Revista Colombiana de Matemáticas
Volumen 49(2015)1, páginas 105-138

Time Dependent Quantum Scattering Theory on Complete Manifolds with a Corner of Codimension 2

Teoría de dispersión cuántica dependiente del tiempo sobre variedades completas con una esquina de codimensión 2

LEONARDO A. CANO G.

Universidad Sergio Arboleda, Bogotá, Colombia

ABSTRACT. We show the existence and orthogonality of wave operators naturally associated to a compatible Laplacian on a complete manifold with a corner of codimension 2. In fact, we prove asymptotic completeness i.e. that the image of these wave operators is equal to the space of absolutely continuous states of the compatible Laplacian. We achieve this last result using time dependent methods coming from many-body Schrödinger equations.

Key words and phrases. Quantum scattering theory, Manifolds with corners, Wave operators, Many-body Schrödinger equations.

2010 Mathematics Subject Classification. 53C21, 53C42.

RESUMEN. Demostramos la existencia y ortogonalidad de operadores de onda naturalmente asociados a un Laplaciano compatible sobre una variedad completa con una esquina de codimensión 2. De hecho, probamos su completitud asintótica, es decir que la imagen de esos operadores de onda es igual al espacio de estados absolutamente contínuos del Laplaciano compatible. Logramos esto último usando métodos dependientes del tiempo que provienen del estudio de operadores de Schrödinger de varios cuerpos.

Palabras y frases clave. Teoría de dispersión cuántica, variedades con esquinas, operadores de onda, ecuaciones de Schrödinger de varios cuerpos.

1. Introduction

In this paper we use analytic tools to tackle problems of quantum scattering theory naturally associated to geometric Laplacians, at the same time this makes explicit the interactions between the geometry of the manifold and the quantum dynamics of the Laplacians.

Classical mechanics tells us that the time-asymptotic behavior of n-particles interacting with a pairwise potential of short range can be described by clusters whose centers of mass do not "feel" each other. In the papers [22] and [23] it was proved that a similar phenomenon occurs in quantum mechanics for many-particle Schrödinger operators with short range potentials. These proofs were time-dependent and geometric in nature, and they were initially developed in the papers [7] and [26]. In this article we prove asymptotic completeness for compatible Laplacians on complete manifolds with corners of codimension 2, which we abbreviate c.m.w.c.2 through the text, by adapting the proof of [26] as explained in [11]. Even though the ideas are adapted in a quite direct way, we believe that this article provides a deeper understanding of the spectral theory of compatible Laplacians on c.m.w.c.2 and of the geometric insight behind the proof of the results in [7] and [26], since the spectral analysis of Schrödinger operators and geometric Laplacians are analogous but not exactly the same.

The motivation to study these manifolds is the same as in [3] and [4]: they work as toy models for understanding singularities as those that appear on symmetric spaces of rank greater than 0; they are natural examples of complete manifolds whose spectral theory is well known, since they are a natural geometric generalization of the Cartesian products of complete manifolds with cylindrical ends. This last class of manifolds is very important in the study of the index theorems of the seminal paper [1] and we believe that a deeper understanding of the spectral theory of compatible Laplacians on c.m.w.c.2 (see Section 1.1) will shed light on the nature of the generalization of such theorems, specifically in order to complete the method applied in [16]. Generalizations of the index theorems of [1] to c.m.w.c.2 were obtained in [10] using surgery methods, we believe that these formulas are related to our scattering operator (see (26)). Finally, our work shows a clear analogy between many–particle Schrödinger operators and the compatible Laplacians on c.m.w.c.2, this analogy provides a deeper understanding of the geometric nature of the spectral theory of the former operators.

1.1. Compatible Laplacians on Complete Manifolds with a Corner of Codimension 2

Following [16], we explain the notions of *compact and complete manifolds with a corner of codimension 2* as is done in [3] and [4]. Let X_0 be a compact oriented Riemannian manifold with boundary M and suppose that there exists a hypersurface Y of M that divides M in two manifolds with boundary M_1 and M_2, i.e. $M = M_1 \cup M_2$ and $Y = M_1 \cap M_2$. Assume also that a neighborhood of Y in M is diffeomorphic to $Y \times (-\varepsilon, \varepsilon)$. We say that the manifold X_0 **has a corner of codimension 2** if X_0 is endowed with a Riemannian metric g that is a product metric on small neighborhoods, $M_i \times (-\varepsilon, 0]$ of the M_i's and on a small neighborhood $Y \times (-\varepsilon, 0]^2$ of the corner Y. If X_0 has a corner of codimension 2, we say that X_0 is a **compact manifold with a corner of codimension 2** (see Figure 1).

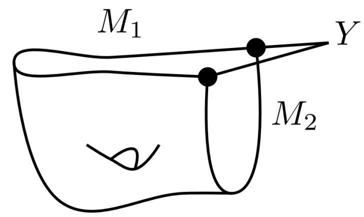

FIGURE 1. Compact manifold with a corner of codimension 2.

Example 1.1. For $i = 1, 2$, let M_i be a compact oriented Riemannian manifold with boundary $\partial M_i \coloneqq Y_i$. Suppose that on a neighborhood $Y_i \times (-\varepsilon, 0]$ of Y_i the Riemannian metric g_i of M_i is a product metric i.e. $g_i \coloneqq g_{Y_i} + du \otimes du$ where u is the coordinate associated to the interval $(-\varepsilon, 0]$ in $Y_i \times (-\varepsilon, 0]$ and g_{Y_i} is a Riemannian metric on Y_i independent of u. Then the Cartesian product $M_1 \times M_2$ is a compact manifold with a corner of codimension 2.

Throughout this article we will denote $\mathbb{R}_+ \coloneqq [0, \infty)$. From the compact manifold with a corner X_0 we construct a complete manifold X. Let $Z_i \coloneqq M_i \cup_Y (\mathbb{R}_+ \times Y)$, i=1,2, where the bottom $\{0\} \times Y$ of the half-cylinder $\mathbb{R}_+ \times Y$ is identified with $\partial M_i = Y$. Then Z_i is a complete manifold with cylindrical end. Let us define the manifolds

$$W_1 \coloneqq X_0 \cup_{M_2} \big(\mathbb{R}_+ \times M_2\big) \quad \text{and} \quad W_2 \coloneqq X_0 \cup_{M_1} \big(\mathbb{R}_+ \times M_1\big).$$

Observe that W_i is an n-dimensional manifold with boundary Z_i that can be equipped with a Riemannian metric compatible with the product Riemannian metric of $\mathbb{R}_+ \times M_2$ and the Riemannian metric of X_0. Let

$$X \coloneqq W_1 \cup_{Z_1} \big(\mathbb{R}_+ \times Z_1\big) = W_2 \cup_{Z_2} \big(\mathbb{R}_+ \times Z_2\big),$$

where we identify $\{0\} \times Z_i$ with Z_i, the boundary of W_i.

Figure 2 is a sketch, in particular the lines that enclose Figure 2 should not be thought as boundaries.

Let $T \geq 0$ be given and set $Z_{i,T} \coloneqq M_i \cup_Y \big([0, T] \times Y\big)$, for $i = 1, 2$, where $\{0\} \times Y$ is identified with Y, the boundary of M_i. $Z_{i,T}$ is a family of manifolds with boundary which exhausts Z_i. Next we attach to X_0 the manifold $[0, T] \times M_1$ by identifying $\{0\} \times M_1$ with M_1. The resulting manifold $W_{2,T}$ is a compact manifold with a corner of codimension 2, whose boundary is the union of M_1 and $Z_{2,T}$. The manifold X has associated a natural exhaustion given by

$$X_T \coloneqq W_{2,T} \cup_{Z_{2,T}} \big([0, T] \times Z_{2,T}\big), \quad T \geq 0, \tag{1}$$

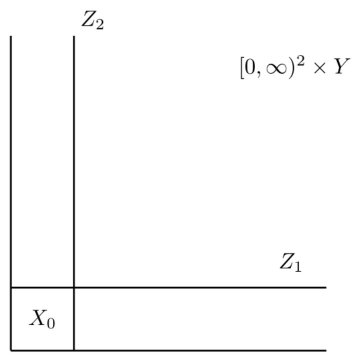

FIGURE 2. Sketch of a complete manifold with a corner of codimension 2.

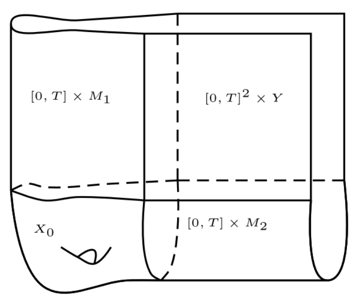

FIGURE 3. X_T, element of the exhaustion of X.

where we identify $Z_{2,T}$ with $\{0\} \times Z_{2,T}$ (see Figure 3).

For each $T \in [0,\infty)$, X has two submanifolds with cylindrical ends, namely $(\{T\} \times M_i) \cup (\{T\} \times [0,\infty) \times Y)$, for $i = 1, 2$. Here we are considering that the T is related with the coordinate u_i and the interval $[0,\infty)$ with the coordinate u_j for $i, j \in \{1, 2\}$, $i \neq j$ (see Remark 1.2 below). All these submanifolds are isometric in the Riemannian sense to Z_i and we identify their disjoint union with the Cartesian product $Z_i \times [0,\infty)$.

Let E be a Hermitian vector bundle over a c.m.w.c.2, X. Let Δ be a generalized Laplacian acting on $C^\infty(X,E)$, the sections of the vector bundle E. The operator Δ is a **compatible Laplacian** over X if the following properties are satisfied:

i) There exists a Hermitian vector bundle E_i over Z_i such that $E|_{\mathbb{R}_+ \times Z_i}$ is the pullback of E_i under the projection $\pi : \mathbb{R}_+ \times Z_i \to Z_i$, for $i = 1, 2$. We suppose also that the Hermitian metric of E is the pullback of the Hermitian metric of E_i. On $\mathbb{R}_+ \times Z_i$, we have $\Delta = -\frac{\partial^2}{\partial u_i^2} + \Delta_{Z_i}$, where Δ_{Z_i} is a compatible Laplacian acting on $C^\infty(Z_i, E_i)$.

ii) There exists a Hermitian vector bundle S over Y such that $E|_{\mathbb{R}^2_+\times Y}$ is the pullback of S under the projection $\pi : \mathbb{R}^2_+ \times Y \to Y$. We assume also that the Hermitian product on $E|_{\mathbb{R}^2_+\times Y}$ is the pullback of the Hermitian product on S. Finally we suppose that the operator Δ restricted to $\mathbb{R}^2_+ \times Y$ satisfies $\Delta = -\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + \Delta_Y$, where Δ_Y is a generalized Laplacian acting on $C^\infty(Y, S)$.

Examples of compatible Laplacians are given by the Laplacian acting on forms and Laplacians associated to compatible Dirac operators (see [16]), they satisfy conditions i) and ii) due to the product structure of the Riemannian metric on the submanifolds $Y \times \mathbb{R}^2_+$ and $Z_i \times \mathbb{R}_+$. Since X is a manifold with bounded geometry and the vector bundle E has bounded Hermitian metric, the operator $\Delta : C_c^\infty(X, E) \subset L^2(X, E) \to L^2(X, E)$ is essentially self-adjoint (see [21, Corollary 4.2]). Similarly $\Delta_{Z_i} : C_c^\infty(Z_i, E_i) \subset L^2(Z_i, E_i) \to L^2(Z_i, E_i)$ is also essentially self-adjoint for $i = 1, 2$.

***Remark* 1.2.** If $j, k \in \{1, 2\}$ and $j \neq k$, then we will denote by u_j the coordinate in $\mathbb{R}_+$ in the cylinder $Y \times \mathbb{R}_+$ of the complete manifold with cylindrical end Z_k.

Definition 1.3.

- Let H and $H^{(i)}$ be the self-adjoint extensions of $\Delta : C_c^\infty(X, E) \to L^2(X, E)$ and $\Delta_{Z_i} : C_c^\infty(Z_i, E_i) \to L^2(Z_i, E_i)$ respectively.
- Let b_i be the self-adjoint extension of $-\frac{d^2}{du_i^2} : C_c^\infty\big(\mathbb{R}_+\big) \to L^2\big(\mathbb{R}_+\big)$ obtained by imposing Dirichlet boundary conditions at 0.
- Let H_i be the self-adjoint operator $b_i \otimes Id + Id \otimes H^{(i)}$ acting on $L^2\big(\mathbb{R}_+\big) \otimes L^2(Z_i, E_i)$.
- Let $H^{(3)}$ be the self-adjoint operator associated to the essentially self-adjoint operator $\Delta_Y : C^\infty(Y, S) \subset L^2(Y, S) \to L^2(Y, S)$ and let H_3 be the self-adjoint operator $H_3 \coloneqq b_1 \otimes Id \otimes Id + Id \otimes b_2 \otimes Id + Id \otimes Id \otimes H^{(3)}$ acting on $L^2\big(\mathbb{R}_+\big) \otimes L^2\big(\mathbb{R}_+\big) \otimes L^2(Y, S)$.
- The operators H_i are called **channel operators** for $i = 1, 2, 3$.

The self-adjoint operators H_1 and H_2 have a free channel of dimension 1 (associated to b_1 and b_2, respectively); the operator H_3 has a free channel of dimension 2 (associated to $b_1 \otimes Id \otimes Id + Id \otimes b_2 \otimes Id$). In some parts of this text we make an abuse of notation by denoting H, H_i, and $H^{(i)}$ the Laplacians acting on distributions and the self-adjoint operators previously defined.

It is known that the compatible Laplacian $H^{(k)}$ decomposes the Hilbert space $L^2(Z_k, E)$ into the orthogonal $H^{(k)}$-invariant subspaces $L^2_{pp}(Z_k, E)$ and

$L^2_{ac}(Z_k, E)$ associated to pure point states and absolutely continuous states (see [9] [13]). We have $H^{(k)} = H^{(k)}_{pp} \oplus H^{(k)}_{ac}$ on $L^2(Z_k, E) = L^2_{pp}(Z_k, E) \oplus L^2_{ac}(Z_k, E)$ where $H^{(k)}_{pp}$ and $H^{(k)}_{ac}$ are self-adjoint operators acting on $L^2_{pp}(Z_k, E)$ and $L^2_{ac}(Z_k, E)$. We define the self-adjoint operators $H_{k,pp} := b_k \otimes 1 + 1 \otimes H^{(k)}_{pp}$ acting on $L^2\big(\mathbb{R}_+\big) \otimes L^2_{pp}(Z_k, E_k)$, for $k = 1, 2$, that together with H will define important wave-operators in this article. We notice that the operators $H_{k,pp}$ and $H^{(k)}_{pp}$ are different operators; to see that we observe that they act in different Hilbert spaces, $H_{k,pp}$ has only absolutely continuous spectrum and $H^{(k)}_{pp}$ has only pure point spectrum. Similarly, we define the self-adjoint operators $H_{k,ac} := b_k \otimes 1 + 1 \otimes H^{(k)}_{ac}$ acting on $L^2\big(\mathbb{R}_+\big) \otimes L^2_{ac}(Z_k, E_k)$. The operators $H_{k,ac}$ together with H define important wave–operators (see Theorem 1.4).

1.2. Main Results

Our first result is

Theorem 1.4.

1) For $k = 1, 2$ the following strong limits exist

$$\begin{aligned}
W_\pm\big(H, H_{k,pp}\big) &:= \lim_{t\to\mp\infty} e^{itH} e^{-itH_{k,pp}},\\
W_\pm(H, H_{k,ac}) &:= \lim_{t\to\mp\infty} e^{itH} e^{-itH_{k,ac}},\\
W_\pm(H, H_3) &:= \lim_{t\to\mp\infty} e^{itH} e^{-itH_3},\\
W_\pm(H_{k,ac}, H_3) &:= \lim_{t\to\mp\infty} e^{itH_{k,ac}} e^{-itH_3}.
\end{aligned}$$

2) The images of the operators $W_\pm\big(H, H_{1,pp}\big)$, $W_\pm\big(H, H_{2,pp}\big)$ and $W_\pm(H, H_3)$ are pairwise orthogonal.

We call the operators defined in part 1) of the theorem **wave operators**.

Definition 1.5. We say that the wave operators, $W_\pm(H, H_{1,pp})$, $W_\pm(H, H_{2,pp})$ and $W_\pm(H, H_3)$, are **asymptotically complete** if for all $\psi \in L^2_{ac}(X, E)$ there exists $\varphi_k \in L^2_{pp}(Z_k, E_k) \otimes L^2\big(\mathbb{R}_+\big)$, for $k = 1, 2$, and $\varphi_3 \in L^2(Y, S) \otimes L^2\big(\mathbb{R}^2_+\big)$ such that

$$\psi = W_\pm(H, H_3)\varphi_3 + \sum_{k=1}^{2} W_\pm\big(H, H_{k,pp}\big)\varphi_k. \tag{2}$$

Our second result is

Theorem 1.6. *The wave operators $W_\pm\big(H, H_{1,pp}\big)$, $W_\pm\big(H, H_{2,pp}\big)$ and $W_\pm(H, H_3)$ are asymptotically complete.*

Section 2 provides the first relation between the quantum dynamics of the compatible Laplacian and the geometry of X; Theorem 1.4 is proved in Section 3 using stationary phase methods. We prove Theorem 1.6 in Section 5 based on the methods of [26]. In Appendix A we give a summary of the stationary phase methods used in Section 3.

1.3. Related Literature

The literature about quantum scattering theory on open manifolds is large. For that reason we restrict our bibliography to some recent articles on the subject, where the reader can find references to classic or basic articles, or to articles that we consider directly related to the topics of this article. Articles on quantum scattering theory on manifolds with cylindrical ends are [6, 18, 20]; on manifolds asymptotically Euclidean [15]; on $SL(3)/SO(3)$ [14]; on homogeneous spaces associated to finite groups on [2]; connections between scattering theory on compact asymptotically Einstein manifolds and conformal geometry are studied in [8]; quantum scattering theory on more general open manifolds can be found in [5, 17]. Relations between the geometry of manifolds with corners and the quantum dynamics of many–particle Schrödinger operators has been treated also in [25] but the topics are different to ours, and in particular the operators studied there are many–particle Schrödinger operators that are essentially perturbations via potentials of the Laplacian on $\mathbb{R}^n$; here we treat perturbations associated to the geometry and not to a potential. In [16] the spectral theory of compatible Laplacians on c.m.w.c.2 is studied near 0 under the hypothesis that the compatible Laplacian on the corner has kernel 0. In this article we eliminate this hypothesis and study the whole spectrum of the compatible Laplacians.

2. Ruelle's Theorem

In this section we formulate Ruelle's theorem in the context of compatible Laplacians on complete manifolds with a corner of codimension 2. Our aim is to give a first relation between the quantum dynamics of the compatible Laplacian and the geometry of the manifold X.

Let A be a self-adjoint operator acting on a Hilbert space $\mathscr{H}$. We denote $\mathscr{H}_{pp}(A)$ the subspace spanned by all eigenvectors of A, $\mathscr{H}_c(A) \coloneqq (\mathscr{H}_{pp})^{\perp}(A)$, $\mathscr{H}_{ac}(A)$, $\mathscr{H}_{sc}(A)$ will denote the absolutely continuous and singular continuous subspaces of $\mathscr{H}$ associated to A.

Theorem 2.1. (cf. [11, page 3452]) *Let A be a self-adjoint operator acting on $L^2(X,E)$ and suppose that A satisfies*

$$\chi_K(A-\lambda)^{-1} \quad \text{is a compact operator, for any compact subset } K \text{ of } X, \tag{3}$$

for each $\chi_K \in C_c^\infty(X)$ such that $\chi_K = 1$ restricted to K. Then:

$$\varphi \in \mathscr{H}_{pp}(A) \Leftrightarrow \lim_{R\to\infty} \left\|(1-\chi_R)e^{iAt}\varphi\right\| = 0, \quad \text{uniformly in } 0 \le t < \infty.$$

$$\varphi \in \mathscr{H}_{c}(A) \Leftrightarrow \lim_{t\to\infty} t^{-1}\int_0^t \left\|\eta_R e^{iAs}\varphi\right\|^2 ds = 0, \quad \text{for any } R < \infty,$$

where η_R is any function in $C_c^\infty(X)$ that is equal to 1 on X_R, the compact manifold with a corner of codimension 2 defined in (1).

It follows from classical results in global analysis (see for example [21]) that the compatible Laplacian H satisfies (3). Then, intuitively, Theorem 2.1 implies that the continuous states associated to H are moving away of compact sets as $t \to \infty$. Theorems 1.4 and 1.6 describe in more detail the asymptotic behavior of this escape.

3. Existence of the Wave Operators

In this section we prove part 1) of Theorem 1.4 using Cook's criterion as expressed in the following simple lemma of abstract scattering theory. We will make use also of stationary phase methods which are summarized in Appendix A.

Lemma 3.1. [27, page 84] *Let B and B_0 be self-adjoint operators acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{H}_0$ respectively. Let $\mathscr{J} : \mathscr{H}_0 \to \mathscr{H}$ be a bounded operator that takes the domain* $\mathrm{Dom}(B_0)$ *into the domain* $\mathrm{Dom}(B)$*. Suppose that for some $D_0 \subset \mathrm{Dom}(B_0) \cap \mathscr{H}_{0,ac}(B_0)$ dense in $\mathscr{H}_{0,ac}(B_0)$ and for any $f \in D_0$,*

$$\int_0^{\pm\infty} \left\|(B\mathscr{J} - \mathscr{J}B_0)\exp(\mp itB_0)f\right\| dt < \infty. \tag{4}$$

Then $W_\pm(B, B_0, \mathscr{J}) := s - \lim_{t\to\infty} \exp(\pm itB)\mathscr{J}\exp(\mp itB_0)$ exists.

We prove first the existence of $W_\pm\big(H, H_{k,pp}\big)$, for $k \in \{1,2\}$.

Let $\{\varphi_{k,j}\}_{j=1}^{N_k}$ be an orthonormal collection of L^2–eigenfunctions of the operator $H_{pp}^{(k)}$ that generates $L^2_{pp}(Z_k, E_k)$ for $k = 1,2$. Observe that N_1 and N_2 denote the number of L^2–eigenvalues of the Laplacians $H^{(1)}$ and $H^{(2)}$ (counted with multiplicity). As pointed out in [3] and [4], the number of L^2–eigenvalues of a Laplacian on a manifold with a cylindrical end can be 0, finite or infinite. Without lost of generality for our computations we will assume that there are infinite L^2–eigenvalues that is $N_1 = N_2 = \infty$. Given $a \in L^2\big(\mathbb{R}_+\big)$, $\widehat{a}(u) := \int_0^\infty a(v)\sin v\, dv$ will denote the sine transform of a. Let $\kappa \in C^\infty\big(\mathbb{R}_+\big)$ be such that $\kappa(u) = 0$ for $u \le 2$ and $\kappa(u) = 1$ for $u > 3$. Let us define $\kappa_k \in C^\infty\big(Z_k \times \mathbb{R}_+\big)$ by $\kappa_k(z_k, u_k) := \kappa(u_k)$ for $k = 1,2$ and extend it to $C^\infty(X)$ by making it 0 on $X \smallsetminus \big(Z_k \times \mathbb{R}_+\big)$. We will show that we can apply

Lemma 3.1 taking $\mathscr{J} = \kappa_k$, $B_0 = H_{k,pp}$ and $B = H$. It is easy to see that κ_k takes $\mathrm{Dom}(H_{k,pp})$ into $\mathrm{Dom}(H)$. Let us denote by $\mathscr{S}\big((0,\infty)\big)$ the set of C^∞–functions of $[0,\infty)$ whose derivatives decrease faster than any polynomial and such that all their derivatives at 0 are equal to 0. We take

$$D_0 \coloneqq \Big\{ g\varphi_{k,j} : j \in \mathbb{N}, g \in \mathscr{S}\big((0,\infty)\big) \quad \text{and} \quad \widehat{g} \in C_c^\infty\big((0,\infty)\big) \Big\}.$$

Since $\mathscr{S}\big((0,\infty)\big)$ is dense in $L^2\big(\mathbb{R}_+\big)$, it is easy to see that the set D_0 is dense in $\mathrm{Dom}\,\big(H_{k,pp}\big)$.

To prove (4) of Lemma 3.1 observe that for $f \in D_0$

$$\begin{aligned} &\big\|\big(H\kappa_k - \kappa_k H_{k,pp}\big) e^{\mp itH_{k,pp}} f\big\| \\ &\qquad \le \left\| \frac{\partial^2}{\partial u_k^2}(\kappa_k) e^{\mp itH_{k,pp}} f \right\| + 2 \left\| \frac{\partial}{\partial u_k}(\kappa_k) \frac{\partial}{\partial u_k} e^{\mp itH_{k,pp}} f \right\|. \end{aligned} \tag{5}$$

If $f = g\varphi_{k,j} \in D_0$, we have

$$\left\| \frac{\partial^2}{\partial u_k^2}(\kappa_k) e^{\mp itH_{k,pp}} f \right\| = \left\| \frac{d^2}{du_k^2}(\kappa_k) e^{\mp itb_k} g \right\|_{L^2\left(\mathbb{R}_+\right)}. \tag{6}$$

We can use Appendix A to see $\int_{-\infty}^{\infty} \left\| \frac{d^2}{du_k^2}(\kappa_k) e^{\mp itb_k} g \right\|_{L^2\left(\mathbb{R}_+\right)} dt < \infty$. To estimate $\frac{\partial}{\partial u_k}(\kappa_k) \frac{\partial}{\partial u_k} e^{\mp itH_{k,pp}} f$, observe that

$$\left\| \frac{\partial}{\partial u_k}(\kappa_k) \frac{\partial}{\partial u_k} e^{\mp itH_{k,pp}} f \right\| = \left\| \frac{d}{du_k}(\kappa_k) e^{\mp itb_k} \frac{d}{du_k} g \right\|_{L^2\left(\mathbb{R}_+\right)}, \tag{7}$$

then we can apply again the methods of Appendix A. Finally, Lemma 3.1 proves the existence of $W_\pm\big(H, H_{k,pp}, \kappa_k\big)$.

Proposition 3.2. *$W_\pm\big(H, H_{k,pp}\big)$ exists and $W_\pm\big(H, H_{k,pp}, \kappa_k\big) = W_\pm\big(H, H_{k,pp}\big)$.*

***Proof*.** Observe that for $f = g\varphi_{k,j} \in D_0$, we have $\left\| e^{itH}(1-\kappa_k) e^{itH_{k,pp}} f \right\| = \left\| (1-\kappa_k) e^{itb_k} g \right\|_{L^2(\mathbb{R}_+, du_k)}$. Since $1-\kappa_k$ as a function of u_k has compact support, Appendix A implies $s - \lim_{t\to\infty} e^{itH}(1-\kappa_k) e^{itH_{k,pp}} = 0$. ☑

To prove the existence of $W(H, H_3)$ and $W(H, H_{k,ac})$ we proceed analogously. Let $\{\phi_n\}_{n=0}^\infty$ be an orthonormal collection of L^2–eigenfunctions of the operator $H^{(3)}$ that generates $L^2(Y, S)$. We take as dense sets

$$\begin{aligned} &D_{0,H_3} \coloneqq \\ &\quad \Big\{ fg\phi_n : f \in \mathscr{S}\big((0,\infty)_{u_1}\big), g \in \mathscr{S}\big((0,\infty)_{u_2}\big) \quad \text{and} \quad \widehat{f}, \widehat{g} \in C_c^\infty\big((0,\infty)\big) \Big\}, \end{aligned}$$

and

$$D_{0,H_{k,ac}} := \Big\{ f(z_k)g(u_k) : f \in \operatorname{Dom}\Big(H_{ac}^{(k)}\Big), g \in \mathscr{S}\big((0,\infty)\big) \quad \text{and} \quad \widehat{g} \in C_c^\infty\big((0,\infty)\big)\Big\}.$$

It is easy to see that (5)–(7)) generalize and we can apply Lemma 3.1 to prove the existence of $W(H, H_3, \kappa_1\kappa_2)$ and $W(H, H_{k,ac}, \kappa_k)$. Finally, there are natural generalizations of Proposition 3.2 that show the existence of $W(H, H_3)$ and $W(H, H_{k,ac})$.

The existence of $W_\pm(H_{1,ac}, H_3)$ follows from the existence of $W_\pm\Big(H_{ac}^{(1)}, b_2 + H^{(3)}\Big)$ (see [9]) and the following equality

$$W_\pm\Big(b_1 + H_{ac}^{(1)}, b_1 + b_2 + H^{(3)}\Big) = Id_{L^2(\mathbb{R}_+, du_1)} \otimes W_\pm\Big(H_{ac}^{(1)}, b_2 + H^{(3)}\Big).$$

4. Orthogonality of the Wave Operators

We prove part 2) of Theorem 1.4.

4.1. Orthogonality of $W(H, H_{1,pp})$ and $W(H, H_{2,pp})$

In this section we prove that for all $f_k \in L^2_{pp}(Z_k, E_k) \otimes L^2\big(\mathbb{R}_+\big)$, $k = 1, 2$, the following equality holds

$$\big\langle W_\pm\big(H, H_{1,pp}\big) f_1, W_\pm\big(H, H_{2,pp}\big) f_2\big\rangle_{L^2(X,E)} = 0. \tag{8}$$

We observe that

$$\big\langle W_\pm\big(H, H_{1,pp}\big) f_1, W_\pm\big(H, H_{2,pp}\big) f_2\big\rangle_{L^2(X,E)} = \lim_{t\to\infty} \big\langle e^{\mp itH_{1,pp}} f_1, e^{\mp itH_{2,pp}} f_2\big\rangle_{L^2(X,E)},$$

hence (8) is satisfied as a consequence of the following lemma.

Lemma 4.1. *For all $f_k \in L^2_{pp}(Z_k, E_k) \otimes L^2\big(\mathbb{R}_+\big)$, $k = 1, 2$,*

$$\lim_{t\to\infty} \big\langle e^{\mp itH_{1,pp}} f_1, e^{\mp itH_{2,pp}} f_2\big\rangle_{L^2(X,E)} = 0.$$

Proof. By continuity of the bilinear form

$$(f_1, f_2) \mapsto \big\langle W_\pm(H, H_{1,pp}) f_1, W_\pm(H, H_{2,pp}) f_2\big\rangle_{L^2(X,E)},$$

it is enough to prove the lemma for the dense set of functions of the form $f_k = a_k\varphi_k$, where $\varphi_k \in L^2(Z_k, E_k)$ is an L^2-eigenfunction of $H^{(k)}$ with eigenvalue γ_k, $a_k \in \mathscr{S}\big((0,\infty)\big)$ and $\widehat{a}_k \in C_c^\infty\big((0,\infty)\big)$.

In the next computation we use the notation given in definition 1.3 and explained in Remark 1.2,

$$\begin{aligned}\Big|\big\langle e^{\mp itH_{1,pp}} f_1, e^{\mp itH_{2,pp}} f_2\big\rangle_{L^2(X,E)}\Big| \\ \leq \int \bigg| \bigg\langle \int_0^\infty \varphi_1(u_2,y){\bullet} e^{\pm itb_2}(a_2)(u_2)\,du_2, \\ \int_0^\infty \varphi_2(u_1,y){\bullet} e^{\pm itb_1}(a_1)(u_1)du_1 \bigg\rangle\bigg|\, dvol(y), \quad (9)\end{aligned}$$

where the Hermitian product inside the integrals on the right–hand side of the inequality is the Hermitian product of the vector bundle $S \to Y$. It is well known that there exists $C \in \mathbb{R}$ such that $\left|e^{\pm itb_k}(a_k)(u_k)\right| \leq Ct^{-1/2}$, for all $t > 1$ and for all $u_k \in \mathbb{R}_+$ (see [19, Corollary, page 41]). Cauchy–Schwartz applied to the last term of (9) and the fact $|\varphi_k(u_j,y)| \leq Ce^{-cu_1}$ for some $c > 0$ (see [13, Lemma 1.36]) finish the proof of the lemma. ☑

4.2. Im $\big(W_\pm(H,H_3)\big)$ is orthogonal to Im $\big(W_\pm(H,H_{k,pp})\big)$

Without lost of generality we prove the orthogonality of Im $\big(W_\pm(H,H_3)\big)$ and Im $\big(W_\pm\big(H,H_{1,pp}\big)\big)$. Let $\phi \in L^2(Y,S)$, $\varphi \in L^2_{pp}(Z_1,E_1)$, $c \in L^2\big(\mathbb{R}_+,du_1\big)$ and $a_i \in L^2\big(\mathbb{R}_+,du_i\big)$ for $i = 1,2$. It is enough to prove that

$$\big\langle W_\pm(H,H_3)(a_1a_2\phi), W_\pm\big(H,H_{1,pp}\big)(c\varphi)\big\rangle_{L^2(X,E)} = 0. \quad (10)$$

We have

$$\begin{aligned}\Big|\langle e^{\pm itH_3}\big(a_1a_2\phi\big), e^{\pm itH_{1,pp}}(c\varphi)\rangle_{L^2(X,E)}\Big| \\ \leq \bigg|\Big\langle e^{\pm it\left(b_2+H^{(3)}\right)}(a_2\phi), e^{\pm itH^{(1)}_{pp}}(\varphi)\Big\rangle_{L^2(Z_1,E_1)}\bigg|. \quad (11)\end{aligned}$$

Since the wave operator $W_\pm\big(H^{(1)}, b_2 + H^{(3)}\big)$ is complete, we can find $\psi \in L^2_{ac}(Z_1,E_1)$ such that

$$\lim_{t\to\infty} \Big\| e^{\pm it\left(b_2+H^{(3)}\right)}(a_2\phi) - e^{\pm itH^{(1)}}\psi \Big\|_{L^2(Z_1,E_1)} = 0.$$

This together with (11) imply (10), since $L^2_{ac}(Z_1,E_1)$ is orthogonal to $L^2_{pp}(Z_1,E_1)$.

5. Asymptotic Clustering: A Time Dependent Approach

In this section we prove asymptotic completeness (Theorem 1.6) using a time dependent approach. We follow closely [11], and as in this article our main tools will be Mourre's inequality and the Yafaev functions (see Section 5.2) properly adapted to the context of compatible Laplacians on c.m.w.c.2.

5.1. Mourre Estimate for Compatible Laplacians

First we state Mourre's inequality which will be used to prove asymptotic completeness. It was developed in [4] and used to prove the absence of singular continuous spectrum of compatible Laplacians on c.m.w.c.2 and also to prove that the pure point spectrum of these operators accumulates only at thresholds. Let $\kappa \in C^\infty(\mathbb{R}_+)$ be such that $\kappa(u) = 0$ for $u \leq 2$ and $\kappa(u) = 1$ for $u > 3$. Let us define $\kappa_k \in C^\infty(Z_k \times \mathbb{R}_+)$ by $\kappa_k(z_k, u_k) := \kappa(u_k)$ for $k = 1, 2$ and the function $r^2 \in C^\infty(\mathbb{R}^2_+)$ by $r^2(u_1, u_2) := \kappa(u_1)u_1^2 + \kappa(u_2)u_2^2$. The function r^2 induces a function on $Y \times \mathbb{R}^2_+$ by $(y, u_1, u_2) \mapsto r^2(u_1, u_2)$ and this function extends naturally to X by making it 0 out of $Y \times \mathbb{R}^2_+$, by an abuse of notation we denote this new function by r^2 too. We extend κ_1 and κ_2 to X similarly by making them 0 out of $Z_1 \times \mathbb{R}_+$ and $Z_2 \times \mathbb{R}_+$ respectively. Let us define the first order differential operator A by

$$A := i[H, r^2].$$

We define **the set of thresholds of** H, $\tau(H)$, by

$$\tau(H) := \sigma_{pp}\Big(H^{(1)}\Big) \cup \sigma_{pp}\Big(H^{(2)}\Big) \cup \sigma_{pp}\Big(H^{(3)}\Big).$$

Let $\Sigma := \min \tau(H)$, such a minimum exists because $H^{(1)}, H^{(2)}$ and $H^{(3)}$ are bounded from below (see [13, Satz 1.27]) and hence the three sets on the right are discrete and with a minimum. For $\lambda \in \mathbb{R}$, define the number

$$\theta(\lambda) := \begin{cases} 0, & \text{for} \quad \lambda \leq \Sigma; \\ \inf\{\lambda - \gamma : \gamma \in \tau(H), \gamma < \lambda\}, & \text{for} \quad \lambda > \Sigma. \end{cases}$$

The next theorem is a generalization of Mourre's inequality to c.m.w.c.2 that was developed in [4].

Theorem 5.1. [4, theorem 5] *Given $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, there exist an open interval $I \ni \lambda$, and an H-compact operator K such that*

$$E_I(H)\, i[H, A]\, E_I(H) \geq \big(\theta(\lambda) - \varepsilon\big)\, E_I(H) + K,$$

where $E_I(H)$ denotes the spectral projection of the operator H on the interval $I \subset \mathbb{R}$.

5.2. Graf-Yafaev Functions

Consider the Schrödinger operators $\sum_{i=1}^2 \left(-\frac{\partial^2}{\partial u_i^2} + V_i\right)$ acting on $L^2(\mathbb{R}^2)$ where $V_i \in C^\infty(\mathbb{R}^2)$, V_i depends only of the variable u_i and is compactly supported in this variable. Our Graf-Yafaev functions are constructed in analogy to the Graf–Yafaev functions associated to these Schrödinger operators following [11, 12] and [26]. In this section we will omit some proofs, because we consider that the analogy is direct once the Graf–Yafaev functions are constructed.

Given $\epsilon > 0$, we take $\varepsilon_0^- := l_0 \leq \varepsilon_0 \leq l_0 + \epsilon =: \varepsilon_0^-$, $\varepsilon_3^- := 2\epsilon^2 < \varepsilon_3 < 3\epsilon^2 =: \varepsilon_3^+$, and $\varepsilon_i^- := 2\epsilon < \varepsilon_i < 3\epsilon := \varepsilon_i^+$ for $i = 1, 2$. We call the vectors $\varepsilon := (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ ϵ–**admissible**. From now on we will denote $\big|(u_1, u_2)\big| := \sqrt{u_1^2 + u_2^2}$.

Let χ be the characteristic function of the interval $[0, \infty)$. The functions $g^{(1)}, g^{(2)}$ and $g^{(3)}$ defined below are analogous to the functions $m^{(a)}$ in [26, equation 3.9].

$$g^{(0)}(\varepsilon, x) := \begin{cases} \varepsilon_0 \chi\big(\varepsilon_0 - \max\big\{ (1+\varepsilon_1)u_1, (1+\varepsilon_2)u_2, (1+\varepsilon_3)|u|\big\}\big), \\ \qquad\qquad\qquad \text{for} \quad x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2; \\ \varepsilon_0, \qquad\qquad\quad \text{if} \quad x = (u_1, z_1) \in \big[0, \frac{\varepsilon_0}{1+\varepsilon_1}\big] \times Z_{1,0}; \\ \varepsilon_0, \qquad\qquad\quad \text{if} \quad x = (u_2, z_2) \in \big[0, \frac{\varepsilon_0}{1+\varepsilon_2}\big] \times Z_{2,0}; \\ \varepsilon_0, \qquad\qquad\quad \text{if} \quad x \in X_0; \\ 0, \qquad\qquad\quad\ \ \text{otherwise.} \end{cases}$$

$$g^{(1)}(\varepsilon, x) :=$$
$$\begin{cases} (1+\varepsilon_1)u_1\chi\big((1+\varepsilon_1)\, u_1 - \max\big\{\varepsilon_0, (1+\varepsilon_2)u_2, (1+\varepsilon_3) \mid (u_1, u_2)|\big\}\big), \\ \qquad\qquad\qquad \text{for} \quad x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2; \\ (1+\varepsilon_1)u_1, \qquad \text{for} \quad x = (z_1, u_1) \in Z_{1,0} \times \big[\frac{\varepsilon_0}{1+\varepsilon_1}, \infty\big); \\ 0, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

$$g^{(2)}(\varepsilon, x) :=$$
$$\begin{cases} (1+\varepsilon_2)u_2\chi\big((1+\varepsilon_2)\, u_2 - \max\big\{\varepsilon_0, (1+\varepsilon_1)u_1, (1+\varepsilon_3)|(u_1, u_2)|\big\}\big), \\ \qquad\qquad\qquad \text{for} \quad x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2; \\ (1+\varepsilon_2)u_2, \qquad \text{for} \quad x = (z_2, u_2) \in Z_{2,0} \times \big[\frac{\varepsilon_0}{1+\varepsilon_2}, \infty\big); \\ 0, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

$$g^{(3)}(\varepsilon, x) :=$$
$$\begin{cases} (1+\varepsilon_3)|(u_1, u_2)|\chi\big((1+\varepsilon_3)\, |(u_1, u_2)| - \max\big\{\varepsilon_0, (1+\varepsilon_2)u_2, (1+\varepsilon_1)u_1\big\}\big), \\ \qquad\qquad\qquad \text{for} \quad x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2; \\ 0, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

The functions $g^{(i)}(x, \varepsilon)$ could be defined more directly in our case, for example for $(y, u_1, u_2) \in Y \times \mathbb{R}_+^2$, ε ϵ–admissible and ϵ small enough,

$g^{(1)}(\varepsilon, y, u_1, u_2) = (1+\varepsilon_1)u_1$ if and only if $(1+\varepsilon_1)u_1$ is greater or equal than ε_0, $(1+\varepsilon_2)u_2$ and $(1+\varepsilon_3)|(u_1,u_2)|$, and $g^{(1)}(\varepsilon, y, u_1, u_2) = 0$ otherwise. However we used the previous definitions because we would like to point out that the Graf–Yafaev method could generalize to manifolds with corners of higher codimension. Let us define the function

$$g(x,\varepsilon) := \begin{cases} (1+\varepsilon_i)u_i, & \text{for} \quad x = (z_i, u_i) \in Z_{i,0} \times \mathbb{R}_+; \\ \max\big\{\varepsilon_0, (1+\varepsilon_1)u_1, (1+\varepsilon_2)u_2, (1+\varepsilon_3)|u|\big\}, & \\ & \text{for} \quad x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2; \\ \varepsilon_0, & \text{for} \quad x \in X_0. \end{cases}$$

We observe that

$$g(x,\varepsilon) = \sum_{i=0}^{3} g^{(i)}(x,\varepsilon). \tag{12}$$

The next functions will be important in the description of the functions g and $g^{(i)}$.

$$k_1(\varepsilon_1, \varepsilon_3) := \frac{1+\varepsilon_3}{\sqrt{(1+\varepsilon_1^2) - (1+\varepsilon_3)^2}},$$
$$k_2(\varepsilon_1, \varepsilon_2) := \frac{1+\varepsilon_2}{1+\varepsilon_1},$$

and

$$k_3(\varepsilon_2, \varepsilon_3) := \frac{\sqrt{(1+\varepsilon_2)^2 - (1+\varepsilon_3)^2}}{1+\varepsilon_3}.$$

The next proposition is a consequence of the following limits: $\lim_{\epsilon\to 0} k_1(\varepsilon_1, \varepsilon_3) = \infty$, $\lim_{\epsilon\to 0} k_2(\varepsilon_1, \varepsilon_2) = 1$, and $\lim_{\epsilon\to 0} k_3(\varepsilon_2, \varepsilon_3) = 0$.

Proposition 5.2. *Let $\epsilon > 0$ be small enough and let $\varepsilon := (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be an ϵ–admissible vector. Then*

$$k_1(\varepsilon_1, \varepsilon_3) \geq k_2(\varepsilon_1, \varepsilon_2) \geq k_3(\varepsilon_2, \varepsilon_3).$$

Proposition 5.2 implies that $(1+\varepsilon_1)u_1 \geq \max\big\{\varepsilon_0, (1+\varepsilon_2)u_2, (1+\varepsilon_3)|(u_1,u_2)|\big\}$ if and only if $u_1 \geq \frac{\varepsilon_0}{1+\varepsilon_1}$ and $u_1 > k_1(\varepsilon_1, \varepsilon_3)u_2$. Reasoning in this way we obtain the sketch of the function $g(x,\varepsilon)$ given in Figure 4.

Let $\varphi_i \geq 0$, $\varphi_i \in C_c^\infty(\mathbb{R}_+)$, $\operatorname{supp}\varphi_i \subset [\varepsilon_i^-, \varepsilon_i^+]$ and $\int_0^\infty \varphi_i(\varepsilon_i)\, d\varepsilon_i = 1$, for $i = 1,2,3$. Let $\varphi_0 \in C^\infty(\mathbb{R}_+)$ be a real function with support in the interval $(l_0, l_0+\epsilon)$ for some $l_0 > 0$, that satisfies also $\int_0^\infty \varphi_0(\varepsilon_0)\, d\varepsilon_0 = 1$. We regularize the function $g^{(i)}$ averaging over the ϵ–compatible vectors ε

$$g^{(i)}(x) := \int_{-\infty}^{\infty} g^{(i)}(x,\varepsilon)\Pi_{i=0}^{3}\big(\varphi_i(\varepsilon_i)\, d\varepsilon_i\big). \tag{13}$$

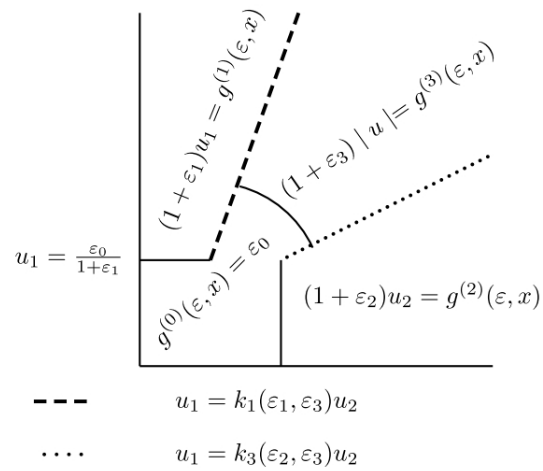

FIGURE 4. Sketch of the Graf–Yafaev function $g(x, \varepsilon)$.

Definition (13) is inspired by [26, definition 3.12]. For $i = 0, 1, 2, 3$, define $\Phi_i(\xi) \coloneqq \int_0^\xi \varphi_i(\varepsilon_i)\, d\varepsilon_i$. An easy computation shows that

$$g^{(1)}(x) = \int_{-\infty}^{\infty} (1+\varepsilon_1)u_1\varphi_1(\varepsilon_1)\Phi_0\big((1+\varepsilon_1)u_1\big)\Phi_2\big((1+\varepsilon_1)u_1u_2^{-1} - 1\big)\bullet \\ \Phi_3\big((1+\varepsilon_1)u_1|(u_1,u_2)|^{-1} - 1\big)\, d\varepsilon_1, \quad (14)$$

for $x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2$ or $x = (z_1, u_1) \in Z_1 \times \mathbb{R}_+$. We observe that $g^{(1)}(x) = 0$ on $X \smallsetminus \big(Z_1 \times \mathbb{R}_+\big)$. There is a similar formula for $g^{(2)}(x)$. For $g^{(3)}$ and $g^{(0)}$ we have

$$g^{(3)}(x) = \int (1+\varepsilon_3)|u|\varphi_3(\varepsilon_3)\Phi_0\big((1+\varepsilon_3)|u|\big)\Phi_1\left(\frac{(1+\varepsilon_3)|u|}{u_1} - 1\right)\bullet \\ \Phi_2\left(\frac{(1+\varepsilon_3)|u|}{u_2} - 1\right) d\varepsilon_3, \quad (15)$$

$$g^{(0)}(x) = \int \varepsilon_0\varphi_0(\varepsilon_0)\Phi_1\left(\frac{\varepsilon_0}{u_1} - 1\right)\Phi_2\left(\frac{\varepsilon_0}{u_2} - 1\right)\Phi_3\left(\frac{\varepsilon_0}{|u|} - 1\right) d\varepsilon_0, \quad (16)$$

for $x = (y, u_1, u_2) \in Y \times \mathbb{R}_+^2$.

We define g, the regularization of the function $g(x, \varepsilon)$, by taking the average on ε of $g(x, \varepsilon)$.

$$g(x) \coloneqq \int \max\big\{\varepsilon_0, (1+\varepsilon_1)u_1, (1+\varepsilon_2)u_2, (1+\varepsilon_3)|u|\big\} \\ \varphi_0(\varepsilon_0)\varphi_1(\varepsilon_1)\varphi_2(\varepsilon_2)\varphi_3(\varepsilon_3)\, d\varepsilon_0\, d\varepsilon_1\, d\varepsilon_2\, d\varepsilon_3.$$

Let us define $\mu_0 := \int \varepsilon_0 \varphi_0(\varepsilon_0)\, d\varepsilon_0$ and $\mu_i := \int (1+\varepsilon_i)\varphi_i(\varepsilon_i)\, d\varepsilon_i$ for $i = 1, 2, 3$. We observe that the maximum of the function k_1 in $[2\epsilon, 3\epsilon] \times \left[2\epsilon^2, 3\epsilon^2\right]$ is attained in $(\varepsilon_1, \varepsilon_2) = \left(2\epsilon, 3\epsilon^2\right)$ and its minimum is attained in $(\varepsilon_1, \varepsilon_2) = \left(3\epsilon, 2\epsilon^2\right)$. The maximum of the function k_2 in $[2\epsilon, 3\epsilon] \times \left[2\epsilon^2, 3\epsilon^2\right]$ is attained in $(\varepsilon_2, \varepsilon_3) = \left(3\epsilon, 2\epsilon^2\right)$ and its minimum is obtained in $(\varepsilon_1, \varepsilon_2) = (2\epsilon, 3\epsilon)$. Based on these observations and Proposition 5.2 we obtain Figure 5, a sketch of the Yafaev function. The arcs in this figure are part of the circles $\left|(u_1, u_2)\right| = \frac{l_0}{1+3\epsilon^2}$ and $\left|(u_1, u_2)\right| = \frac{l_0+\epsilon}{1+2\epsilon^2}$.

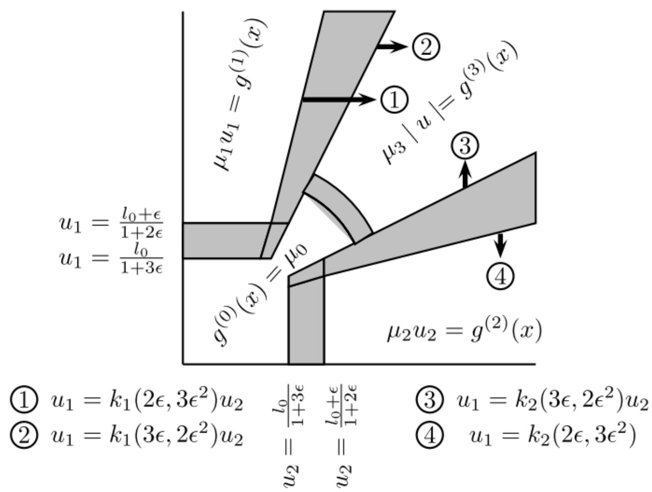

FIGURE 5. Sketch of the Graf–Yafaev functions.

The next lemma summarize the main properties of g that we will use in this article.

Lemma 5.3. (cf. [26, page 538]) *g satisfies the following properties:*

1) $g \in C^\infty(X)$ and $g(x)$ is real homogeneous of degree 1 in the sense that:

$$g(tu_1, z_1) = tg(u_1, z_1), \quad \text{for} \quad z_1 \in Z_1, u_1 \geq 4; \quad \text{and,}$$
$$g(tu_1, tu_2, y) = tg(u_1, u_2, y), \quad \text{for} \quad (y, u_1, u_2) \in Y \times [4, \infty)^2,$$

for $t \geq 0$.

2) $g(x) \geq 1$ for $x \in X \setminus X_4$.

3) $g(x)$ is convex in the sense that for (y, u_1, u_2) and (y, v_1, v_2) $Y \times \mathbb{R}^2_+$:

$$g\big((y, t(u_1, u_2) + s(v_1, v_2)\big) \leq sg\big((y, u_1, u_2)\big) + tg(y, v_1, v_2),$$

for $s, t \in [0, 1], s + t = 1$.

4) The functions $g^{(i)}$'s are related to g by the equality $g(x) = \sum_{i=0}^{3} g^{(i)}(x)$.

Proof. (14), (15) and (16) prove that the functions $g^{(i)}$ are smooth. 4) follows from (12) and these results imply that g is smooth and it is convex because it is the integral of the maximum of convex functions. The other properties follow from direct calculations. ☑

Definition 5.4. A function g satisfying Properties 1), 2), 3) and 4) of the above lemma is called **Yafaev function**.

Let $f : X \to \mathbb{R}$ be a C^∞-function. Let us denote by f'' the matrix valued function

$$f'' := \begin{bmatrix} \frac{\partial^2}{\partial u_1^2} f & \frac{\partial^2}{\partial u_1 \partial u_2} f \\ \frac{\partial^2}{\partial u_1 \partial u_2} f & \frac{\partial^2}{\partial u_2^2} f \end{bmatrix}, \tag{17}$$

defined on $Y \times \mathbb{R}_+^2$. We observe that the matrix of functions $\big(g^{(i)}\big)''$ can be extended to X making it 0 out of $Y \times \mathbb{R}_+^2$; we will make this type of natural extension without to explicitly mention them for other functions. We remark that $(f)''$ is not the Hessian of f.

According to the previous lemma, the functions $g^{(i)}$ are Yafaev functions, but they do not satisfy 3), yet in any case they are bounded by suitable convex functions, as it is shown in the next lemma.

Lemma 5.5. (cf. [11, lemma 7.4]) *For each $i \in \{1, 2, 3\}$ there exists $\widetilde{g}_i$ a Yafaev function such that $\big(\widetilde{g}^{(i)}\big)''(x) \geq \big(g^{(i)}\big)''(x)$, for all $x \in X$.*

Proof. We prove the lemma for $i = 1$, the cases $i = 2$ and $i = 3$ can be treated similarly. Let us define the set

$$\Gamma := \Big\{ (y, u_1, u_2) \in Y \times \mathbb{R}_+^2 : \frac{l_0}{1 + 2\epsilon} \leq u_1 \leq \frac{l_0 + \epsilon}{1 + 3\epsilon} \quad \text{and} \\ k_1\big(3\epsilon, 2\epsilon^2\big) u_2 \leq u_1 \leq k_1\big(2\epsilon, 3\epsilon^2\big) u_2 \Big\}.$$

Let $x_0 \in \Gamma$ and let δ_{x_0} be a positive function in $C_c^\infty(\mathbb{R})$ such that $\delta_{x_0}\big(g(x_0)\big) \neq 0$. Taking ϵ small enough and l_0 suitable we can find a Yafaev function g such that $(g)''(x_0) = (r)''(x_0) > 0$. Let us define the function

$$\widetilde{g}_{x_0}(x) := \int_{g(x)}^{\infty} s \delta_{x_0}(s)\, ds + g(x) \int_{-\infty}^{g(x)} \delta_{x_0}(s)\, ds.$$

We have

$$\widetilde{g}''_{x_0}(x) = g''(x)\int_{-\infty}^{g(x)} \delta_{x_0}(s)\,ds + \delta_{x_0}\big(g(x)\big)\begin{bmatrix} \frac{\partial}{\partial u_1}(g)^2 & \frac{\partial}{\partial u_1}(g)\frac{\partial}{\partial u_2}(g) \\ \frac{\partial}{\partial u_1}(g)\frac{\partial}{\partial u_2}(g) & \frac{\partial}{\partial u_2}(g)^2 \end{bmatrix}(x).$$

Since $(g)''(x)$ is positive and $\int_{-\infty}^{g(x)} \delta_{x_0}(s)\,ds > 0$ for $x \in \Gamma$ near enough to x_0, we have that

$$\left\langle \widetilde{g}''_{x_0}(x_0)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle > \delta_{x_0}\big(g(x_0)\big)\left\langle \begin{bmatrix} \frac{\partial}{\partial u_1}(g)^2 & \frac{\partial}{\partial u_1}(g)\frac{\partial}{\partial u_2}(g) \\ \frac{\partial}{\partial u_1}(g)\frac{\partial}{\partial u_2}(g) & \frac{\partial}{\partial u_2}(g)^2 \end{bmatrix}(x_0)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle \geq 0.$$

This proves $\widetilde{g}''_{x_0}(x)$ is strictly positive in an open ball U_{x_0} around x_0 and multiplying $\widetilde{g}''_{x_0}$ by a constant, if it is necessary, we have $\widetilde{g}''_{x_0}(x) \geq g''^{(i)}(x)$, for all $x \in U_{x_0}$. Since Γ is compact there exists a finite covering $\{U_{x_i}\}_{i=1}^N$ of Γ, with associated functions $\{\widetilde{g}_{x_i}\}_{i=1}^N$. Let us define $\widetilde{g} \coloneqq \sum_{i=1}^N \widetilde{g}_{x_i}$. To see that $\widetilde{g}$ satisfies the lemma, it is enough to prove it for x in the set $A \coloneqq \{(y,u_1,u_2) \in Y \times \mathbb{R}^2_+ : k_1(3\epsilon, 2\epsilon^2)u_2 \leq u_1 \leq k_1(2\epsilon, 3\epsilon^2)u_2\}$. Observe that for $x \in \Gamma$, it follows by construction of $\widetilde{g}$. Let $(y,u_1,u_2) \in A$, then there exists $\lambda \in (0,\infty)$, such that $(y,\lambda u_1,\lambda u_2) \in \Gamma$. Then, by homogeneity, $\big(g^{(1)}\big)''\big((y,u_1,u_2)\big) = 1/\lambda\big(g^{(1)}\big)''\big((y,\lambda u_1,\lambda u_2)\big) \leq 1/\lambda\widetilde{g}''(y,\lambda u_1,\lambda u_2) = \widetilde{g}''(y,u_1,u_2)$. ☑

5.3. Propagation Observables

Let g be a Yafaev function. All our propagation observables are derived from the following scaling of g, defined for $t > 0$ and $0 < \delta < 1$,

$$g_t(x) \coloneqq \begin{cases} t^\delta g\big(y, t^{-\delta}u_1, t^{-\delta}u_2\big), & x = (y,u_1,u_2) \in Y \times \mathbb{R}^2_+; \\ t^\delta g\big(z_i, t^{-\delta}u_i\big), & x = (z_i,u_i) \in Z_{i,0} \times \mathbb{R}_+; \\ t^\delta \int \varepsilon_0\varphi_0(\varepsilon_0)\,d\varepsilon_0, & x \in X_0. \end{cases}$$

We will be more precise about the value of δ later on. The next results about the derivatives of g_t are the basis of forthcoming estimates of propagation observables.

Lemma 5.6. (cf. [11, equation 7.18]) *For each* $\big((k_1,k_2),l\big)\in\mathbb{N}^2\times\mathbb{N}$ *and* $t>0$ *large enough, there exist* $C_1>0$ *and* $C_2>0$ *such that:*

$$\kappa(u_1)\kappa(u_2)\frac{\partial^{k_1}}{\partial u_1^{k_1}}\frac{\partial^{k_2}}{\partial u_2^{k_2}}g_t^{(j)}(x)\le C_1t^{\delta(1-|k|)},\quad \textit{and}$$
$$\frac{\partial^l}{\partial t^l}g_t^{(j)}(x)\le C_2t^{\delta-l},$$

for $j=1,2,3$*, for all* $x\in X$ *and* $k_1\ge 1$ *or* $k_2\ge 1$.

Proof. We prove the lemma for $g^{(1)}$. The functions $g^{(2)}$ and $g^{(3)}$ are treated in a similar form. Observe that the integrand of (14) has support in $[2\epsilon,3\epsilon]$. Using Lebesgue dominated convergence theorem, it is easy to see that there exists a $C>0$ depending only on k_1 and k_2 such that

$$\begin{aligned}\left|\frac{\partial^{k_2}}{\partial u_2^{k_2}}\frac{\partial^{k_1}}{\partial u_1^{k_1}}\big(g^{(1)}\big)\right|\le C\sum_{|(j,s)|=k_1+k_2}\Bigg|\int_{2\epsilon}^{3\epsilon}&\frac{\partial^{j_1}}{\partial u_1^{j_1}}\big((1+\varepsilon_1)u_1\big)\varphi_1(\varepsilon_1)\,\bullet\\ \frac{\partial^{j_0}}{\partial u_1^{j_0}}\big(\Phi_0\big((1+\varepsilon_1)t^{-\delta}u_1\big)\big)\frac{\partial^{s_1}}{\partial u_2^{s_1}}\frac{\partial^{j_2}}{\partial u_1^{j_2}}&\bigg(\Phi_2\bigg((1+\varepsilon_1)\frac{u_1}{u_2}-1\bigg)\bigg)\bigg)\,\bullet\\ &\frac{\partial^{s_2}}{\partial u_2^{s_2}}\frac{\partial^{j_3}}{\partial u_1^{j_3}}\bigg(\Phi_3\bigg((1+\varepsilon_1)\frac{u_1}{|u|}-1\bigg)\bigg)\,d\varepsilon_1\Bigg|.\end{aligned}\tag{18}$$

We notice that the sum on the right-hand side of the above inequality runs over the finite set of multi-indexes $(j,s)\in\mathbb{N}^3\times\mathbb{N}^2$ such that $|(j,s)|=k_1+k_2$, where $|(j,s)|\coloneqq j_0+j_1+j_2+s_1+s_2$. We will denote by $B_{j,s}$ the terms of that sum and we will show that they are uniformly bounded by $t^{\delta(1-k_1-k_2)}$. Since $g^{(1)}(z_1,u_1)=0$ for $u_1\le\frac{l_0t^\delta}{1+3\epsilon}$, the term $B_{j,s}(y,u_1,u_2)=0$. Out of $k_1\big(3\epsilon,2\epsilon^2\big)u_2\le u_1\le k_1\big(2\epsilon,3\epsilon^2\big)u_2$ and $u_1\ge\frac{l_0t^\delta}{1+3\epsilon}$, the function $g^{(1)}$ is constant or linear and the lemma follows easily. Hence we estimate the terms $B_{j,s}$ only for $(y,u_1,u_2)\in Y\times\mathbb{R}_+\times\mathbb{R}_+$ such that $k_1\big(3\epsilon,2\epsilon^2\big)u_2\le u_1\le k_1\big(2\epsilon,3\epsilon^2\big)u_2$ and $u_1\ge\frac{l_0t^\delta}{1+3\epsilon}$.

A direct computation shows that there exists a constant $C(j_0)$ such that

$$\frac{\partial^{j_0}}{\partial u_1^{j_0}}\big(\Phi_0\big((1+\varepsilon_1)t^{-\delta}u_1\big)\big)\le C(j_0)t^{-j_0\delta}.\tag{19}$$

We use above that φ_0 has compact support and hence all its derivatives are bounded in $\mathbb{R}$. Observe that taking $h(u_1,u_2)\coloneqq(1+\varepsilon_1)\frac{u_1}{u_2}-1$ and $f(v)\coloneqq\frac{d^{j_2}}{dv^{j_2}}(\varphi_2)(v)$, one obtains

$$\frac{(1+\varepsilon_1)^{j_2}}{u_2^{j_2}}f\circ h(u_1,u_2)=\frac{\partial^{j_2}}{\partial u_1^{j_2}}\bigg(\Phi_2\bigg((1+\varepsilon_1)\frac{u_1}{u_2}-1\bigg)\bigg).$$

Let $l \in \mathbb{N}$, let us define $M_l \coloneqq \big\{(k_1, \dots, k_l) \in \mathbb{N}^l : \sum_{i=1}^{l} ik_i = l\big\}$. We can conclude from Faà di Bruno's formula that for all $l \in \mathbb{N}$ and $\alpha \in M_l$ there exist constants $a_{l,k,\alpha} \in \mathbb{R}$ and $C > 0$ such that

$$\begin{aligned}
\left|\frac{\partial^{s_1}}{\partial u_2^{s_1}}\left(\frac{1}{u_2^{j_2}} f \circ h\right)(u_1, u_2)\right| &\leq C \sum_{l=0}^{s_1} \left|\frac{\partial^l}{\partial u_2^l}(f \circ h)(u_1, u_2)\frac{1}{u_2^{j_2+s_1-l}}\right| \\
\leq C \sum_{l=0}^{s_1}\sum_{k=0}^{l}\sum_{\alpha\in M_l} &\left|a_{l,k,\alpha}\big(\partial^k(f)\circ h\big)(u_1,u_2)\prod_{i=0}^{l}\left(\frac{\partial^i}{\partial u_2^i}(h)\right)^{\alpha_i}(u_1,u_2)\frac{1}{u_2^{j_2+s_1-l}}\right| \\
\leq C \sum_{l=0}^{s_1}\sum_{k=0}^{l}\sum_{\alpha\in M_l} &\left|a_{l,k,\alpha}\big(\partial^k(f)\circ h\big)(u_1,u_2)\prod_{i=0}^{l}\frac{u_1^{\alpha_i}}{u_2^{(i+1)\alpha_i}}\frac{1}{u_2^{j_2+s_1-l}}\right|. \qquad (20)
\end{aligned}$$

For $u_1 \geq \frac{l_0 t^\delta}{1+3\epsilon}$ and $u_1 \leq k_1\big(2\epsilon, 3\epsilon^2\big)u_2$, there exists a constant $C(s_1, j_2) > 0$ such that the last term of (20) is lower or equal than

$$\begin{aligned}
C \sum_{l=0}^{s_1}\sum_{k=0}^{l}\sum_{\alpha\in\mathbb{N}^n} \big(\partial^k(f)\circ h\big)(u_1,u_2)\prod_{i=0}^{l}\frac{1}{u_2^{i\alpha_i+j_2+s_1-l}}& \\
\leq C(s_1, j_2)t^{-\delta(j_2+s_1)},& \qquad (21)
\end{aligned}$$

where we obtain the last inequality, since $j_2 + s_1 - l + \sum_{i=0}^{l} i\alpha_i = j_2 + s_1$ because the vectors $(\alpha_i) \in M_l$ and the functions $\partial^k(f)$ have compact support.

Similar estimates can be done to obtain

$$\left|\frac{\partial^{s_2}}{\partial u_2^{s_2}}\frac{\partial^{j_3}}{\partial u_1^{j_3}}\left(\Phi_3\left((1+\varepsilon_1)\frac{u_1}{|u|}-1\right)\right)\right| \leq Ct^{-\delta(j_3+s_2)}. \qquad (22)$$

We observe that (19), (20), (21) and (22) together with (5.3) imply the first estimate of the lemma for the function $g^{(1)}$.

Next we will prove the second estimate of the lemma for the function $g^{(1)}$. Let $\mathbb{N} \ni j \geq 1$; we proceed by induction in j. The basis case, $j = 1$, follows easily deriving with respect to t the scaling of expression (14). For $j \geq 1$, one uses Faà di Bruno's formula for $f = \varphi_0$ and $g(v) = (1+\varepsilon_1)t^{-\delta}v$, in a similar way as it was used in (20). One can adapt the proof of the lemma for $g^{(1)}$ to the functions $g^{(2)}$ and $g^{(3)}$. ☑

We define the Heisenberg derivative of a function $h \in C^\infty\big(\mathbb{R}_+ \times X\big)$ by

$$D_t h \coloneqq i[H, h] + \frac{\partial}{\partial t}h. \qquad (23)$$

Now we estimate the first Heisenberg derivative γ_t of g_t i.e.

$$\gamma_t \coloneqq D_t g_t = i[H, g_t] + \frac{\partial}{\partial t}g_t.$$

We will denote $\mathscr{W}_1(X,E)$ the domain of the self–adjoint operator $|H|^{1/2}$. Using an interpolation argument one can see that $\mathscr{W}_1(X,E)$ coincides with the first Sobolev space (see [24, Chapter 2]). Let us define the first order differential operator $p := i\begin{bmatrix}\frac{\partial}{\partial u_1}\\ \frac{\partial}{\partial u_2}\end{bmatrix}$ acting on sections $f \in C^\infty\big(Y \times \mathbb{R}_+ \times \mathbb{R}_+, S\big)$ by $pf := i\begin{bmatrix}\frac{\partial}{\partial u_1}f\\ \frac{\partial}{\partial u_2}f\end{bmatrix}$. We will denote p^T the operator $i\big(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\big)$. The next lemma shows that the asymptotic behavior of γ_t is described by the matrix function $g_t''(x)$ defined in (17), it is a consequence of lemma 5.6.

Lemma 5.7. (cf. [11, equation (7.22)]) *For all $2 > \delta > 0$ and all $\psi \in \mathscr{W}_1(X,E)$*

$$\Big\langle D_t\Big(\gamma_t - 2\frac{\partial}{\partial t}g_t\Big)\psi_t, \psi_t\Big\rangle_{L^2(X,E)} = \Big\langle\Big(- 4p^T g_t'' p + O\big(t^{-3\delta}\big) + O\big(t^{\delta-2}\big)\Big)\psi_t, \psi_t\Big\rangle_{L^2(X,E)}$$

Proof. Observe that $\frac{\partial}{\partial t}[H,g_t] = \big[H, \frac{\partial}{\partial t}g_t\big]$, hence $D_t\big(\gamma_t - 2\frac{\partial}{\partial t}g_t\big) = -\big[H,[H,g_t]\big] - \frac{\partial^2}{\partial t^2}g_t$. Using Leibnitz rule for Laplacians and straightforward computations

$$\big[H,[H,g_t]\big] = 4p^T g_t''(x)p + \sum_{i,j=1}^{2}\partial_{jjii}(g_t).$$

According to Lemma 5.6, $\partial_{jjii}(g_t) = O\big(t^{-3\delta}\big)$ and $\frac{\partial^2}{\partial t^2}g_t \le t^{\delta-2}$, which implies the lemma. ☑

The next lemma is consequence of Lemma 5.7.

Lemma 5.8. (cf. [11, theorem 7.5]) *For $1 > \delta > 1/3$ there exists $C > 0$ such that*

$$\left|\int_1^\infty \Big\langle p^T g_t'' p\psi_t, \psi_t\Big\rangle_{L^2(X,E)} dt\right| \le C\|\psi\|_1^2,$$

for all $\psi \in \mathscr{W}_1(X,E)$.

Proof. Using lemma 5.7 we have that

$$\left|\int_1^\infty \Big\langle p^T g_t'' p\psi_t, \psi_t\Big\rangle_{L^2(X,E)} dt\right| \le \left|\int_1^\infty \Big\langle D_t\Big(\gamma_t - 2\frac{\partial}{\partial t}g_t\Big)\psi_t, \psi_t\Big\rangle_{L^2(X,E)} dt\right| + K\|\psi\|^2_{L^2(X,E)}$$

where $K>0$ is a constant. Next we estimate the first term in the right side of the above inequality,

$$\begin{aligned}\left|\int_1^{t_0}\Big\langle D_t\Big(\gamma_t-2\frac{\partial}{\partial t}g_t\Big)\psi_t,\psi_t\Big\rangle_{L^2(X,E)}dt\right|\\ &=\left|\Big\langle\Big(\gamma_t-2\partial_t g_t\Big)\psi_t,\psi_t\Big\rangle_{L^2(X,E)}\right|_{t=1}^{t_0}\\ &\le\left|\big\|(\gamma_t-2\partial_t g_t)\psi_t\big\|_{L^2(X,E)}\right|_{t=1}^{t_0}\bullet\|\psi\|_{L^2(X,E)}\le C\|\psi\|_1^2,\end{aligned}$$

where the last inequality is true because Lemma 5.6 implies that the first order differential operator $\gamma_t-2\frac{\partial}{\partial t}g_t$ has bounded coefficients for $t\in[1,\infty)$ and hence it is continuous from $L^2(X,E)$ to $\mathscr{W}_1(X,E)$. Since the above inequality is true for arbitrary t_0 we have proved the lemma. ☑

We introduce and recall some notation:

$$g_{i,t}(x)\coloneqq t^{\delta}g^{(i)}\big(t^{-\delta}x\big),\qquad g_t\coloneqq\sum_{i=0}^{3}g_{i,t},\qquad \gamma_{i,t}\coloneqq D_t g_{i,t},\qquad \gamma_t=\sum_{i=0}^{3}\gamma_{i,t},$$

where D_t denotes the Heisenberg derivative defined in (23).

From part 3) of Lemma 5.3, it is easy to see that $g_t''(x)$ is a positive matrix for all $t\in[1,\infty)$ and $x\in X$. Therefore the matrix $B(x,t)\coloneqq\sqrt{g_t''(x)}$ is well defined. It is straightforward to prove the following proposition.

Proposition 5.9. *For $\varphi,\psi\in\mathscr{W}_1(X,E)$, the following equality holds*

$$\int_X\Big\langle p^T g_t'' p\psi,\varphi\Big\rangle(x)\,dvol(x)=\int_X\Big\langle Bp\psi,Bp\varphi\Big\rangle(x)\,dvol(x).$$

Let $\mathrm{Dom}(r)$ be the maximal domain in $L^2(X,E)$ of the operator defined by multiplication by the function r defined at the beginning of Section 5.1.

Proposition 5.10. *The domain $\mathscr{W}_1(X,E)\cap\mathrm{Dom}(r)$ is invariant under the action of e^{iHt}.*

Proof. Let $\varphi\in\mathscr{W}_1(X,E)\cap\mathrm{Dom}(r)$. Since e^{iHt} and $H^{1/2}$ commute, $e^{iHt}\varphi\in\mathscr{W}_1(X,E)$, for all $t\in\mathbb{R}$. We have to show that $re^{iHt}\varphi\in L^2(X,E)$. Let $\chi_n\in C_c^\infty(X)$ be such that $\chi_n(x)=1$ for $x\in X_n$, and such that its gradient $\nabla(\chi_n)$

and Laplacian $\Delta(\chi_n)$ are bounded uniformly. We have

$$\int_X \Big\langle e^{iHt}\chi_n r^2 e^{-iHt}\varphi,\varphi\Big\rangle(x)\,dvol(x) = \\ i\int_X\int_0^t \Big\langle e^{iHs}\big[H,\chi_n r^2\big]e^{-iHs}\varphi,\varphi\Big\rangle(x)\,ds\,dvol(x) + \\ \int_X \chi_n r^2\langle\varphi,\varphi\Big\rangle(x)\,dvol(x).$$

Let us see that the last integral is finite. By hypothesis $r\varphi \in L^2(X,E)$, hence we can apply Lebesgue convergence theorem to obtain

$$i\lim_{n\to\infty}\int_X \chi_n r^2\langle\varphi,\varphi\rangle(x)\,dvol(x) = \int_X r^2\langle\varphi,\varphi\rangle(x)\,dvol(x) < \infty.$$

Using that $\big[H,\chi_n r^2\big]$ is a first order differential operator with uniformly bounded coefficients and Fubini's Theorem we can prove that

$$i\int_X\int_0^t \Big\langle e^{iHs}\big[H,\chi_n r^2\big]e^{-iHs}\varphi,\varphi\Big\rangle(x)\,ds\,dvol(x) \le Ct\|\varphi\|_1.$$

Lebesgue convergence theorem implies

$$i\int_X\int_0^t \Big\langle e^{iHs}\big[H,r^2\big]e^{-iHs}\varphi,\varphi\Big\rangle(x)\,ds\,dvol(x) \\ = \lim_{n\to\infty} i\int_X\int_0^t \Big\langle e^{iHs}\big[H,\chi_n r^2\big]e^{-iHs}\varphi,\varphi\Big\rangle(x)\,ds\,dvol(x) < \infty. \quad ☑$$

The above proposition shows that the Heisenberg observables $e^{iHt}\gamma_t e^{-iHt}$ and $e^{iHt}g_t e^{-iHt}$ are defined in the dense domain $\mathscr{W}_1(X,E)\cap \mathrm{Dom}(r)$.

Theorem 5.11. (cf. [11, theorem 7.6])

(1) The strong limits

$$\gamma^+ := s-\lim_{t\to\infty} e^{iHt}\gamma_t e^{-iHt},\\ \gamma_k^+ := s-\lim_{t\to\infty} e^{iHt}\gamma_{k,t} e^{-iHt}$$

exist on $\mathscr{W}_1(X,E)$ with respect to L^2-norm.

(2) γ^+ and γ_k^+ have the following properties:

$$\gamma_0^+ = [\gamma^+,H] = [\gamma_k^+,H] = 0,\\ \gamma^+ = s-\lim_{t\to\infty}\frac{e^{iHt}g_t e^{-iHt}}{t} \ge 0,\\ \gamma_k^+ = s-\lim_{t\to\infty}\frac{e^{iHt}g_{k,t} e^{-iHt}}{t} \ge 0,$$

and

$$\gamma^+ = \sum_k \gamma_k^+$$

where the last strong limits are taken over $\mathscr{W}_1(X,E)\cap\mathrm{Dom}(r)$ *with respect to the norm* $\|\bullet\|_{L^2(X,E)}$.

(3) γ^+ *and* γ_k^+ *are independent of* $\delta\in(1/3,1)$. *Moreover, we have*

$$\gamma^+ = s-\lim_{t\to\infty} e^{iHt}\frac{g(x)}{t}e^{-iHt},$$

where the strong limit is taken over $\mathscr{W}_1(X,E)\cap\mathrm{Dom}(r)$, *and where* $g(x)$ *is the unscaled Graf-Yafaev function (similar roles play the functions* $g^{(k)}$ *for the operators* γ_k^+*).*

Theorem 5.11 will be proved later on. We observe for the moment that from Property 2) we can deduce $\gamma_0^+=0$. Intuitively, the importance of the operators γ_1^+, γ_2^+ and γ_3^+ is that they allow us to localize the absolutely continuous states of H into the regions $Z_1\times\mathbb{R}_+$, $Z_2\times\mathbb{R}_+$ and $Y\times\mathbb{R}_+^2$ associated with the domains of the operators H_1, H_2 and H_3.

We will use the following proposition to prove the existence of γ^+.

Proposition 5.12. *If one of the following limits exists, then*

$$s-\lim_{t\to\infty} e^{iHt}\gamma_t e^{-iHt}(H-\lambda)^{-2} = s-\lim_{t\to\infty}(H-\lambda)^{-1}e^{iHt}\gamma_t e^{-iHt}(H-\lambda)^{-1}$$

Proof. We have that

$$\begin{aligned}(H-\lambda)^{-1}e^{iHt}\gamma_t e^{-iHt}(H-\lambda)^{-1} &= e^{iHt}(H-\lambda)^{-1}\gamma_t(H-\lambda)^{-1}e^{-iHt}\\ &= e^{iHt}\gamma_t(H-\lambda)^{-2}e^{-iHt} - e^{iHt}(H-\lambda)^{-1}[\gamma_t,H]e^{-iHt}(H-\lambda)^{-2}.\end{aligned}$$

Then, to prove the proposition it is enough to prove

$$s-\lim_{t\to\infty} e^{iHt}(H-\lambda)^{-1}[\gamma_t,H]e^{-iHt}(H-\lambda)^{-2}=0. \tag{24}$$

By Lemma 5.6, $\left\|\frac{\partial}{\partial t}(g_t)\right\|_{0,0}=O\left(t^{\delta-1}\right)$, where $\|\bullet\|_{0,0}$ denotes the norm of the bounded linear operators acting in $L^2(X,E)$. Then we have

$$\begin{aligned}&s-\lim_{t\to\infty} e^{iHt}(H-\lambda)^{-1}[\gamma_t,H]e^{-iHt}(H-\lambda)^{-2}\\ &\qquad = s-\lim_{t\to\infty} e^{iHt}\left[(H-\lambda)^{-1},\gamma_t-\frac{\partial}{\partial t}(g_t)\right]e^{-iHt}(H-\lambda)^{-1}.\end{aligned}$$

Let $\psi \coloneqq (H-\lambda)^{-1}\varphi$, for $\varphi \in L^2(X,E)$. We have

$$\begin{aligned}\left\|\left[(H-\lambda)^{-1}, \gamma_t - \frac{\partial}{\partial t}(g_t)\right]\psi\right\| &= \left\|\left[(H-\lambda)^{-1}, [H, g_t]\right]\right\|_{L^2(X,E)} \\ &\leq \left\|(H-\lambda)^{-1}\left[H, \sum_{i=1}^{2}\left\{-\partial_{ii}(g_t) - 2\partial_i(g_t)\partial_i\right\}\right](H-\lambda)^{-1}\psi\right\|_{L^2(X,E)} \\ &\leq \sum_{j,i=1}^{2}\left\|(H-\lambda)^{-1}\left\{\partial_{jjii}(g_t) + 2\partial_{ji}(g_t)\partial_{ij}\right\}(H-\lambda)^{-1}\psi\right\|_{L^2(X,E)}.\end{aligned}$$

Using Lemma 5.6, one can prove $\left\|\partial_{iijj}(g_t)\right\|_{0,0} \leq Ct^{-3\delta}$; that implies

$$\left\|(H-\lambda)^{-1}\partial_{jjii}(g_t)(H-\lambda)^{-1}\psi\right\|_{L^2(X,E)} \leq Ct^{-3\delta}.$$

Now we analyze the term $\left\|(H-\lambda)^{-1}\partial_{ji}(g_t)\partial_{ij}(H-\lambda)^{-1}\psi\right\|_{L^2(X,E)}$. Since $\partial_{ji}(g_t)\partial_{ij}$ is a second order differential operator with coefficients bounded uniformly in $x \in X$ and $t \in \mathbb{R}$, it defines a continuous operator from $\mathscr{W}_2(X,E)$ to $L^2(X,E)$. Hence

$$\begin{aligned}&\left\|(H-\lambda)^{-1}\partial_{ji}(g_t)\partial_{ij}(H-\lambda)^{-1}\psi\right\|_{L^2(X,E)} \\ &\qquad \leq \left\|(H-\lambda)^{-1}\right\|_{0,2}\bullet\left\|\partial_{ij}(g_t)\partial_{ij}\right\|_{2,0}\bullet\left\|(H-\lambda)^{-1}\right\|_{0,2}\bullet\|\psi\|_{L^2(X,E)},\end{aligned}$$

where $\|\bullet\|_{k,l}$ denotes the operator norm from $\mathscr{W}_k(X,E)$ to $\mathscr{W}_l(X,E)$. We observe that, by Lemma 5.6, we have $\left\|\partial_{ij}(g_t)\partial_{ij}\right\|_{2,0} \leq Ct^{\delta-1}$. This finishes the proof of the proposition. ☑

Proof of Theorem 5.11.

1. Existence of $\boldsymbol{\gamma^+}$ and $\boldsymbol{\gamma_k^+}$: Lemma 5.6 implies that $\left([H,g_t]\right)_{t\in\mathbb{R}_+}$ and $\left(\frac{\partial}{\partial t}(g_t)\right)_{t\in\mathbb{R}_+}$ have coefficients bounded uniformly in $t \in \mathbb{R}_+$ and $x \in X$ and then we can deduce the inequalities

$$\begin{aligned}\left\|e^{iHt}\frac{\partial}{\partial t}(g_t)e^{-iHt}\varphi\right\|_{L^2(X,E)} &\leq Ct^{\delta-1}\|\varphi\|_{L^2(X,E)} \\ \left\|e^{iHt}[H,g_t]e^{-iHt}\varphi\right\|_{L^2(X,E)} &\leq C\|\varphi\|_1,\end{aligned}$$

for all $\varphi \in \mathscr{W}_4(X,E) \subset \mathscr{W}_1(X,E)$. The previous estimates show that, assuming the existence of the limit, the function $\varphi \mapsto \lim_{t\to\infty} e^{iHt}\gamma_t e^{-iHt}\varphi$ would be a continuous linear map (as a function from $\mathscr{W}_1(X,E)$

to $L^2(X,E)\big)$. Since $\mathscr{W}_2(X,E) \subset \mathscr{W}_1(X,E)$ is dense with respect to the first Sobolev norm $\|\bullet\|_1$, it is enough to prove that the limit $\lim_{t\to\infty} e^{iHt}\gamma_t e^{-iHt}(H-\lambda)^{-2}$ exists and hence, by Proposition 5.12, it is enough to prove the existence of the limit $s-\lim_{t\to\infty}(H-\lambda)^{-1}e^{iHt}\gamma_t e^{-iHt}$ $(H-\lambda)^{-1}$ with respect to the norm $\|\bullet\|_{L^2(X,E)}$. Since $\big\|\frac{\partial}{\partial t}g_t\big\|_{0,0} = O(t^{\delta-1})$, we have

$$s - \lim_{t\to\infty}(H-\lambda)^{-1}e^{iHt}\gamma_t e^{-iHt}(H-\lambda)^{-1} = \\ s - \lim_{t\to\infty}(H-\lambda)^{-1}e^{iHt}\left(\gamma_t - 2\frac{\partial}{\partial t}(g_t)\right)e^{-iHt}(H-\lambda)^{-1}.$$

We will show the existence of the last limit with respect to the L^2-norm. We denote $\widetilde{\gamma}_t \coloneqq \gamma_t - 2\frac{\partial}{\partial t}(g_t)$.

Define $\varphi_t \coloneqq (H-\lambda)^{-1}e^{iHt}\widetilde{\gamma}_t e^{-iHt}(H-\lambda)^{-1}\psi$ for $\psi \in L^2(X,E)$. We will prove that $\int_1^\infty \big\|\frac{\partial}{\partial t}\varphi_t\big\|_{L^2(X,E)}\,dt$ is finite. Observe that

$$\frac{\partial}{\partial t}\varphi_t \coloneqq (H-\lambda)^{-1}e^{iHt}D_t\widetilde{\gamma}_t e^{-iHt}(H-\lambda)^{-1}\psi.$$

From Lemma 5.7, for $\delta > 1/3$, we can deduce

$$D_t\widetilde{\gamma}_t = p^T g_t'' p + L^2\text{-norm integrable in } t \text{ terms}.$$

Therefore it remains to prove that $u_t \coloneqq (H-\lambda)^{-1}e^{iHt}p^T g_t'' p e^{-iHt}$ $(H-\lambda)^{-1}\psi$ is L^2-norm integrable in $[1,\infty)$. We use Cauchy–Schwarz inequality and Proposition 5.9 to prove

$$\begin{aligned}\int_1^s \|u_t\|^2_{L^2(X,E)}\,dt &= \int_1^s \sup_{\|v\|_{L^2(X,E)}=1} \big|\langle v, u_t\rangle_{L^2(X,E)}\big|^2\,dt \\ &\leq \sup_{\|v\|_{L^2(X,E)}=1}\int_1^s \big\|B_t p e^{-iHt}(H-\overline{\lambda})^{-1}v\big\|^2_{L^2(X,E)}\,dt \,\bullet \\ &\qquad \int_1^s \big\|B_t p e^{-iHt}(H-\lambda)^{-1}\psi\big\|^2_{L^2(X,E)}dt.\end{aligned}$$

By Lemma 5.8 the last two integrals are bounded; hence u_t is L^2–norm integrable in t. We have proved the existence of γ^+, the existence of γ_k^+ is proved following a very similar reasoning.

2. Proof of parts 2) and 3) of Theorem 5.11: Since γ^+ exists on $\mathscr{W}_1(X,E)$, it follows from (24) that $\gamma^+(H-\lambda)^{-1} = (H-\lambda)^{-1}\gamma^+$. Hence $[\gamma^+, H] = (H+\lambda)\big\{(H+\lambda)^{-1}\gamma^+ - \gamma^+(H+\lambda)^{-1}\big\}(H+\lambda) = 0$. A similar proof applies for γ_k^+.

Now we prove that $\lim_{t\to\infty} \frac{e^{iHt} g_t e^{-iHt}}{t}\varphi = \gamma^+\varphi$ for $\varphi \in \mathrm{Dom}(r) \cap \mathscr{W}_1(X,E)$ and where the limit is considered in the L^2-norm. Using that $e^{iHt}\gamma_t e^{-iHt} = \frac{\partial}{\partial t} e^{iHt} g_t e^{-iHt}$, we have

$$\gamma^+ = s - \lim_{t\to\infty} \frac{1}{t}\int_1^t \frac{\partial}{\partial s} e^{iHs} g_s e^{-iHs} ds = s - \lim_{t\to\infty} \frac{e^{iHt} g_t e^{-iHt}}{t} \geq 0.$$

Finally we prove part 3) of Theorem 5.11. Observe that $g_t = g$ for $x \in X \setminus X_R$ and for $R > \frac{l_0+\epsilon}{1+2\epsilon^2}$; hence $t^{-1}\|g_t - g\|_{L^2(X,E)} \leq Ct^{\delta-1}$. Part 3) follows from part 2) of the theorem and this fact. ☑

5.4. Propagation Observables and Mourre's Inequality

Next we discuss the connection between the operator γ^+ and Mourre's inequality enunciated in theorem 5.1.

Definition 5.13. (cf. [11, (6.17)])

A finite, open interval $I \subset \mathbb{R}$ will be called **a Mourre interval** if for all $\psi \in E_I(H) \cap \mathrm{Dom}(r)$,

$$\left\langle E_I(H) i\big[H, i\big[H, r^2\big]\big] E_I(H)\psi, \psi\right\rangle_{L^2(X,E)} \geq C\langle\psi,\psi\rangle_{L^2(X,E)}, \quad \text{for some } C > 0.$$

Lemma 5.14. (cf. [11, lemma 7.7]) *Let $\mathscr{H}_I := E_I(H)$ be the spectral subspace of H associated to a Mourre interval I. Then ${\gamma^+}^2$ reduces to a strictly positive operator $\mathscr{H}_I \to \mathscr{H}_I$. In particular $\mathscr{H}_I \subset \mathrm{Im}(\gamma^+)$.*

Proof. According to Theorem 5.11, γ^+ is H-bounded and commutes with H, then it reduces to $\mathscr{H}_I \to \mathscr{H}_I$. Let $\psi \in \mathscr{H}_I$. By Theorem 5.11 we have

$$\begin{aligned}\left\langle \psi, {\gamma^+}^2\psi\right\rangle_{L^2(X,E)} &= \lim_{t\to\infty}\frac{1}{t^2}\left\langle e^{2iHt}\psi, g_t^2 e^{2iHt}\psi\right\rangle_{L^2(X,E)} \\ &\geq \lim_{t\to\infty}\frac{1}{t^2}\left\langle e^{2iHt}\psi, r^2 e^{2iHt}\psi\right\rangle_{L^2(X,E)}.\end{aligned}$$

Define the function $h(t) := \left\langle e^{2iHt}\psi, r(x)^2 e^{2iHt}\psi\right\rangle_{L^2(X,E)}$. Since I is a Mourre interval, there exists $c > 0$ such that $h''(t) \geq c > 0$. Then, there exist $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$ such that $h(t) \geq ct^2 + c_1 t + c_2$ and

$$\lim_{t\to\infty}\frac{1}{t^2}\left\langle e^{2iHt}\psi, r(x)^2 e^{2iHt}\psi\right\rangle_{L^2(X,E)} \geq c > 0. \qquad ☑$$

As a consequence of Theorem 5.1 we have that if $\lambda \in \mathbb{R}$ is not an L^2–eigenvalue nor a threshold of H, then λ belongs to some Mourre interval I. In [4] and [3] it is proved by different methods that the set of L^2–eigenvalues of H is countable and it accumulates only in the set of thresholds $\sigma_{pp}(H^{(1)}) \cup \sigma_{pp}(H^{(2)}) \cup \sigma_{pp}(H^{(3)})$. The next corollary follows from these facts.

Corollary 5.15. (cf. [11, page 3480]) *The sum of eigenspaces $E_I(H)$, associated to Mourre intervals I, is a L^2-dense set on the absolutely continuous part of H.*

5.5. Deift-Simon Wave Operators

The proof of the following theorem follows the same lines of the proof of the existence of γ^+ and γ_k^+ in Theorem 5.11 and the proof of similar facts given in [11, page 3492], because of this we omit the proof here.

Theorem 5.16. (cf. [11, page 3492]) *For $k = 1, 2, 3$, the Deift-Simon wave operators,*

$$\omega_k := s - \lim_{t\to\infty} e^{iH_k t}\gamma_{k,t} e^{-iHt},$$

exist, with respect to the L^2-norm, on $\mathscr{W}_1(X, E)$ for δ satisfying $\min(3\delta, 2 - \delta) < 1$.

As we explained below Theorem 5.11, intuitively the importance of the operators $\gamma_{k,t}$ is that they allow us to localize in the domains of the operators H_k the absolutely continuous states of H. In Theorem 5.16 we find states whose dynamics under H_k behave asimptotically as the dynamic of these localizations under H. We will formalize these intuitions in the next section.

5.6. Proof of Asymptotic Clustering

In this section we prove asymptotic clustering witch finishes the proof of Theorem 1.6. We say that $\psi \in L^2_{ac}(X, E)$ clusters asymptotically, if there exist $\varphi_k \in L^2_{pp}(Z_k, E_k) \otimes L^2(\mathbb{R}_+)$ for $k = 1, 2$ and $\varphi_3 \in L^2(Y \times \mathbb{R}^2_+, E)$ such that (2) holds.

Let $\psi \in E_I(H) \cap \mathscr{W}_2(X, E)$ for I a Mourre interval as defined in definition 5.13. By Lemma 5.14 and Theorem 5.11, we have

$$\psi = \sum_{k=1}^{3} \gamma_k^+ \varphi \approx \sum_{k=1}^{3} e^{iHt}\gamma_{k,t} e^{-iHt}\varphi,$$

where $\approx$ means that the difference of the two related expressions vanishes in L^2-norm as $t \to \infty$. Theorem 5.16 implies

$$\psi_t \approx \sum_{i=1}^{3} e^{-iH_k t}\varphi_k, \qquad \text{for} \quad \varphi_k := \omega_k^+\varphi; \tag{25}$$

that, with Corollary 5.15, imply that the wave operators $W_\pm(H_k, H)$ exist, for $k = 1, 2, 3$.

Proposition 5.17. *For all $\psi \in L^2_{ac}(X,E)$ there exist $\varphi_k \in L^2\big(Z_k \times \mathbb{R}_+, E\big)$, for $k=1,2,3$, such that*

$$\lim_{t\to\infty}\left\| e^{\pm iHt}\psi - \sum_{k=1}^{3} e^{\pm iH_k t}\varphi_k \right\|_{L^2(X,E)} = 0.$$

Proposition 5.17 is a kind of asymptotic completeness, however the sum of the wave operators $W_{\pm}(H,H_k)$ $(k=1,2,3)$ is not a direct sum, since their images are not necessarily orthogonal.

For $k=1,2$ and the φ_k's of (25), we have $\varphi_k = \Pi_{k,pp}\varphi_k + \Pi_{k,ac}\varphi_k$, where $\Pi_{k,pp}$ and $\Pi_{ac,d}$ denote the orthogonal projection over the closed subspaces of $L^2(X,E)$, $L^2_{pp}(Z_k,E_k)\otimes L^2\big(\mathbb{R}_+\big)$ and $L^2_{ac}(Z_k,E_k)\otimes L^2\big(\mathbb{R}_+\big)$. It is easy to see that $e^{\pm itH_k}\varphi_k = e^{\pm itH_{k,pp}}\Pi_{k,pp}\varphi_k + e^{\pm itH_{k,ac}}\Pi_{k,ac}\varphi_k$.

Since $\Pi_{k,ac}\varphi_k \in L^2_{ac}(Z_k,E_k)\otimes L^2\big(\mathbb{R}_+\big)$ and $W_{\pm}(H_k,H_3)$ is an isometry, there exists $\widetilde{\varphi}_k \in L^2\big(Y\times\mathbb{R}^2_+,E\big)$ such that $\Pi_{k,ac}\varphi_k = W_{\pm}(H_k,H_3)\widetilde{\varphi}_k$. We conclude

$$\begin{aligned} &e^{\pm iHt}\psi - \sum_{k=1}^{3} e^{\pm iH_k t}\varphi_k = \\ &\quad e^{\pm iHt}\psi - \sum_{k=1}^{2}\Big\{ e^{\pm iH_{k,ac}t}W_{\pm}(H_k,H_3)\widetilde{\varphi}_k - e^{\pm iH_{k,pp}t}\Pi_{k,pp}\varphi_k \Big\} - e^{i\pm H_3 t}\varphi_3. \end{aligned}$$

Observe that

$$\lim_{t\to\infty}\left\| e^{\pm iH_{k,ac}t}W_{\pm}(H_k,H_3)\widetilde{\varphi}_k - e^{\pm iH_3 t}\widetilde{\varphi}_k \right\|_{L^2(X,E)} = 0,$$

for $k=1,2$. The above computations imply

Proposition 5.18. *For all $\psi \in L^2_{ac}(X,E)$ there exist $\phi_k \in L^2_{pp}(Z_k,E_k)\otimes L^2\big(\mathbb{R}_+\big)$, for $k=1,2$, and $\varphi \in L^2\big(Y\times\mathbb{R}^2_+,E\big)$, such that*

$$\lim_{t\to\infty}\left\| e^{\pm iHt}\psi - e^{\pm iH_3 t}\varphi - \sum_{k=1}^{2} e^{\pm iH_{k,pp}}\phi_k \right\|_{L^2(X,E)} = 0.$$

Let $\varphi \in L^2\big(Y\times\mathbb{R}^2_+,E\big)$, we can calculate

$$\begin{aligned} e^{\pm iH_3 t}\varphi &= e^{\pm iH_3 t}\varphi - e^{\pm iH_{1,ac}}W_{\pm}(H_1,H_3)\varphi + e^{\pm iH_3 t}\varphi - \\ &\qquad e^{\pm iH_{2,ac}}W_{\pm}(H_2,H_3)\varphi - e^{\pm iH_3 t}\varphi + \sum_{k=1}^{2} e^{\pm iH_{k,ac}}W_{\pm}(H_k,H_3)\varphi. \end{aligned}$$

Observe that for all $\varphi \in L^2\big(Y \times \mathbb{R}^2_+, E\big)$, we have

$$\lim_{t\to\infty} \left\| e^{\pm iH_3 t}\varphi - e^{\pm iH_{k,ac}} W_\pm(H_k, H_3)\varphi \right\|_{L^2(X,E)} = 0,$$

for $k = 1, 2$. Hence,

$$e^{\pm iH_3 t}\varphi \approx -e^{\pm iH_3 t}\varphi + \sum_{k=1}^{2} e^{\pm iH_{k,ac}} W_\pm(H_k, H_3)\varphi.$$

Proposition 5.18 and the previous computation imply asymptotic clustering and hence Theorem 1.6.

Let us denote

$$\mathscr{W}_\pm := W_\pm(H, H_3) \oplus \bigoplus_{k=1}^{2} W_\pm\big(H, H_{k,pp}\big),$$

acting from $L^2\big(Y \times \mathbb{R}^2_+\big) \oplus \bigoplus \big(L^2_{pp}(Z_k, E_k) \otimes L^2\big(\mathbb{R}_+\big)\big)$ to $L^2_{ac}(X, E)$. We define the scattering operator

$$\mathscr{S} := \big(\mathscr{W}_-\big)^{-1}\mathscr{W}_+ \tag{26}$$

In a forthcoming article, we plan to study how the scattering operator $\mathscr{S}$ encodes geometric information, particularly we would like to generalize the approach of [16] to prove a signature formula that would be closely related with the formulas of [10].

A. Stationary Phase Methods

Let $V \in C_c^\infty(\mathbb{R})$. In this appendix we prove $\int_{-\infty}^{\infty} \left\| V e^{it\widetilde{b}} u \right\| dt < \infty$ where $\widetilde{b}$ is the self–adjoint operator associated to $-\frac{d^2}{dx^2} : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$. We use stationary phase methods as explained in [19]. Let $u \in \mathscr{S}(\mathbb{R})$ be such that $\widehat{u}$ has compact support contained in an interval $[a, d]$. Here $\widehat{u}$ denotes the Fourier transform of u. Let

$$u_t(x) := e^{it\widetilde{b}}u = (\frac{1}{2\pi})^{1/2} \int \exp\big[it(xk - tk^2)\big]\widehat{u}(k)\, dk.$$

From [19, Corollary, page 38] we see that for all m there exists a c depending on m, u and the interval $[a, b]$ such that

$$|u_t(x)| \le c(1 + |x| + t)^{-m}$$

for all x, t such that $x/t \notin [a, d]$. From this we deduce that

$$\left(\int_{-\infty}^{at} + \int_{dt}^{\infty}\right) \big|V(x)\big|^2 \big|u_t(x)\big|^2\, dx \le c(1 + t)^{-2}. \tag{27}$$

[19, Corollary, page 41] proves $\left|u_{\pm t}(x)\right|^2 \leq Ct^{-1}$ for $t > 1$, then

$$\int_1^\infty \left(\int_{at}^{dt} \left|V(x)\right|^2 \left|u_t(x)\right|^2 dx \right) dt \leq c \int_1^\infty t^{-1/2} \left(\int_{at}^{dt} \left|V(x)\right|^2 dx \right) dt.$$

Making the change of variables $x = xt$ we obtain that, for all $m \in \mathbb{N}$, there exists a C such that $\left| \int_{at}^{dt} \left|V(x)\right|^2 dx \right| \leq \frac{Ct}{1+t^m}$. Then

$$\int_{at}^{dt} \left|V(x)\right|^2 \left|u_t(x)\right|^2 dx \leq C \frac{t^{1/2}}{1+t^5}. \tag{28}$$

(27) and (28) show that $\int_{-\infty}^{\infty} \left\| V e^{it\widetilde{b}} u \right\| dt < \infty$.

Next we make some classical comments in order to extend the previous estimates to $\left\| V e^{itb} \varphi \right\|$ where b is the self–adjoint operator associated to $-\frac{d^2}{dx^2}$: $C_c^\infty(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with Dirichlet boundary conditions at 0. We observe that for all $u \in \mathscr{S}\big((0,\infty)\big)$ such that $\widehat{u} \in C^\infty\big((0,\infty)\big)$, the function

$$\widetilde{u}(x) := \begin{cases} u(x), & x \in (0,\infty), \\ 0, & x = 0; \\ -u(-x), & \text{otherwise.} \end{cases}$$

is an odd function in $\mathscr{S}(\mathbb{R})$ such that $\widehat{\widetilde{u}}$ has compact support. Since $\widetilde{u}$ is odd, $\widehat{\widetilde{u}} = 2i \int_0^\infty \sin(xy) u(y)\, dy$. From these observations, (27) and (28), we deduce $\int_{-\infty}^{\infty} \left\| V e^{itb} u \right\| dt < \infty$.

Acknowledgements. This article is based on results obtained during the doctoral studies of the author, realized under the supervision of Werner Müller, at the University of Bonn as well as work carried out at Universidad de los Andes during the author's postdoc. He wants to thank both institutions for the nice environments they provided and Professor Müller for his continuous support.

References

[1] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral Asymmetry and Riemannian Geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[2] Ulrich Bunke and Martin Olbrich, *Scattering Theory for Geometrically Finite Groups*, Geometry, analysis and topology of discrete groups, Adv. Lect. Math. (ALM), vol. 6, Int. Press, Somerville, MA, 2008, pp. 40–136.

[3] L. Cano, *Analytic Dilation on Complete Manifolds With Corners of Codimension 2*, Ph.D. thesis, Bonn University, 2011, pp. 1–117.

[4] ______, *Mourre Estimates for Compatible Laplacians on Complete Manifolds with Corners of Codimension 2*, Ann. Glob. Anal. Geom. **43** (2013), 75–97.

[5] G. Carron, *Déterminant relatif et la fonction Xi*, Amer. J. Math. **124** (2002), no. 2, 307–352 (fr).

[6] C. Gordon, P. Perry, and D. Schueth, *Isospectral and Isoscattering Manifolds: A Survey of Techniques and Examples*, Geometry, spectral theory, groups, and dynamics, Contemp. Math., vol. 387, Amer. Math. Soc., Providence, USA, 2005, pp. 157–179.

[7] G. M. Graf, *Asymptotic Completeness for N-Body Short-Range Quantum Systems: A New Proof*, Comm. Math. Phys. **132** (1990), no. 1, 73–101.

[8] C. R. Graham and M. Zworski, *Scattering Matrix in Conformal Geometry*, Invent. Math. **152** (2003), no. 1, 89–118.

[9] L. Guillopé, *Théorie spectrale de quelques variétés à bouts*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 137–160 (fr).

[10] A. Hassell, R. Mazzeo, and R. B. Melrose, *A Signature Formula for Manifolds With Corners of Codimension Two*, Topology **36** (1997), no. 5, 1055–1075.

[11] W. Hunziker and I. M. Sigal, *The Quantum N-Body Problem*, J. Math. Phys. **41** (2000), no. 6, 3448–3510.

[12] ______, *Time-Dependent Scattering Theory of N-Body Quantum Systems*, Rev. Math. Phys. **12** (2000), no. 8, 1033–1084.

[13] R. Husseini, *Zur Spektraltheorie Verallgemeinerter Laplace-Operatoren Auf Mannigfaltigkeiten Mit Zylindrischen Enden*, Diplomarbeit, Rheinischen Friedrich-Wilhelms-Universität Bonn (2005) (de).

[14] R. Mazzeo and A. Vasy, *Scattering Theory on $SL(3)/SO(3)$: Connections with Quantum 3-Body Scattering*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 3, 545–593.

[15] R. B. Melrose, *Spectral and Scattering Theory for the Laplacian on Asymptotically Euclidian Spaces*, Spectral and scattering theory (Sanda, 1992), Lecture Notes in Pure and Appl. Math., vol. 161, Dekker, New York, USA, pp. 85–130.

[16] W. Müller, *On the L^2-Index of Dirac Operators on Manifolds with Corners of Codimension Two I*, J. Differential Geom. **44** (1996), no. 1, 97–177.

[17] W. Müller and G. Salomonsen, *Scattering Theory for the Laplacian on Manifolds with Bounded Curvature*, J. Funct. Anal. **253** (2007), no. 1, 158–206.

[18] W. Müller and A. Strohmaier, *Scattering at Low Energies on Manifolds with Cylindrical Ends and Stable Systoles*, Geom. Funct. Anal. **20** (2010), no. 3, 741–778.

[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. III*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, USA, 1979.

[20] S. Richard and R. Tiedra de Aldecoa, *Spectral Analysis and Time-Dependent Scattering Theory on Manifolds with Asymptotically Cylindrical Ends*, Rev. Math. Phys. **25** (2013), no. 2, 1350003, 40.

[21] M. A. Shubin, *Spectral Theory of Elliptic Operators on Non-Compact Manifolds*, Paper on lectures Summer School on Semiclassical Methods, Nantes (1991).

[22] I. M. Sigal and A. Soffer, *The N-Particle Scattering Problem: Asymptotic Completeness for Short-Range Systems*, Ann. of Math. (2) **126** (1987), no. 1, 35–108.

[23] ______, *Long-Range Many-Body Scattering. Asymptotic Clustering for Coulomb-Type Potentials*, Invent. Math. **99** (1990), no. 1, 115–143.

[24] M. E. Taylor, *Partial Differential Equations I. Basic Theory*, second ed., Applied Mathematical Sciences, vol. 115, Springer, New York, USA, 2011.

[25] A. Vasy, *Geometry and Analysis in Many-Body Scattering*, Inside out: inverse problems and applications, Math. Sci. Res. Inst. Publ., vol. 47, Cambridge Univ. Press, Cambridge, UK, 2003, pp. 333–379.

[26] D. Yafaev, *Radiation Conditions and Scattering Theory for N-Particle Hamiltonians*, Comm. Math. Phys. **154** (1993), no. 3, 523–554.

[27] D. R. Yafaev, *Mathematical Scattering Theory*, Translations of Mathematical Monographs, vol. 105, American Mathematical Society, Providence, USA, 1992, General theory, Translated from the Russian by J. R. Schulenberger.

(Recibido en agosto de 2014. Aceptado en febrero de 2015)

Departamento de Matemáticas
Universidad Sergio Arboleda
Facultad de Ciencias
Carrera 14, calle 74
Bogotá, Colombia
e-mail: leonardo.cano@usa.edu.co