y close to X.

Proof. See [9]. ☑

Let σ be a Lorenz-like singularity of a C^1 vector field X in $\mathcal{X}^1(M)$, and Σ^t, Σ^b be a singular cross-section of σ. Thus for σ we recall that,

$$\begin{aligned} \dim\big(W^{ss}_X(\sigma)\big) &= s, \quad \text{then} \\ \dim\big(W^{s}_X(\sigma)\big) &= s+1, \quad \text{and} \quad \dim\big(W^{u}_X(\sigma)\big) = n-s-1, \\ \dim(\Sigma^*) &= s + (n-s-1) = n-1. \end{aligned} \tag{2}$$

Since $\Sigma^* = B^u[0,1] \times B^{ss}[0,1]$, we will set up a family of singular cross-sections as follows: Given $0 < \Delta \leq 1$ small, we define $\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]$, such that

$$l^* \subset \Sigma^{*,\Delta} \subset \Sigma^*, \quad \text{i.e.}$$

$$\big(l^* = \{0\} \times B^{ss}[0,1]\big) \subset \big(\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]\big) \subset \big(\Sigma^* = B^u[0,1] \times B^{ss}[0,1]\big),$$

where we fix a coordinate system (x^*, y^*) in Σ^* $(* = t, b)$. We will assume that $\Sigma^* = \Sigma^{*,1}$.

We will use this notation throughout of next lemma and the Theorem 1.4 proof.

Lemma 3.7. *Let X be a transitive sectional-Anosov flow C^1 of M. Let σ be a singularity of X in $M(X)$. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Let O_n be a periodic orbit of Y^n such that the sequence $\{O_n : n \in \mathbb{N}\}$ accumulates on σ. If $0 < \Delta \leq 1$ and Σ^t, Σ^b is a singular cross-section of σ, then there is n such that either*

$$O_n \cap \operatorname{int}\big(\Sigma^{t,\Delta}\big) \neq \varnothing$$

or

$$O_n \cap \operatorname{int}\left(\Sigma^{b,\Delta}\right) \neq \varnothing.$$

Proof. Since O_n accumulates on $\sigma \in M(X)$ and $M(X)$ is maximal invariant, we have that $O_n \subset M(X)$ for all n large (recall $Y^n \to X$ as $n \to \infty$). Let us fix a fundamental domain D_ϵ of the vector field's flow X_t restricted to the local stable manifold $W^s_{loc}(\sigma)$ ([12]) for $\epsilon > 0$ as follows (See Figure 1):

$$D_\epsilon = S_\epsilon \cup S_{-\epsilon} \cup C_\epsilon,$$

where

$$S_\epsilon = \left\{x \in \mathbb{R}^{s+1} : \Sigma_{i=1}^s x_i^2 + (x_{s+1} - \epsilon)^2 = 1,\ \wedge,\ x_{s+1} \geq \epsilon\right\},$$
$$S_{-\epsilon} = \left\{x \in \mathbb{R}^{s+1} : \Sigma_{i=1}^s x_i^2 + (x_{s+1} - \epsilon)^2 = 1,\ \wedge,\ x_{s+1} \leq -\epsilon\right\},$$
$$C_\epsilon = \left\{x \in \mathbb{R}^{s+1} : \Sigma_{i=1}^s x_i^2 = 1,\ \wedge,\ x_{s+1} \in [-\epsilon, \epsilon]\right\}.$$

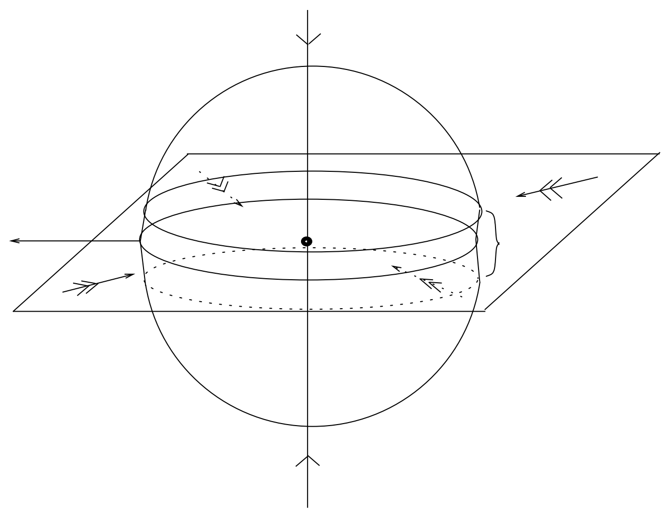

FIGURE 1. The fundamental domain.

As $W^s_{loc}(\sigma)$ is $(s+1)$-dimensional and D_ϵ is homeomorphic to the sphere (s)-dimensional, by construction D_ϵ intersects $W^{ss}_X(\sigma)$ in $C_\epsilon|_{x_{s+1}=0}$ that is a sphere $(s-1)$-dimensional. Note that the orbits of all point in $C_\epsilon|_{x_{s+1}=0}$ together with σ yields $W^{ss}_X(\sigma)$. In particular, $C_\epsilon|_{x_{s+1}=0} \notin M(X)$ by Theorem 3.3. Also note that for all ϵ, D_ϵ is a fundamental domain.

Let $\widetilde{D_\epsilon}$ be a cross section of X such that $W^s_{loc}(\sigma) \cap \widetilde{D_\epsilon} = D_\epsilon$. It follows that $\widetilde{D_\epsilon}$ is a $(n-1)$-cylinder, and so we can consider a system coordinated (x, s) with $x \in D_\epsilon$ and $s \in I^u$. Thus, by using this system coordinate we can construct a family of singular cross-sections Σ^t_δ, Σ^b_δ (for all $\delta \in [-\epsilon, \epsilon]$) by setting

$$\Sigma_\delta^t = \{(x,s) \in \widetilde{D_\epsilon} : x \in S_\delta, s \in I^u\},$$
$$\Sigma_\delta^b = \{(x,s) \in \widetilde{D_\epsilon} : x \in S_{-\delta}, s \in I^u\}.$$

Due to the smooth variation of $W_Y^{ss}(\sigma(Y))$ with respect to Y close to X we can assume that $\sigma(Y) = \sigma$ and that $W_{loc,Y}^{ss}(\sigma(Y)) = W_{loc}^{ss}(\sigma)$ for every Y close to X. By choosing D_ϵ so close to σ we can further assume that $\widetilde{D_\epsilon}$ is a cross-section of Y, for every Y close to X. We claim that there is $\delta > 0$ such that the conclusion of the lemma holds for $\Sigma^t = \Sigma_\delta^t$ and $\Sigma^b = \Sigma_\delta^b$. Indeed, we first note that under the cylindrical coordinate system (x,s) one has $\Sigma^{*,\Delta} = \Sigma_\Delta^*$ for all $0 < \Delta \leq \delta$ $(* = t, b)$. Otherwise, if the conclusion of the claim fails, it implies that O_n intersects $\widetilde{D_\epsilon} \setminus (\Sigma_\Delta^t \cup \Sigma_\Delta^b)$ for all $\Delta > 0$ small. Further, we would find a sequence of periodic points such that $p_n \in O_n$ (for all n large) and $p_n = (x_n, s_n)$ with $x_n \in C_\Delta$ and $s_n \to 0$ as $n \to \infty$. As Δ is arbitrary and $s_n \to 0$, we conclude that p_n converges to a point in $C_\Delta|_{x_{s+1}=0}$ by passing to a subsequence if necessary, since if $s_n \to 0$, it implies that the intersection tends to (s)-dimensional sphere D_ϵ.

As $O_n \subset M(Y^n)$, $Y^n \to X$ and $M(Y^n)$ is arbitrarily close to $M(X)$ (indeed, in the C^1-topology, these sets are at least to a distance ϵ) for all n $(n \in \mathbb{N})$, by using the above arguments, there exists a point $z \in (C_\epsilon|_{x_{s+1}=0})$ such that $z \in M(X)$. This contradicts Theorem 3.3 and the proof follows. ☑

4. Proof of Theorem 1.4

We prove the theorem by contradiction. Let X be a transitive sectional-Anosov flow C^1 of M. Then, we suppose that there exists a sequence $X^n \xrightarrow{C^1} X$ such that every X^n exhibits a non-singular attractor $A^n \in M(X^n)$ arbitrarily close to $M(X)$ and since A^n also is arbitrarily close to $M(X)$, we can assume that A^n belongs to $M(X)$ for all n. It follows from the definition of attractor that each A^n is compact, invariant and nonempty. As A^n is non-singular by hypothesis, then the Lemma 3.1 and the Lemma 3.2 imply the following:

A^n is a hyperbolic attractor of type saddle of X^n for all n, and since A^n is non-singular for all n, obviously A^n is not a singularity of X^n for all n. (3)

We denote by $Sing(X)$ the set of singularities of X, $Cl(A)$ the closure of A, $A \subset M$. Moreover, given $\delta > 0$ and $A \subset M$, we define $B_\delta(A) = \{x \in M : d(x,A) < \delta\}$ where $d(\bullet, \bullet)$ is the metric on M.

Now, let us consider the following lemma that, as in [9], is useful for the higher dimension case. The lemma gives one description on behavior of the attractors.

Lemma 4.1. *The sequence of attractors A^n accumulate on $Sing(X)$, i.e.*

$$Sing(X)\bigcap Cl\left(\bigcup_{n\in\mathbb{N}} A^n\right)\neq\varnothing.$$

Proof. We prove the lemma by contradiction. With that purpose we suppose that there is $\delta>0$, such that

$$B_\delta\big(Sing(X)\big)\bigcap\left(\bigcup_{n\in\mathbb{N}} A^n\right)=\varnothing. \tag{4}$$

In the same way as in [9], we define

$$H=\bigcap_{t\in\mathbb{R}} X_t\big(M\setminus B_{\delta/2}(Sing(X))\big).$$

Note that by definition H is a invariant set and $Sing(X)\cap H=\varnothing$. Additionally H is a compact set as it is Λ and therefore H is a nonempty compact set (see [9]). By using the Lemma 3.1 we conclude that H is hyperbolic set. So, we denote by $E^s\oplus E^X\oplus E^u$ the corresponding hyperbolic splitting (see Definition 2.1).

By the stability of hyperbolic sets we can fix a neighborhood W of H and $\epsilon>0$ such that if Y is a vector field C^r close to X and H_Y is a compact invariant set of Y in W then:

H_Y is hyperbolic and its hyperbolic splitting $E^{s,Y}\oplus E^Y\oplus E^{u,Y}$. $\dim(E^u)=\dim(E^{u,Y})$, $\dim(E^s)=\dim(E^{s,Y})$. The manifolds $W^{uu}_Y(x,\epsilon)$, $x\in H_Y$, have uniform size ϵ. (5)

As $X^n\to X$, we have that:

$\bigcap_{t\in\mathbb{R}} X^n_t\big(M\setminus B_{\delta/2}(Sing(X)\big)\subset W$, for all n large;
$A^n\subset M\setminus B_{\delta/2}\big(Sing(X)\big)$ for all n, and $A^n\subset W$ for all n large;
If $x^n\in A^n$ so that x^n converges to some $x\in M$, then $x\in H$;
If $w\in W^{uu}_{X^n}(x^n,\epsilon)$, the tangent vectors of $W^{uu}_{X^n}(x^n,\epsilon)$ in this point are in E^{u,X^n}_w; (6)
As $x^n\to x$, $W^{uu}_{X^n}(x^n,\epsilon)\to W^{uu}_X(x,\epsilon)$ in the sense of C^1 submanifolds [13] and $\angle(E^{u,X^n},E^u)\longrightarrow 0$, if $n\to\infty$ [9].

Thus, we fix an open set $U\subset W^{uu}_X(x,\epsilon)$ containing the point x.

By (3), it follows that the periodic orbits of X^n in A^n are dense in A^n (Anosov closing Lemma). Particularly, we can assume that each x^n is a periodic point of A^n. As $M(X)\cap Sing(X)\neq\varnothing$ and $M(X)$ is a transitive set, it follows from Lemma 3.2 that there exists $q\in U$, $0<\delta_1<\delta_2<\frac{\delta}{2}$ and $T>0$ such that $X_T(q)\in B_{\delta_1}\big(Sing(X)\big)$.

By [12, Tubular Flow Box Theorem], there is an open set V_q containing q such that $X_T(V_q) \subset B_{\delta_1}\big(Sing(X)\big)$ and as $X^n \to X$ it follows that

$$X^n_T(V_q) \subset B_{\delta_2}\big(Sing(X)\big) \tag{7}$$

for all n large (see Figure 2).

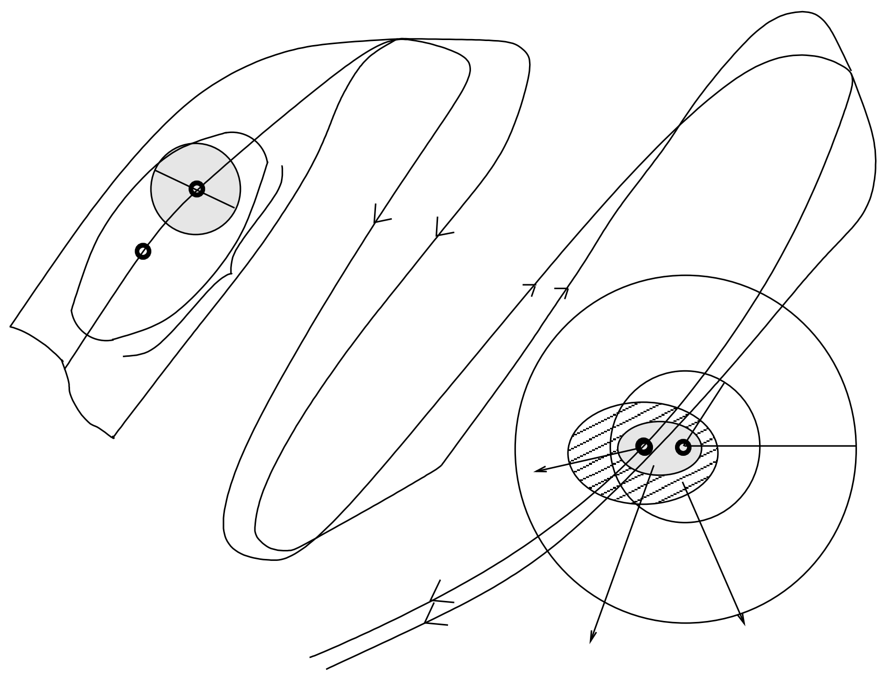

FIGURE 2. Tubular Flow Box Theorem for $X_T(V_q)$.

In addition, $W^{uu}_{X^n}(x^n,\epsilon) \cap V_q \neq \varnothing$ for n large enough, since $W^{uu}_{X^n}(x^n,\epsilon) \to W^{uu}_X(x,\epsilon)$ and $q \in U \subset W^{uu}_X(x,\epsilon)$. Applying (7) to X^n for n large we have

$$X^n_T\big(W^{uu}_{X^n}(x^n,\epsilon)\big) \cap B_{\delta_2}\big(Sing(X)\big) \neq \varnothing.$$

In particular $W^{uu}_{X^n}(x^n,\epsilon) \subset W^u_{X^n}(x^n)$. Then the invariance of $W^u_{X^n}(x^n)$ implies

$$W^u_{X^n}(x^n) \cap B_{\delta/2}\big(Sing(X)\big) \neq \varnothing.$$

Observe that $W^u_{X^n}(x^n) \subset A^n$ since $x^n \in A^n$ and A^n is an attractor. We conclude that

$$A^n \cap B_\delta\big(Sing(X)\big) \neq \varnothing.$$

This contradicts (4) and the proof follows. ☑

Proof of Theorem 1.4. By Lemma 4.1 there exists $\sigma \in M(X)$ such that

$$\sigma \in Sing(X) \bigcap Cl\left(\bigcup_{n\in\mathbb{N}} A^n\right).$$

By Theorem 3.3 we have that σ is Lorenz-like and satisfies

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

By Proposition 3.6, we can choose Σ^t, Σ^b, singular-cross section for σ and $M(X)$ such that

$$M(X) \cap \left(\partial^h \Sigma^t \cup \partial^h \Sigma^b\right) = \varnothing.$$

As $X^n \to X$ we have that Σ^t, Σ^b is singular-cross section of X^n too, thus we can assume that $\sigma(X^n) = \sigma$ and $l^t \cup l^b \subset W^s_{X^n}(\sigma)$ for all n (Implicit function theorem).

We have that the splitting $E^s \oplus E^c$ persists by small perturbations of X [8]. The dominance condition (Definition 1.1-(2)) together with [6, Proposition 2.2] imply that for $* = t, b$ one has

$$T_x\Sigma^* \cap \left(E^s_x \oplus E^X_x\right) = T_x l^*,$$

for all $x \in l^*$.

Denote by $\angle(E, F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle\big(T_x\Sigma^* \cap E^c_x, T_x l^*\big) > \rho,$$

for all $x \in l^*$ $(* = t, b)$. In this way, since $E^{c,n} \to E^c$ as $n \to \infty$ we have for n large enough that

$$\angle\big(T_x\Sigma^* \cap E^{c,n}_x, T_x l^*\big) > \frac{\rho}{2}, \tag{8}$$

for all $x \in l^*$ $(* = t, b)$.

As in the previous section we fix a coordinate system $(x, y) = (x^*, y^*)$ in Σ^* such that

$$\Sigma^* = B^u[0, 1] \times B^{ss}[0, 1], \qquad l^* = \{0\} \times B^{ss}[0, 1]$$

with respect to (x, y). Also, given $\Delta > 0$ we define $\Sigma^{*,\Delta} = B^u[0, \Delta] \times B^{ss}[0, 1]$.

Hereafter $\Pi^* : \Sigma^* \to B^u[0, 1]$ will be the projection such that $\Pi^*(x, y) = x$. We will denote the line field in Σ^{*,Δ_0} by F^n, where

$$F^n_x = T_x\Sigma^* \cap E^{c,n}_x, \qquad x \in \Sigma^{*,\Delta_0}.$$

***Remark* 4.2.** The continuity of $E^{c,n}$ and (8) imply that there is $\Delta_0 > 0$ such that for every n large the line F^n is *transverse to* Π^*. By this we mean that $F^n(z)$ is *not tangent to the curves* $(\Pi^*)^{-1}(c)$, *for every* $c \in B^u[0, \Delta_0]$.

Now recall that A^n is a hyperbolic attractor of type saddle of X^n for all n (see (3)) and that the periodic orbits of X^n in A^n are dense in A^n [13]. As $\sigma \in Cl\big(\cup_{n\in\mathbb{N}} A^n\big)$, we can find a sequence of periodic orbits $(O_n)_{n\in\mathbb{N}}$ such that $O_n \in A^n$ and accumulating on σ. It follows from Lemma 3.7 applied to $Y^n = X^n$ that there exists $n_0 \in \mathbb{N}$ such that either

$$O_{n_0} \cap \operatorname{int}\left(\Sigma^{t,\Delta_0}\right) \neq \varnothing \qquad \text{or} \qquad O_{n_0} \cap \operatorname{int}\left(\Sigma^{b,\Delta_0}\right) \neq \varnothing.$$

As $O_{n_0} \subset A_{n_0}$ we conclude that either

$$A^{n_0} \cap \operatorname{int}\left(\Sigma^{t,\Delta_0}\right) \neq \varnothing \qquad \text{or} \qquad A^{n_0} \cap \operatorname{int}\left(\Sigma^{b,\Delta_0}\right) \neq \varnothing.$$

We shall assume that $A^{n_0} \cap \operatorname{int}\left(\Sigma^{t,\Delta_0}\right) \neq \varnothing$ (Analogous proof for the case $* = b$). Note that $\partial^h \Sigma^{t,\Delta_0} \subset \partial^h \Sigma^t$ by definition. Then, by Proposition 3.6 one has

$$A \cap \partial^h \Sigma^{t,\Delta_0} = \varnothing.$$

As A^{n_0} and Σ^{t,Δ_0} are compact non-empty sets, it follows that $A^{n_0} \cap \Sigma^{t,\Delta_0}$ is a compact nonempty subset of Σ^{t,Δ_0}, and thus there exists $p \in \Sigma^{t,\Delta_0} \cap A^{n_0}$ such that

$$\operatorname{dist}\left(\Pi^t\big(\Sigma^{t,\Delta_0} \cap A^{n_0}\big), 0\right) = \operatorname{dist}\left(\Pi^t(p), 0\right),$$

where dist denotes the distance in $B^u[0,\Delta_0]$. Note that $\operatorname{dist}\left(\Pi^t(p), 0\right)$ is the minimum distance of $\Pi^t\big(\Sigma^{t,\Delta_0} \cap A^{n_0}\big)$ to 0 in $B^u[0,\Delta_0]$.

As $p \in A^{n_0}$, we have that $W^u_{X^{n_0}}(p)$ is a well defined submanifold, since that A^{n_0} is hyperbolic set (see (3)), and $\dim(E^c) = \dim(E^{c,n_0})$ (see (5)).

By domination Definition 1.1-(2), $T_z\big(W^u_{X^{n_0}}(p)\big) = E^{c,n_0}_z$ for every $z \in W^u_{X^{n_0}}(p)$ and hence $\dim\big(W^u_{X^{n_0}}(p)\big) = (n - s - 1)$ (2). Next, we can ensure that

$$T_z\big(W^u_{X^{n_0}}(p)\big) \cap T_z\Sigma^{t,\Delta_0} = E^{c,n_0}_z \cap T_z\Sigma^{t,\Delta_0} = F^{n_0}_z$$

for every $z \in W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$.

Note that the last equality shows that $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$ is transversal, and therefore there exists some compact submanifold inside of $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$. We denote this compact submanifold by K^{n_0}. Thus by construction $p \in K^{n_0}$ (see (3)) and K^{n_0} is tangent to F^{n_0}, since $K^{n_0} \subset W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$.

***Remark* 4.3.** By construction we have that $\dim\big(B^u[0,\Delta_0]\big) = (n - s - 1)$, since $\dim(E^{c,n_0}) = \dim\big(W^u_{X^{n_0}}(p)\big) = (n - s - 1)$.

We have that $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$ is a submanifold of M, since $W^u_{X^{n_0}}(p) \cap \Sigma^{t,\Delta_0}$ is transversal and nonempty and $W^u_{X^{n_0}}(p)$, Σ^{t,Δ_0} are submanifolds of M. Note that $\dim\big(W^u_{X^{n_0}}(p)\big) + \dim(\Sigma^{t,\Delta_0}) \geq n$.

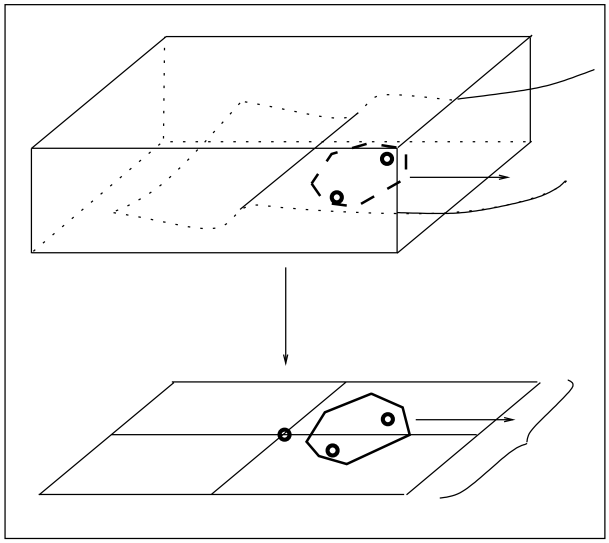

FIGURE 3. The projection $\Pi^t(K^{n_0}) = K_1^{n_0}$.

Since F^{n_0} is *transverse* to Π^t, one has that K^{n_0} is *transverse* to Π^t (i.e. K^{n_0} is transverse to the curves $(\Pi^t)^{-1}(c)$, for every $c \in B^u[0,\Delta_0]$). Let us denote the image of K^{n_1} by the projection Π^t in $B^u[0,\Delta_0]$ by $K_1^{n_1}$, i.e., $\Pi^t(K^{n_1}) = K_1^{n_1}$. Note that $K_1^{n_1} \subset B^u[0,\Delta_0]$ and $\Pi^t(p) \in \operatorname{int}(K_1^{n_1})$ (See Figure 3).

As $\dim(K_1^{n_0}) = \dim\big(B^u[0,\Delta_0]\big)$ (By Remark 4.3), there exists $z_0 \in K^{n_0}$ such that

$$\operatorname{dist}\big(\Pi^t(z_0),0\big) < \operatorname{dist}\big(\Pi^t(p),0\big).$$

It follows from the property of attractor that $W^{uu}_{X^{n_0}}(p,\epsilon) \subset W^u_{X^{n_0}}(p) \subset A_{n_0}$. Thus, $K^{n_0} \subset \Sigma^{t,\Delta_0} \cap A^{n_0}$ and $p \in A^{n_0}$.

Since, by Proposition 3.6 $A^{n_0} \cap \ \partial^h\Sigma^{t,\Delta_0} = \varnothing$ and, by Remark 4.3 $\dim\big(K_1^{n_0}\big) = \dim\big(B^u[0,\Delta_0]\big)$, we conclude that

$$\operatorname{dist}\big(\Pi^t\big(\Sigma^{t,\Delta_0} \cap A^{n_0}\big),0\big) = 0.$$

Given that A^{n_0} is closed, this last equality implies

$$A^{n_0} \cap l^t \neq \varnothing.$$

Since $l^t \subset W^s_{X^{n_0}}(\sigma)$ and A^{n_0} is closed invariant set for X^{n_0} we conclude that $\sigma \in A^{n_0}$. We have proved that A^{n_0} contains a singularity of X^{n_0}. But A^{n_0} is a hyperbolic attractor of X^{n_0} by the Property (3) and this leads to $A^{n_0} = \{\sigma\}$. Finally, using the Property (3) we obtain a contradiction and the proof follows. ☑

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