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Grüss Type Inequalities for Complex Functions Defined on Unit Circle with Applications for Unitary Operators in Hilbert Spaces

Desigualdades de tipo Grüss para funciones complejas definidas sobre el círculo unitario con aplicaciones para operadores en espacios Hilbert

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Abstract. Some Grüss type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of integrators are given. Natural applications for functions of unitary operators in Hilbert spaces are provided.

Key words and phrases. Grüss type inequalities, Riemann-Stieltjes integral inequalities, Unitary operators in Hilbert spaces, Spectral theory, Quadrature rules.

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Resumen. Se proporcionan algunas desigualdades tipo Grüss para la integral de Riemann-Stieltjes de integrandos de valores continuos complejos definidos sobre el circulo unitario complejo $\mathcal{C}(0,1)$ y varias subclases de integradores son dados. Aplicaciones naturales para funciones de operadores unitarios en espacios de Hilbert son proporcionadas.

Palabras y frases clave. Desigualdades de tipo Grüss, desigualdades integrales de Riemann-Stieltjes, Operadores unitarios en espacios de Hilbert, teoria espectral, reglas de cuadratura.

1. Introduction

In [4], in order to extend the Grüss inequality to *Riemann-Stieltjes integral*, the author introduced the following *Čebyšev functional*:

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)du(t) \times \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t), \quad (1)$$

where f, g are *continuous* on $[a,b]$ and u is of *bounded variation* on $[a,b]$ with $u(b) \neq u(a)$.

The following result that provides sharp bounds for the Čebyšev functional defined above was obtained in [4].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{C}$ be continuous functions on $[a,b]$ and $u : [a,b] \to \mathbb{C}$ with $u(a) \neq u(b)$. Assume also that there exists the real constants γ, Γ such that*

$$\gamma \leq f(t) \leq \Gamma \quad \text{for each} \quad t \in [a,b]. \quad (2)$$

a) If u is of bounded variation on $[a,b]$, then we have the inequality:

$$\left|T(f,g;u)\right| \leq \frac{1}{2} \times \frac{\Gamma-\gamma}{|u(b)-u(a)|}\left\|g - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u), \quad (3)$$

where $\bigvee_a^b(u)$ denotes the total variation of u in $[a,b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller quantity.

b) If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then one has the inequality:

$$\left|T(f,g;u)\right| \leq \frac{1}{2} \times \frac{\Gamma-\gamma}{u(b)-u(a)}\int_a^b \left|g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t). \quad (4)$$

The constant $\frac{1}{2}$ is sharp.

c) Assume that f, g are Riemann integrable functions on $[a,b]$ and f satisfies the condition (2). If $u : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant L, then we have the inequality

$$\left|T(f,g;u)\right| \leq \frac{1}{2} \times \frac{L(\Gamma-\gamma)}{|u(b)-u(a)|}\int_a^b \left|g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt. \quad (5)$$

The constant $\frac{1}{2}$ *is best possible in* (5).

For some recent inequalities for Riemann-Stieltjes integral see [1]- [2], [3]-[5] and [8].

For continuous functions $f, g : \mathcal{C}(0,1) \to \mathbb{C}$, where $\mathcal{C}(0,1)$ is the unit circle from $\mathbb{C}$ centered in 0 and $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$ with $u(a) \neq u(b)$, we can define the following functional as well:

$$S_{\mathcal{C}}\big(f,g;u,a,b\big) := \frac{1}{u(b)-u(a)} \int_a^b f\big(e^{it}\big) g\big(e^{it}\big)\, du(t) - \frac{1}{u(b)-u(a)} \int_a^b f\big(e^{it}\big)\, du(t) \frac{1}{u(b)-u(a)} \int_a^b g\big(e^{it}\big)\, du(t). \quad (6)$$

In this paper we establish some bounds for the magnitude of $S_{\mathcal{C}}\big(f,g;u,a,b\big)$ when the *integrands* $f, g : \mathcal{C}(0,1) \to \mathbb{C}$ satisfy some *Hölder's type conditions* on the circle $\mathcal{C}(0,1)$ while the *integrator* u is of bounded variation.

It is also shown that this functional can be naturally connected with continuous functions of unitary operators on Hilbert spaces to obtain some Grüss type inequalities for two functions of such operators.

We recall here some basic facts on unitary operators and spectral families that will be used in the sequel.

We say that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [7, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \le E_\mu$ for $0 \le \lambda \le \mu \le 2\pi$;

b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);

c) $E_{\lambda+0} = E_\lambda$ for $0 \le \lambda < 2\pi$;

d) $U = \int_0^{2\pi} e^{i\lambda}\, dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0,2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\big(e^{i\lambda}\big)\, dE_\lambda \quad (7)$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\left\langle f(U)x, y\right\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\left\langle E_\lambda x, y\right\rangle \tag{8}$$

and

$$\left\|f(U)x\right\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\left\|E_\lambda x\right\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\left\langle E_\lambda x, x\right\rangle, \tag{9}$$

for any $x, y \in H$.

Examples of such functions of unitary operators are

$$\exp(U) = \int_0^{2\pi} \exp\left(e^{i\lambda}\right) dE_\lambda$$

and

$$U^n = \int_0^{2\pi} e^{in\lambda} \, dE_\lambda$$

for n an integer.

We can also define the *trigonometric functions* for a unitary operator U by:

$$\sin(U) = \int_0^{2\pi} \sin\left(e^{i\lambda}\right) dE_\lambda \qquad \text{and} \qquad \cos(U) = \int_0^{2\pi} \cos\left(e^{i\lambda}\right) dE_\lambda$$

and the *hyperbolic functions* by

$$\sinh(U) = \int_0^{2\pi} \sinh\left(e^{i\lambda}\right) dE_\lambda \qquad \text{and} \qquad \cosh(U) = \int_0^{2\pi} \cosh\left(e^{i\lambda}\right) dE_\lambda$$

where

$$\sinh(z) := \frac{1}{2}\left[\exp z - \exp(-z)\right] \quad \text{and} \quad \cosh(z) := \frac{1}{2}\left[\exp z + \exp(-z)\right], \quad z \in \mathbb{C}.$$

2. Inequalities for Riemann-Stieltjes Integral

We say that the complex function $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies an *H-r-Hölder's type condition* on the circle $\mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given, if

$$\left|f(z) - f(w)\right| \le H|z - w|^r \tag{10}$$

for any $w, z \in \mathcal{C}(0,1)$. If $r = 1$ and $L = H$ then we call it of *L-Lipschitz type.*

Consider the power function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = z^m$ where m is a nonzero integer. Then, obviously, for any z, w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality

$$\left|f(z) - f(w)\right| \le |m||z - w|$$

which shows that f is Lipschitzian with constant $L = |m|$ on the circle $\mathcal{C}(0,1)$.

For $a \neq \pm 1, 0$ real numbers, consider the function $f : \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. Observe that

$$\left|f_a(z) - f_a(w)\right| = \frac{|a||z-w|}{|1-az||1-aw|} \tag{11}$$

for any $z, w \in \mathcal{C}(0,1)$. If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$\begin{aligned} |1-az|^2 &= 1 - 2a\operatorname{Re}(\overline{z}) + a^2|z|^2 = 1 - 2a\cos t + a^2 \\ &\geq 1 - 2|a| + a^2 = (1-|a|)^2. \end{aligned}$$

Therefore,

$$\frac{1}{|1-az|} \leq \frac{1}{\left|1-|a|\right|} \qquad \text{and} \qquad \frac{1}{|1-aw|} \leq \frac{1}{\left|1-|a|\right|} \tag{12}$$

for any $z, w \in \mathcal{C}(0,1)$.

Utilizing (11) and (12) we deduce

$$\left|f_a(z) - f_a(w)\right| \leq \frac{|a|}{\left(1-|a|\right)^2}|z-w| \tag{13}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

For other examples of Lipschitzian functions that can be constructed for power series on Banach algebras, see [6].

The following result holds:

Theorem 2.1. *Assume that* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is of* H-r-Hölder's *type and* $g : \mathcal{C}(0,1) \to \mathbb{C}$ *is of* K-q-*Hölder's type. If* $u : [a,b] \subseteq [0, 2\pi] \to \mathbb{C}$ *is a function of bounded variation with* $u(a) \neq u(b)$, *then*

$$\left|S_{\mathcal{C}}(f,g;u,a,b)\right| \leq HKB_{r,q}(a,b)\left[\frac{1}{|u(b)-u(a)|}\bigvee_a^b(u)\right]^2 \tag{14}$$

where

$$B_{r,q}(a,b) := 2^{r+q-1}\max_{(s,t)\in[a,b]^2}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r+q}. \tag{15}$$

Proof. We have the following identity

$$S_{\mathcal{C}}(f,g;u,a,b) = \frac{1}{2\left[u(b)-u(a)\right]^2}\times \int_a^b\left(\int_a^b\left[f\left(e^{it}\right) - f\left(e^{is}\right)\right]\left[g\left(e^{it}\right) - g\left(e^{is}\right)\right]du(s)\right)du(t). \tag{16}$$

It is known that if $p : [c, d] \to \mathbb{C}$ is a continuous function and $v : [c, d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left| \int_c^d p(t)\,dv(t) \right| \le \max_{t\in[c,d]} \left| p(t) \right| \bigvee_c^d (v). \tag{17}$$

Applying this property twice, we have

$$\begin{aligned}
&\left| S_{\mathcal{C}}(f, g; u, a, b) \right| \\
&\quad = \frac{1}{2\left|u(b) - u(a)\right|^2} \times \\
&\qquad \left| \int_a^b \left(\int_a^b \left[f\left(e^{it}\right) - f\left(e^{is}\right) \right] \left[g\left(e^{it}\right) - g\left(e^{is}\right) \right] du(s) \right) du(t) \right| \\
&\quad \le \frac{1}{2\left|u(b) - u(a)\right|^2} \times \\
&\qquad \max_{t\in[a,b]} \left| \int_a^b \left[f\left(e^{it}\right) - f\left(e^{is}\right) \right] \left[g\left(e^{it}\right) - g\left(e^{is}\right) \right] du(s) \right| \bigvee_a^b (u) \\
&\quad \le \frac{1}{2\left|u(b) - u(a)\right|^2} \times \\
&\qquad \max_{(t,s)\in[a,b]^2} \left| \left[f\left(e^{it}\right) - f\left(e^{is}\right) \right] \left[g\left(e^{it}\right) - g\left(e^{is}\right) \right] \right| \left[\bigvee_a^b (u) \right]^2 .
\end{aligned} \tag{18}$$

Utilizing the properties of f and g we have

$$\left| \left[f\left(e^{it}\right) - f\left(e^{is}\right) \right] \left[g\left(e^{it}\right) - g\left(e^{is}\right) \right] \right| \le HK \left| e^{is} - e^{it} \right|^{r+q} \tag{19}$$

for any $s, t \in [a, b]$.

Since

$$\begin{aligned}
\left| e^{is} - e^{it} \right|^2 &= \left| e^{is} \right|^2 - 2\,\mathrm{Re}\left(e^{i(s-t)} \right) + \left| e^{it} \right|^2 \\
&= 2 - 2\cos(s - t) = 4 \sin^2 \left(\frac{s - t}{2} \right)
\end{aligned}$$

for any $t, s \in \mathbb{R}$, then

$$\left| e^{is} - e^{it} \right|^{r+q} = 2^{r+q} \left| \sin\left(\frac{s - t}{2} \right) \right|^{r+q} \tag{20}$$

for any $t, s \in \mathbb{R}$.

Utilizing (18) and (20) we deduce the desired result (14). ☑

***Remark* 2.2.** If $b = 2\pi$ and $a = 0$ then obviously there are $s, t \in [0, 2\pi]$ such that $s - t = \pi$ showing that

$$\max_{(s,t)\in[0,2\pi]^2} \left| \sin\left(\frac{s-t}{2}\right) \right|^{r+q} = 1.$$

In this situation we have

$$\left| S_{\mathcal{C}}(f, g; u, 0, 2\pi) \right| \le 2^{r+q-1} HK \left[\frac{1}{\left| u(2\pi) - u(0) \right|} \bigvee_0^{2\pi}(u) \right]^2. \tag{21}$$

Moreover, if f and g are Lipschitzian with constants L and N, respectively the inequality (21) becomes

$$\left| S_{\mathcal{C}}(f, g; u, 0, 2\pi) \right| \le 2LN \left[\frac{1}{\left| u(2\pi) - u(0) \right|} \bigvee_0^{2\pi}(u) \right]^2. \tag{22}$$

***Remark* 2.3.** For intervals smaller than π, i.e. $0 < b - a \le \pi$ then for all $t, s \in [a, b] \subseteq [0, 2\pi]$ we have $\frac{1}{2}|t - s| \le \frac{1}{2}(b - a) \le \frac{\pi}{2}$. Since the function sin is increasing on $\left[0, \frac{\pi}{2}\right]$, then we have successively that

$$\max_{(t,s)\in[a,b]^2} \left| \sin\left(\frac{s-t}{2}\right) \right| = \sin\left(\max_{(t,s)\in[a,b]^2} \frac{1}{2}|t - s| \right) = \sin\left(\frac{b-a}{2}\right). \tag{23}$$

In this case we get the inequality

$$\left| S_{\mathcal{C}}(f, g; u, a, b) \right| \le HKB_{r,q}(a, b) \left[\frac{1}{\left| u(b) - u(a) \right|} \bigvee_a^b(u) \right]^2 \tag{24}$$

where

$$B_{r,q}(a, b) := 2^{r+q-1} \left| \sin\left(\frac{b-a}{2}\right) \right|^{r+q}. \tag{25}$$

Moreover, if f and g are Lipschitzian with constants L and N, respectively then

$$B(a, b) := B_{1,1}(a, b) = 2\sin^2\left(\frac{b-a}{2}\right)$$

and the inequality (24) becomes

$$\left| S_{\mathcal{C}}(f, g; u, a, b) \right| \le 2LN \sin^2\left(\frac{b-a}{2}\right) \left[\frac{1}{\left| u(b) - u(a) \right|} \bigvee_a^b(u) \right]^2. \tag{26}$$

We also have:

Theorem 2.4. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is of H-r-Hölder's type and $g : \mathcal{C}(0,1) \to \mathbb{C}$ is of K-q-Hölder's type. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a M-Lipschitzian function with $u(a) \neq u(b)$, then*

$$\left|S_{\mathcal{C}}(f,g;u,a,b)\right| \le 2^{p+q-1}\frac{M^2HK}{\left|u(b)-u(a)\right|^2}C_{p,q}(a,b) \tag{27}$$

where

$$\begin{aligned} C_{r,q}(a,b) &:= \int_a^b\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right|^{r+q} ds\,dt \\ &\le \frac{1}{2^{r+q-1}(r+q+1)(r+q+2)}(b-a)^{r+q+2}. \end{aligned} \tag{28}$$

Proof. It is well known that if $p : [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds:

$$\left|\int_a^b p(t)\,dv(t)\right| \le M\int_a^b \left|p(t)\right| dt. \tag{29}$$

Utilizing this property and the identity (16) we have

$$\begin{aligned} &\left|S_{\mathcal{C}}(f,g;u,a,b)\right| \\ &\quad = \frac{1}{2\left|u(b)-u(a)\right|^2} \times \\ &\qquad \left|\int_a^b\left(\int_a^b\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]du(s)\right)du(t)\right| \\ &\quad \le \frac{M}{2\left|u(b)-u(a)\right|^2} \times \\ &\qquad \int_a^b\left|\int_a^b\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]du(s)\right|dt \\ &\quad \le \frac{M^2}{2\left|u(b)-u(a)\right|^2} \times \\ &\qquad \int_a^b\int_a^b\left|\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]\right|ds\,dt. \end{aligned} \tag{30}$$

Utilizing the properties of f and g we have

$$\left|\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]\right| \le HK\left|e^{is}-e^{it}\right|^{r+q}$$

for any $s, t \in [a, b]$, which implies that

$$\int_a^b \int_a^b \left| \left[f\left(e^{it}\right) - f\left(e^{is}\right) \right] \left[g\left(e^{it}\right) - g\left(e^{is}\right) \right] \right| ds\, dt$$
$$\leq HK \int_a^b \int_a^b \left| e^{is} - e^{it} \right|^{r+q} ds\, dt = 2^{r+q} HK \int_a^b \int_a^b \left| \sin\left(\frac{s-t}{2}\right) \right|^{r+q} ds\, dt.$$

Utilizing the well known inequality

$$|\sin x| \leq |x|, \qquad \text{for any } x \in \mathbb{R}$$

we have

$$\begin{aligned}
\int_a^b \int_a^b \left| \sin\left(\frac{s-t}{2}\right) \right|^{r+q} ds\, dt
&\leq \frac{1}{2^{r+q}} \int_a^b \int_a^b |s-t|^{r+q}\, ds\, dt \\
&= \frac{1}{2^{r+q}} \int_a^b \left[\int_a^t (t-s)^{r+q}\, ds + \int_t^b (s-t)^{r+q}\, ds \right] dt \\
&= \frac{1}{2^{r+q}} \int_a^b \frac{(t-a)^{r+q+1} + (b-t)^{r+q+1}}{r+q+1}\, dt \\
&= \frac{2(b-a)^{r+q+2}}{2^{r+q}(r+q+1)(r+q+2)} \\
&= \frac{1}{2^{r+q-1}(r+q+1)(r+q+2)} (b-a)^{r+q+2}
\end{aligned}$$

and the bound in (28) is proved. ☑

The case of Lipschitzian integrators is of importance and can be stated as follows:

Corollary 2.5. *Assume that $u : [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ is a M-Lipschitzian function with $u(a) \neq u(b)$. If f and g are Lipschitzian with constants L and N, respectively then*

$$\left| S_{\mathcal{C}}(f, g; u, a, b) \right| \leq \frac{4M^2 NL}{\left| u(b) - u(a) \right|^2} \left[\left(\frac{b-a}{2} \right)^2 - \sin^2\left(\frac{b-a}{2} \right) \right]. \tag{31}$$

Proof. We have to calculate

$$\begin{aligned} C_{1,1}(a,b) &= \int_a^b \int_a^b \sin^2\left(\frac{s-t}{2}\right) ds\, dt \\ &= \int_a^b \int_a^b \frac{1-\cos(s-t)}{2}\, ds\, dt \\ &= \frac{1}{2}\left[(b-a)^2 - \int_a^b \big[\sin(b-t) - \sin(a-t)\big]\, dt\right]. \end{aligned}$$

Since

$$\int_a^b \sin(b-t)\, dt = 1 - \cos(b-a)$$

and

$$\int_a^b \sin(a-t)\, dt = \cos(b-a) - 1$$

then

$$\begin{aligned} C_{1,1}(a,b) &= \frac{1}{2}\left[(b-a)^2 - 2\big(1-\cos(b-a)\big)\right] \\ &= \frac{1}{2}\left[(b-a)^2 - 4\sin^2\left(\frac{b-a}{2}\right)\right] \\ &= 2\left[\left(\frac{b-a}{2}\right)^2 - \sin^2\left(\frac{b-a}{2}\right)\right] \end{aligned}$$

and the inequality (31) then follows from (27). ☑

The case of monotonic nondecreasing integrators is as follows:

Theorem 2.6. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is of H-r-Hölder's type and $g : \mathcal{C}(0,1) \to \mathbb{C}$ is of K-q-Hölder's type. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{R}$ is a monotonic nondecreasing function with $u(a) < u(b)$, then*

$$\big|S_{\mathcal{C}}(f,g;u,a,b)\big| \le \frac{2^{p+q-1}HK}{\big[u(b)-u(a)\big]^2} D_{r,q}(a,b) \tag{32}$$

where

$$\begin{aligned} D_{r,q}(a,b) &:= \int_a^b \int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right|^{r+q} du(s)\, du(t) \\ &\le \max_{s,t\in[a,b]^2} \left|\sin\left(\frac{s-t}{2}\right)\right|^{r+q} \big[u(b)-u(a)\big]^2. \end{aligned} \tag{33}$$

Proof. It is well known that if $p : [a,b] \to \mathbb{C}$ is a continuous function and $v : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds:

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b \left|p(t)\right| dv(t). \tag{34}$$

Utilizing this property and the identity (16) we have

$$\begin{aligned}
&\left|S_{\mathcal{C}}(f,g;u,a,b)\right| \\
&\quad = \frac{1}{2\left[u(b)-u(a)\right]^2} \times \\
&\qquad \left|\int_a^b \left(\int_a^b \left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right] du(s)\right) du(t)\right| \\
&\quad \le \frac{1}{2\left[u(b)-u(a)\right]^2} \times \\
&\qquad \int_a^b \left|\int_a^b \left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right] du(s)\right| du(t) \\
&\quad \le \frac{1}{2\left[u(b)-u(a)\right]^2} \times \\
&\qquad \int_a^b \int_a^b \left|\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]\right| du(s)\,du(t).
\end{aligned} \tag{35}$$

Since

$$\left|\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]\right| \le HK\left|e^{is}-e^{it}\right|^{r+q}$$

for any $s,t \in [a,b]$, then

$$\begin{aligned}
&\int_a^b \int_a^b \left|\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]\left[g\left(e^{it}\right)-g\left(e^{is}\right)\right]\right| du(s)\,du(t) \\
&\qquad \le HK \int_a^b \int_a^b \left|e^{is}-e^{it}\right|^{r+q} du(s)\,du(t) \\
&\qquad\qquad = 2^{r+q} HK \int_a^b \int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right|^{r+q} du(s)\,du(t)
\end{aligned}$$

and the inequality (32) is proved.

The bound (33) for $D_{r,q}(a,b)$ is obvious. ☑

The Lipschitzian case is of interest due to many examples that can be provided as follows:

Corollary 2.7. *If f and g are Lipschitzian with constants L and N, respectively and $u : [a,b] \subseteq [0,2\pi] \to \mathbb{R}$ is a monotonic nondecreasing function with $u(a) < u(b)$, then*

$$\left|S_{\mathcal{C}}(f,g;u,a,b)\right| \leq \frac{LN}{\left[u(b)-u(a)\right]^2} \times \left[\left[u(b)-u(a)\right]^2 - \left(\int_a^b \cos s\, du(s)\right)^2 - \left(\int_a^b \sin s\, du(s)\right)^2\right]. \quad (36)$$

Proof. We have to calculate

$$\begin{aligned} D_{1,1}(a,b) &= \int_a^b \int_a^b \sin^2\left(\frac{s-t}{2}\right) du(s)\, du(t) \\ &= \int_a^b \int_a^b \frac{1-\cos(s-t)}{2}\, du(s)\, du(t) \\ &= \frac{1}{2}\left[\left(u(b)-u(a)\right)^2 - J(a,b)\right] \end{aligned}$$

where

$$J(a,b) := \int_a^b \int_a^b \cos(s-t)\, du(s)\, du(t).$$

Since

$$\cos(s-t) = \cos s \cos t + \sin s \sin t$$

then

$$J(a,b) = \left(\int_a^b \cos s\, du(s)\right)^2 + \left(\int_a^b \sin s\, du(s)\right)^2.$$

Utilizing (32) we deduce the desired result (36). ☑

***Remark* 2.8.** Utilizing the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^b \cos s\, du(s) = u(b)\cos b - u(a)\cos a + \int_a^b u(s)\sin s\, ds$$

and

$$\int_a^b \sin s\, du(s) = u(b)\sin b - u(a)\sin a - \int_a^b u(s)\cos s\, ds.$$

If we insert these values in the right hand side of (36) we can get some expressions containing only Riemann integrals. However they are complicated and will not be presented here.

3. Applications for Functions of Unitary Operators

We have the following vector inequality for functions of unitary operators.

Theorem 3.1. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is of H-r-Hölder's type and $g : \mathcal{C}(0,1) \to \mathbb{C}$ is of K-q-Hölder's type. If the operator $U : H \to H$ on the Hilbert space H is unitary, then*

$$\left|\langle x, y\rangle\big\langle f(U)g(U)x, y\big\rangle - \big\langle f(U)x, y\big\rangle\big\langle g(U)x, y\big\rangle\right| \leq 2^{r+q-1}HK\left[\bigvee_0^{2\pi}\big\langle E_{(\bullet)}x, y\big\rangle\right]^2 \leq 2^{r+q-1}HK\|x\|^2\|y\|^2 \quad (37)$$

for any $x, y \in H$. In particular, if f and g are Lipschitzian with constants L and N, respectively then

$$\left|\langle x, y\rangle\big\langle f(U)g(U)x, y\big\rangle - \big\langle f(U)x, y\big\rangle\big\langle g(U)x, y\big\rangle\right| \leq 2LN\left[\bigvee_0^{2\pi}\big\langle E_{(\bullet)}x, y\big\rangle\right]^2 \leq 2LN\|x\|^2\|y\|^2 \quad (38)$$

for any $x, y \in H$.

Proof. For given $x, y \in H$, define the function $u(\lambda) := \big\langle E_\lambda x, y\big\rangle$, $\lambda \in [0, 2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) := \bigvee_0^{2\pi}\left(\big\langle E_{(\bullet)}x, y\big\rangle\right) \leq \|x\|\|y\|. \quad (39)$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then, the following inequality is a *generalization of the Schwarz inequality* in H:

$$\big|\langle Px, y\rangle\big|^2 \leq \langle Px, x\rangle\langle Py, y\rangle, \quad (40)$$

for any $x, y \in H$.

Now, if $d : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 2\pi$ is an arbitrary partition of the interval $[0, 2\pi]$, then we have by Schwarz's inequality for nonnegative operators (40)) that

$$\bigvee_0^{2\pi}\left(\big\langle E_{(\bullet)}x, y\big\rangle\right) = \sup_d\left\{\sum_{i=0}^{n-1}\left|\Big\langle\big(E_{t_{i+1}} - E_{t_i}\big)x, y\Big\rangle\right|\right\}$$
$$\leq \sup_d\left\{\sum_{i=0}^{n-1}\left[\Big\langle\big(E_{t_{i+1}} - E_{t_i}\big)x, x\Big\rangle^{1/2}\Big\langle\big(E_{t_{i+1}} - E_{t_i}\big)y, y\Big\rangle^{1/2}\right]\right\} := I. \quad (41)$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$I \le \sup_{d} \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right)x, x \right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right)y, y \right\rangle \right]^{1/2} \right\}$$
$$\le \left[\bigvee_{0}^{2\pi} \left(\langle E_{(\bullet)}x, x\rangle\right) \right]^{1/2} \left[\bigvee_{0}^{2\pi} \left(\langle E_{(\bullet)}y, y\rangle\right) \right]^{1/2} = \|x\|\|y\| \quad (42)$$

for any $x, y \in H$.

Now, from the inequality (21) we have

$$\Bigg| \left(\langle E_{2\pi}x, y\rangle - \langle E_0 x, y\rangle\right) \int_0^{2\pi} f\left(e^{it}\right) g\left(e^{it}\right) d\langle E_t x, y\rangle - \int_0^{2\pi} f\left(e^{it}\right) d\langle E_t x, y\rangle \int_0^{2\pi} g\left(e^{it}\right) d\langle E_t x, y\rangle \Bigg|$$
$$\le 2^{r+q-1} HK \left[\bigvee_{0}^{2\pi} \left(\langle E_{(\bullet)}x, y\rangle\right) \right]^2 \quad (43)$$

for any $x, y \in H$. The proof is complete. ☑

***Remark* 3.2.** If $U : H \to H$ is an unitary operator on the Hilbert space H, then for any integers m, n we have from (38) the power inequalities

$$\left|\langle x, y\rangle \left\langle U^{m+n}x, y\right\rangle - \left\langle U^m x, y\right\rangle \left\langle U^n x, y\right\rangle\right|$$
$$\le 2|mn| \left[\bigvee_{0}^{2\pi} \left\langle E_{(\bullet)}x, y\right\rangle \right]^2 \le 2|mn| \|x\|^2 \|y\|^2 \quad (44)$$

for any $x, y \in H$. In particular, we have

$$\left|\langle x, y\rangle \left\langle U^2 x, y\right\rangle - \left\langle Ux, y\right\rangle^2\right| \le 2 \left[\bigvee_{0}^{2\pi} \left\langle E_{(\bullet)}x, y\right\rangle \right]^2 \le 2\|x\|^2\|y\|^2 \quad (45)$$

and

$$\left|\langle x, y\rangle^2 - \left\langle Ux, y\right\rangle \left\langle x, Uy\right\rangle\right| \le 2 \left[\bigvee_{0}^{2\pi} \left\langle E_{(\bullet)}x, y\right\rangle \right]^2 \le 2\|x\|^2\|y\|^2 \quad (46)$$

for any $x, y \in H$.

For $a \neq \pm 1, 0$ real numbers, consider the function $f : \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. This function is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$. Now, if we take $a, b \neq \pm 1, 0$ and use the inequality (38) then we have

$$\begin{aligned}\Big|\langle x,y\rangle\big\langle(1_H - aU)^{-1}(1_H - bU)^{-1}x,y\big\rangle & - \\ \big\langle(1_H - aU)^{-1}x,y\big\rangle\big\langle(1_H - bU)^{-1}x,y\big\rangle\Big| & \\ \le \frac{2|a||b|}{(1-|a|)^2(1-|b|)^2}\left[\bigvee_0^{2\pi}\big\langle E_{(\bullet)}x,y\big\rangle\right]^2 & \\ \le \frac{2|a||b|}{(1-|a|)^2(1-|b|)^2}\|x\|^2\|y\|^2 & \quad (47)\end{aligned}$$

for any $x, y \in H$. In particular, we have

$$\begin{aligned}\Big|\langle x,y\rangle\big\langle(1_H - aU)^{-2}x,y\big\rangle - \big\langle(1_H - aU)^{-1}x,y\big\rangle^2\Big| & \\ \le \frac{2|a|^2}{(1-|a|)^4}\left[\bigvee_0^{2\pi}\big\langle E_{(\bullet)}x,y\big\rangle\right]^2 & \\ \le \frac{2|a|^2}{(1-|a|)^4}\|x\|^2\|y\|^2 & \quad (48)\end{aligned}$$

for any $x, y \in H$.

Theorem 3.3. *If f and g are Lipschitzian with constants L and N, respectively and $U : H \to H$ is an unitary operator on the Hilbert space H, then*

$$\begin{aligned}\Big|\|x\|^2\big\langle f(U)g(U)x,x\big\rangle - \big\langle f(U)x,x\big\rangle\big\langle g(U)x,x\big\rangle\Big| & \\ \le LN\Big[\|x\|^4 - \big\langle \operatorname{Re}(U)x,x\big\rangle^2 - \big\langle \operatorname{Im}(U)x,x\big\rangle^2\Big] & \\ = LN\Big[\|x\|^4 - \big|\langle Ux,x\rangle\big|^2\Big] & \quad (49)\end{aligned}$$

for any $x \in H$, where

$$\operatorname{Re}(U) := \frac{U+U^*}{2} \qquad \textit{and} \qquad \operatorname{Im}(U) := \frac{U-U^*}{2i}.$$

Proof. From the inequality (36) we have

$$\Big|\big(\langle E_{2\pi}x,x\rangle-\langle E_0x,x\rangle\big)\int_0^{2\pi}f\big(e^{it}\big)g\big(e^{it}\big)\,d\langle E_tx,x\rangle - \int_0^{2\pi}f\big(e^{it}\big)\,d\langle E_tx,x\rangle\int_0^{2\pi}g\big(e^{it}\big)\,d\langle E_tx,x\rangle\Big| \le LN\Bigg[\big(\langle E_{2\pi}x,x\rangle-\langle E_0x,x\rangle\big)^2 - \left(\int_0^{2\pi}\cos t\,d\langle E_tx,x\rangle\right)^2-\left(\int_0^{2\pi}\sin t\,d\langle E_tx,x\rangle\right)^2\Bigg] \quad (50)$$

for any $x,y\in H$.

Since

$$\operatorname{Re}\left(e^{it}\right)=\cos t \qquad \text{and} \qquad \operatorname{Im}\left(e^{it}\right)=\sin t,$$

then we have from the representation (7) that

$$\left(\int_0^{2\pi}\cos t\,d\langle E_tx,x\rangle\right)^2=\big\langle \operatorname{Re}(U)x,x\big\rangle^2$$
$$\left(\int_0^{2\pi}\sin t\,d\langle E_tx,x\rangle\right)^2=\big\langle \operatorname{Im}(U)x,x\big\rangle^2$$

and due to the fact that

$$\big|\langle Ux,x\rangle\big|^2=\big|\big\langle[\operatorname{Re}(U)+i\operatorname{Im}(U)]x,x\big\rangle\big|^2 = \big|\big\langle \operatorname{Re}(U)x,x\big\rangle+i\big\langle \operatorname{Im}(U)x,x\big\rangle\big|^2=\big\langle \operatorname{Re}(U)x,x\big\rangle^2+\big\langle \operatorname{Im}(U)x,x\big\rangle^2$$

we deduce from (50) the desired inequality (49). ☑

Remark **3.4.** If $U:H\to H$ is an unitary operator on the Hilbert space H, then for any integers m,n we have from (49) the power inequalities

$$\Big|\|x\|^2\big\langle U^{n+m}x,x\big\rangle-\big\langle U^nx,x\big\rangle\big\langle U^mx,x\big\rangle\Big|\le|mn|\Big[\|x\|^4-\big|\langle Ux,x\rangle\big|^2\Big] \quad (51)$$

for any $x\in H$. In particular, we have for $n=m=1$

$$\Big|\|x\|^2\big\langle U^2x,x\big\rangle-\langle Ux,x\rangle^2\Big|\le\|x\|^4-\big|\langle Ux,x\rangle\big|^2 \quad (52)$$

for any $x\in H$. If we take $n=1$ and $m=-n$ and take into account that

$$\big\langle U^{-n}x,x\big\rangle=\Big\langle\big(U^n\big)^*x,x\Big\rangle=\big\langle x,U^nx\big\rangle=\overline{\big\langle U^nx,x\big\rangle}$$

for any $x \in H$, then we get from (51) that

$$0 \leq \|x\|^4 - \left|\left\langle U^n x, x\right\rangle\right|^2 \leq n^2 \left[\|x\|^4 - \left|\langle Ux, x\rangle\right|^2\right] \tag{53}$$

for any $x \in H$. Now, if we take $a, b \neq \pm 1, 0$ and use the inequality (49), then we get

$$\begin{aligned} \Big| \|x\|^2 \big\langle (1 - aU)^{-1} (1 - bU)^{-1} x, x \big\rangle &- \\ &\big\langle (1 - aU)^{-1} x, x \big\rangle \big\langle (1 - bU)^{-1} x, x \big\rangle \Big| \\ &\leq \frac{2|a||b|}{\big(1 - |a|\big)^2 \big(1 - |b|\big)^2} \left[\|x\|^4 - \left|\langle Ux, x\rangle\right|^2\right] \end{aligned} \tag{54}$$

for any $x \in H$. In particular, we have

$$\begin{aligned} \Big| \|x\|^2 \big\langle (1 - aU)^{-2} x, x \big\rangle - \big\langle (1 - aU)^{-1} x, x \big\rangle^2 \Big| \\ \leq \frac{2|a|^2}{\big(1 - |a|\big)^4} \left[\|x\|^4 - \left|\langle Ux, x\rangle\right|^2\right] \end{aligned} \tag{55}$$

for any $x \in H$.

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