Revista Colombiana de Matemáticas
Volumen 49(2015)1, páginas 161-170

Convex Lattice-Ordered Subrings of von Neumann Regular f-Rings

Subanillos reticulados convexos de f-anillos von Neumann regulares

Jorge I. Guier

Universidad de Costa Rica, San José, Costa Rica

Abstract. The purpose of this paper is to characterize the lattice-ordered convex subrings of von Neumann regular f-rings. They turn out to be the reduced projectable f-rings satisfying the convexity property, i.e.: for all a, b, if $0 < a < b$ then b divides a. A real closed version of this result can also be stated.

Key words and phrases. Lattice-ordered ring, projectable f-ring, von Neumann regular ring, convex subring, first convexity property, real closed ring, ring of quotients, valuation ring.

2010 Mathematics Subject Classification. 13J25, 06E20.

Resumen. El propósito de este artículo es caracterizar los subanillos convexos de los f-anillos von Neumann regulares. Estos son los f-anillos reducidos, proyectables y que satisfacen la propiedad de convexidad, i.e.: para todo a, b, si $0 < a < b$ entonces b divide a. También se da una versión real cerrada de este resultado.

Palabras y frases clave. Anillo reticulado, f-anillo proyectable, anillo von Neumann regular, subanillo convexo, primera propiedad de convexidad, anillo real cerrado, anillo de cocientes, anillo de valuación.

1. Preliminaries

In the present work, all rings will be commutative with unity. For such a ring A, we denote by qA the total (or classical) ring of quotients of A. This is the ring of quotients $S^{-1}A$ where S is the multiplicative set of all non zero-divisors. Two elements $\frac{a}{s}, \frac{b}{t} \in qA$ (where $a, b \in A$ and s, t are non zero-divisors) are equal if and only if $at - bs = 0$. For these constructions we refer to [10, 1]. It is well known that for any ring A, the ring A is qA if and only if every non-invertible element of A is a zero divisor ([10, Exercise 7, page 42]). With this fact, it is easy to see that $q(qA) = qA$ for any ring A. If $0 \neq \frac{a}{s}$ is a non-invertible element of qA, then a is a zero divisor (otherwise $\frac{s}{a}$ would be an element of qA such that $\frac{a}{s} \cdot \frac{s}{a} = 1$, contradicting the non-invertibility of $\frac{a}{s}$). Then $\frac{a}{s}$ is a zero divisor, because $\frac{a}{s} \cdot \frac{b}{1} = \frac{ab}{s} = \frac{0}{s} = 0$ where $b \in A$ is a nonzero element such that $ab = 0$.

An ordered ring $(A, \leqslant)$ is a lattice-ordered ring (also called an l-ring) if the set $(A, \leqslant)$ is a lattice, i.e.: there exist binary operations $\wedge$ and $\vee$ on A such that $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$, for all $a, b \in A$. A lattice-ordered ring $(A, \leqslant, \wedge, \vee)$ is an f-ring (for function ring), if A is isomorphic to a sub-direct product of a family of totally ordered rings. The reader may refer to [2, chapters 8 and 9] for basic properties concerning lattice ordered rings and f-rings.

In [12], the author mentions that if A is an f-ring then qA is also an f-ring, where the lattice-order structure of qA is given by $\frac{a}{b} \vee 0 = \frac{a \vee 0}{b}$ for any $a \in A$, where $b \in A$ is a non zero-divisor with $b > 0$. In the case of a reduced f-ring A, qA is then a reduced f-ring where A is an f-subring of qA by the embedding $i\colon A \to qA : a \mapsto \frac{a}{1}$. Furthermore, [12, Proposition 1.1] proves that if A is a reduced f-ring then the space πA of minimal prime ideals of A is homeomorphic to the space $\pi(qA)$ of minimal prime ideals of qA, and the homeomorphism is given by $\varphi\colon \pi(qA) \to \pi A : P \mapsto P \cap A = P^c$, the contraction.

Projectability of f-rings was introduced by Keimel in [8] in order to have a representation theorem in terms of continuous sections of Hausdorff sheaves of totally ordered rings over a boolean space. A ring A is *projectable* if $A = a^{\perp} + a^{\perp\perp}$ for all $a \in A$. In the proof of Theorem 2.4 of [12], the author proves that if A is a projectable f-ring then qA is also a projectable f-ring. Moreover, Proposition 1.2 of [12] says that for a reduced f-ring A such that $A = qA$, the ring A is projectable if and only if A is a von Neumann regular ring. The previous discussion and the fact that $q(qA) = qA$ for any commutative ring A gives the next result.

Proposition 1.1. *If A is a reduced projectable f-ring, then qA is a von Neumann regular ring.*

2. Convexity in von Neumann Regular Rings

The following lemma explains how the order of qA is deduced from the definition of the lattice structure on qA, (cf. [12, p. 356]), and it is given with respect to the order of A.

Lemma 2.1. *For any f-ring A, any $a, b \in A$, and any positive non zero-divisors $s, t \in A$, we have that $\frac{a}{s} < \frac{b}{t}$ in qA if and only if $at < bs$ in A.*

Proof. This follows from the equivalences:

$$\begin{aligned}
\frac{a}{s} \leqslant \frac{b}{t} &\iff 0 \leqslant \frac{b}{t} - \frac{a}{s} = \frac{bs - at}{st} \\
&\iff 0 = 0 \wedge \frac{bs-at}{st} = -\left(0 \vee -\frac{bs-at}{st}\right) \\
&\iff 0 = 0 \vee \frac{at-bs}{st} = \frac{0 \vee (at-bs)}{st} \\
&\iff 0 = 0 \vee (at - bs) \\
&\iff at - bs \leqslant 0 \\
&\iff at \leqslant bs.
\end{aligned}$$

Therefore we also have

$$\frac{a}{s} < \frac{b}{t} \iff at < bs. \qquad ☑$$

An ordered ring A satisfies the *first convexity property* if the formula $\forall a \forall b (0 < a < b \to b \mid a)$ is true in A. This definition is a special case of the n-th convexity property in [11] and it also appears in [6] as the divisibility property. The first convexity property may be restated in terms of the embedding of the ring in its total ring of quotients. The next proposition can be deduced from Theorem 7.6 and Proposition 6.10 (i) and (ii) respectively in [9]. We must first observe that a projectable f-ring has dominating units; cf. Definition 2 of Section 7 in [9]. We give here a simpler proof in the case of projectability, but we need first of all the notion of a Boolean product. According to [3] (or [4]), a ring A is a Boolean product of an indexed family $(A_x)_{x\in X}$, $X \neq \varnothing$, of rings if A is a subdirect product of $\prod_{x\in X} A_x$ where X can be endowed with a Boolean space topology so that:

(i) $[\![a = b]\!] = \big\{x \in X : a(x) = b(x)\big\}$ is a clopen set of X, for all $a, b \in A$; and

(ii) if $a, b \in A$ and N is a clopen set of X, then the element $c = a_{\restriction N} \cup b_{\restriction X \smallsetminus N}$ must be in A. The element c is a in N and b in $X \smallsetminus N$.

As usual, we will denote this by $A \in \Gamma^a\big(X, (A_x)_{x\in X}\big)$.

Proposition 2.2. *Let A be an f-ring.*

(i) If A satisfies the first convexity property, then the injective map $\iota\colon A \hookrightarrow qA : a \mapsto \frac{a}{1}$ is convex.

(ii) Suppose that A is a projectable ring. If the injective map $\iota\colon A \hookrightarrow qA : a \mapsto \frac{a}{1}$ is convex, then A satisfies the first convexity property.

Proof.

(i) Let $a, b, c \in A$ be such that $0 < \frac{a}{b} < \frac{c}{1}$, where $b \neq 0$ is a non zero-divisor, and we may suppose that b is positive. By Lemma 2.1, we have $0 < a < bc$. By the first convexity property of A, we have $bc \mid a$, i.e., there is $d \in A$ such that $(bc)d = b(cd) = a$. Therefore $\frac{a}{b} = cd \in A$, proving the convexity of the inclusion map $A \hookrightarrow qA$.

(ii) Let $a, b \in A$ be such that $0 < a < b$.

If b is a non zero-divisor, then we may consider $\frac{a}{b} \in qA$. Since $b > 0$, by Lemma 2.1 we have $0 < \frac{a}{b} < 1$. By convexity of the map ι, we have $\frac{a}{b} = c \in A$. This means $b \mid a$ in A.

If b is a zero divisor, we use the projectability of A and the representation Theorem of Keimel in [8] in order to get $A \in \Gamma^a\big(\pi A, (A/p)_{p\in\pi A}\big)$, where πA is the space of minimal prime ideals of A. Since b is a zero divisor, the clopen set $N = [\![b = 0]\!] = \{p \in \pi A : b(p) = 0\}$ is nonempty. The inequality $0 < a < b$ in A gives $0 \leqslant a(p) \leqslant b(p)$ for all $p \in \pi A$. The element $s = b_{\restriction \pi A \smallsetminus N} \cup 1_{\restriction N}$ belongs to A and is such that $0 < a < s$. Now $s \in A$ is a non zero-divisor. By the previous item we have $s \mid a$ in A, i.e., there exists $c \in A$ such that $sc = a$. We prove $bc = a$. It suffices to see that $b(p)c(p) = a(p)$ for each $p \in \pi A$. If $p \in \pi A \smallsetminus N$, this is clear by the construction of s. If $p \in N$, then $b(p) = 0$ and therefore $a(p) = 0$ by the inequality in A/p; thus it is also clear that $b(p)c(p) = a(p)$. ☑

Proposition 2.3. *If A is a reduced and projectable f-ring satisfying the first convexity property, there exists a von Neumann regular f-ring B such that A is a convex subring of B.*

Proof. It suffices to take $B = qA$ in Propositions 1.1 and 2.2. ☑

It is a well known fact that for a von-Neumann regular ring A, there exists a Boolean space X and a family $(A_x)_{x\in X}$ of fields such that $A \in \Gamma^a\big(X, (A_x)_{x\in X}\big)$. Moreover, X and the family $(A_x)_{x\in X}$ are uniquely determined by A. For an element a in a von-Neumann regular ring A, we call $a^* \in A$ the element defined by

$$a^* = \begin{cases} \frac{1}{a(x)}, & \text{if} \quad a(x) \neq 0; \\ 0, & \text{if} \quad a(x) = 0. \end{cases}$$

Note that $a^* = \big(\frac{1}{a}\big)_{\restriction N} \cup 0_{\restriction X \smallsetminus N}$, where $N = [\![a \neq 0]\!]$; and that aa^* is an idempotent of A.

The converse of Proposition 2.3 is also true, as the next two propositions will prove.

Proposition 2.4. *Let B be a von Neumann regular f-ring. If A is a convex lattice-ordered subring of B, then A is a projectable reduced f-ring.*

Proof. Obviously, A is a reduced f-ring, by [2, §9.1.2]. To prove that A is projectable, we take any $a \in A$ and we show that $A = a^{\perp} + a^{\perp\perp}$. Let $\alpha \in A$ be any element. Considering $a \in B$, there is $a^* \in B$ such that aa^* is an idempotent of B. Then the equality $\alpha = \alpha(aa^*) + \alpha(1 - aa^*)$ holds in B. Observe that $\big|\alpha(aa^*)\big| = |\alpha| \cdot \big|aa^*\big| \leqslant |\alpha|$, see [2, §9.4.17]. Since $\alpha \in A$, then $|\alpha| \in A$ too. Therefore we have $-|\alpha| \leqslant \alpha\big(aa^*\big) \leqslant |\alpha|$. The convexity of A in B proves that $\alpha\big(aa^*\big) \in A$. As $1 - aa^*$ is also an idempotent, we similarly have that $\alpha\big(1 - aa^*\big) \in A$.

It is clear that $c = \alpha\big(1-aa^*\big) \in a^{\perp}$ for $c \cdot a = \alpha\big(1-aa^*\big)a = \alpha\big(a-aa^*a\big) = \alpha(a-a) = \alpha \cdot 0 = 0$. We can now prove that $d = \alpha(aa^*) \in a^{\perp\perp}$. For this, take $y \in a^{\perp}$ (i.e., $ya = 0$) and then $yd = y\big(\alpha\big(aa^*\big)\big) = 0$. Therefore $y \perp d$ for all $y \in a^{\perp}$. We have shown that $\alpha = c + d$ where $c \in a^{\perp}$ and $d \in a^{\perp\perp}$, proving the projectability of A. ☑

Using Proposition 2.4, we can further prove the next result.

Proposition 2.5. *Let A be a lattice-ordered ring. If there exists a von Neumann regular f-ring B such that A is a convex lattice-ordered subring of B, then A satisfies the first convexity property.*

Proof. By Proposition 2.4, A is a projectable reduced f-ring. Therefore qA is a von Neumann regular f-ring. Now we prove that $g\colon qA \to B : \frac{a}{s} \mapsto as^*$, where $a \in A$ and $0 \neq s \in A$ is a non zero-divisor, is a well-defined injective homomorphism of ordered rings.

Observe first that if $0 \neq s \in A$ is a non zero-divisor, then $s^* \in B$ is such that $ss^* = 1$. In a representation of B as continuous sections of a Hausdorff sheaf of fields over the Boolean space $X = \pi B$ of prime ideals of B, we must have $N = \big\{x \in \pi B : s(x) = 0\big\} = \varnothing$; for if $N \neq \varnothing$ then $d = 1_{\restriction N} \cap 0_{\restriction X \smallsetminus N} \in B$ is a nonzero element with $sd = 0$. Then $s(x) \neq 0$ for all $x \in \pi B$, and this shows that $s^*(x) = \frac{1}{s(x)}$ for all $x \in \pi B$. In particular $ss^* = 1$. A similar argument shows that if $s, t \in A$ are non zero-divisors, we must have $(st)^* = s^*t^*$.

The map g is well defined, for if $a, b \in A$ and $s, t \in A$ are two non zero-divisors such that $\frac{a}{s} = \frac{b}{t}$ in qA, then $at = bs$ in A. Multiplying this equation by s^* and t^*, we get $as^* = bt^*$. The map g is clearly a ring homomorphism.

The map g is a homomorphism of *ordered* rings: let $\frac{a}{s} \in qA$ be such that $\frac{a}{s} \geqslant 0$, with $0 \neq s \in A$ a non zero-divisor. As we already know, we may suppose that $s > 0$. Looking at the element $s \in B$ in the representation of B as a Boolean sheaf of fields, we have $s^*(x) = \frac{1}{s(x)} > 0$ for all $x \in \pi B$. Therefore $s^* > 0$. We also know that $\frac{a}{s} \geqslant 0$ means $a \geqslant 0$. Therefore $as^* \geqslant 0$, and thus $g\big(\frac{a}{s}\big) \geqslant 0$.

The map g is injective: let $\frac{a}{s}, \frac{b}{t} \in qA$ with $a, b \in A$ and with $s, t \in A$ non zero-divisors, be such that $g\big(\frac{a}{s}\big) = g\big(\frac{b}{t}\big)$. Then $as^* = bt^*$. Multiplying this equation by st we easily get $at = bs$, i.e., $\frac{a}{s} = \frac{b}{t}$.

We have constructed $g\colon qA \to B$, a well-defined monomorphism of ordered rings. Further, the diagram

$$\begin{tikzcd} qA \arrow[rr, "g"] & & B \\ & A \arrow[ul, "\iota"] \arrow[ur, "j"'] & \end{tikzcd}$$

where $j\colon A \to B$ is the inclusion, is commutative. This is clear, as $\big(g \circ \iota\big)(a) = g\big(\iota(a)\big) = g\big(\frac{a}{1}\big) = a \cdot 1^* = a \cdot 1 = a$, for all $a \in A$.

From the convexity of the inclusion $j\colon A \to B$, it follows that $\iota\colon A \to qA : a \mapsto \frac{a}{1}$ is a convex embedding. If $\frac{a}{s} \in qA$ is such that $0 < \frac{a}{s} < b$, with $a, b \in A$ and $0 \neq s \in A$ a non zero-divisor, then on applying the injective order homomorphism g to this inequality, we get $0 < as^* < b$. By convexity of the inclusion $j\colon A \to B$, we obtain $as^* \in A$. Setting $c := as^* \in A$, we have $cs = \big(as^*\big)s = a\big(ss^*\big) = a1 = a$, and so $\frac{a}{s} = \frac{c}{1} \in A$. By Proposition 2.2, A satisfies the first convexity property. ☑

Putting together Propositions 2.3, 2.4 and 2.5, we arrive at the following result.

Theorem 2.6. *For a lattice ordered ring A, the following are equivalent:*

(i) A is a reduced and projectable f-ring satisfying the first convexity property;

(ii) there exists a von Neumann regular f-ring B such that A is a convex lattice-ordered subring of B.

In fact, [15, Corollary 2.4] proves that the ring B in the previous theorem must be qA.

3. Convexity in von Neumann Regular Real Closed Rings

A "real closed" version of this theorem can be proved. According to [13], a domain A is real closed if the following properties are satisfied: (i) the quotient field $\mathrm{cf}(A)$ of A is a real closed field, (ii) A is integrally closed, and (iii) if $0 \leqslant b \leqslant a$ then $b \mid a^2$ for all $a, b \in A$. A ring A is real closed (cf. [13]) if it has the following properties: (i) A is reduced, (ii) the sum of two radical ideals of A is a radical ideal of A, and, (iii) for every prime ideal p of A, the ring A/p is a real closed domain.

Boolean products of real closed valuation rings and real closed fields turned out to be projectable real closed rings satisfying the first convexity property [7, Proposition 3.4]. We are going to prove that these rings are precisely those whose total ring of quotients is a von Neumann regular real closed ring. We first prove an auxiliary result.

Proposition 3.1. *If A is a reduced and projectable f-ring, then*

$$qA \in \Gamma^a\big(\pi A, \mathrm{cf}(A/p)_{p\in\pi A}\big).$$

Proof. By the proof of [12, Theorem 2.4], by [8] and by Proposition 1.1 above, it suffices to prove that for each (minimal) prime ideal P of qA, there is an ordered-ring isomorphism between qA/P and $\mathrm{cf}(A/p)$, where $p = P \cap A$ is the minimal prime ideal associated to P by contraction.

Consider $\varphi : qA \to \mathrm{cf}(A/p) : \frac{a}{s} \mapsto \frac{a+p}{s+p}$. Since s is a non zero-divisor, $s \notin p = P \cap A$ (by the correspondence of prime ideals of qA and A). Then $s+p \neq 0$ and $\frac{a+p}{s+p} \in \mathrm{cf}(A/p)$ is well defined. The map φ is well defined; for let $\frac{a}{s}, \frac{b}{t} \in qA$ be such that $\frac{a}{s} = \frac{b}{t}$: then $at - bs = 0 \in p$ and $at + p = bs + p$, that is, $\frac{a+p}{s+p} = \frac{b+p}{t+p}$.

Clearly φ is a ring homomorphism. Also, $\ker\varphi = P$, for if $\frac{a}{s} \in qA$, then $\frac{a}{s} \in \ker\varphi$ iff $\frac{a+p}{s+p} = 0$ iff $a + p = 0$ iff $a \in p$ iff $a \in P \cap A$ iff $\frac{a}{1} \in P$ iff $\frac{a}{s} \in P$.

Moreover, φ is a surjective homomorphism. Take $\frac{a+p}{b+p} \in \mathrm{cf}(A/p)$. Then $b + p \neq 0$ with $b \in A$. Using Keimel's representation theorem [8], we get $b(p) \neq 0$. Since b is continuous, there exists a clopen subset $\mathcal{O}$ of πA such that $p \in \mathcal{O}$ and $b_{\restriction\mathcal{O}} \neq 0$. Define $s = b_{\restriction\mathcal{O}} \cup 1_{\restriction\pi A\smallsetminus\mathcal{O}}$ and therefore $s \in A$ is a non zero-divisor (since $s(q) \neq 0$ for all $q \in \pi A$). Since $s_{\restriction\mathcal{O}} = b_{\restriction\mathcal{O}}$ it follows that $s + p = s(p) = b(p) = b + p$ and so $\frac{a+p}{b+p} = \frac{a+p}{s+p} = \varphi\big(\frac{a}{s}\big)$ with $\frac{a}{s} \in qA$.

By the first isomorphism theorem, there is a ring homomorphism $\psi\colon qA/P \to \mathrm{cf}(A/p)$ given by $\frac{a}{s} + P \mapsto \frac{a+p}{s+p}$. We will see that ψ is an order isomorphism. Take $\frac{a}{s} \in qA$ such that $\frac{a}{s} + P > 0$. There exists $\frac{b}{t} \in P$ such that $\frac{a}{s} > \frac{b}{t}$. As usual, suppose that $s > 0$ and $t > 0$. Then $at > bs$ and $(a+p)(t+p) = at+p > bs + p = (b + p)(s + p)$. Since $s > 0$ and $t > 0$ then $s + p > 0$ and $t + p > 0$ and this implies $\frac{a+p}{s+p} > \frac{b+p}{t+p}$. But $\frac{b}{t} \in P$ and so $\frac{b+p}{t+p} = 0$ and therefore $\frac{a+p}{s+p} > 0$ in $\mathrm{cf}(A/p)$. We have shown that $\frac{a}{s} + P > 0$ implies $\psi(\frac{a}{s} + P) > 0$. Similarly, it can be proved that $\frac{a}{s} + P < 0$ implies $\psi(\frac{a}{s} + P) < 0$; and then $\frac{a}{s} + P > 0$ if and only if $\psi(\frac{a}{s} + P) > 0$. This permits us to conclude that ψ is an ordered-ring isomorphism, thereby proving the proposition. ☑

We can now establish the next proposition.

Proposition 3.2. *Let A be an f-ring. If A is a projectable real closed ring satisfying the first convexity property then qA is a von Neumann regular real closed ring.*

Proof. Let A be a projectable real closed ring satisfying the first convexity property. Then A/p is a real closed valuation ring, for every minimal prime ideal p of A ([7]). By Proposition 3.1, we have $qA \in \Gamma^a\big(\pi A, \mathrm{cf}(A/p)_{p\in\pi A}\big)$. And $\mathrm{cf}(A/p)$ are real closed fields, for every minimal prime ideal p of A; see [5]. Therefore qA is a Boolean product of real closed fields, i.e., a von Neumann regular real closed ring. ☑

In [14, Theorems 5.9 and 5.12], the author mentions that if B is a real closed ring and A is a convex subring of B, then A is a real closed ring and B is a ring of quotients of A with respect to the multiplicative set $S = A^+ \cap B^\times$ of positive elements of A invertible in B. With these results, a "real closed" version of Theorem 2.6 can be proved.

Theorem 3.3. *Let A be an f-ring. Then the following statements are equivalent:*

(i) A is a projectable real closed ring satisfying the first convexity property;

(ii) there exists a von Neumann regular real closed ring B such that A is a convex lattice-ordered subring of B.

Proof. The implication *(i)* $\Longrightarrow$ *(ii)* follows easily from Propositions 2.2 and 3.2.

(ii) $\Longrightarrow$ *(i)*: By Theorem 2.6, A is a reduced and projectable f-ring satisfying the first convexity property. By [14, Theorem 5.9], it follows that A is a real closed ring. ☑

By [14, Theorem 5.21], the ring B in Theorem 3.3 must be qA.

There is an easy way to see that A is a real closed ring in the proof *(ii)* $\Longrightarrow$ *(i)* above. First, we remind that an ordered ring $(A, \leqslant)$ satisfies the intermediate value property for one-variable polynomials if for any $p(x) \in A[x]$ a monic polynomial and $a, b \in A$ such that $p(a)p(b) < 0$, there is $c \in (a, b)$ such that $p(c) = 0$. By Propositions 2.4 and 3.4 of [7], it suffices to prove that A satisfies the intermediate value property for one-variable polynomials. Now take $p(x) \in A[x]$ a monic polynomial and $a, b \in A$ such that $p(a)p(b) < 0$. Note that B also satisfies the intermediate value property for one-variable polynomials: see the remarks after Proposition 2.4 of [7]. Therefore there exists $c \in B$ such that $p(c) = 0$ and $a < c < b$. The convexity of A in B then shows that $c \in A$.

References

[1] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, 1969.

[2] A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et anneaux réticulés*, Lectures Notes in Mathematics, vol. 608, Springer, 1977.

[3] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer, Berlin, Germany, 1981.

[4] S. Burris and H. Werner, *Sheaf Constructions and Their Elementary Properties*, Trans. Amer. Math. Soc. **248** (1979), 269–309.

[5] G. Cherlin and M. A. Dickmann, *Real Closed Rings II. Model Theory*, Ann. Pure Appl. Logic **25** (1983), 213–231.

[6] M. A. Dickmann, *Elimination of Quantifiers for Ordered Valuation Rings*, J. Symbolic Logic **52** (1987), 116–128.

[7] J. I. Guier, *Boolean Products of Real Closed Rings and Fields*, Ann. Pure Appl. Logic **112** (2001), 119–150.

[8] K. Keimel, *The Representation of Lattice-Ordered Groups and Rings by Sections of Sheaves*, Lectures Notes in Mathematics (Berlin, Germany), vol. 248, Springer, 1971, pp. 1–98.

[9] M. Knebusch, *Positivity and Convexity in Rings of Fractions*, Positivity **11** (2007), 639–686.

[10] J. Lambek, *Lectures on Rings and Modules*, Chelsea, New York, USA, 1976.

[11] S. Larson, *Convexity Conditions on f-Rings*, Canad. J. Math. **38** (1986), 48–64.

[12] J. Martínez, *The Maximal Ring of Quotient f-Ring*, Algebra Universalis **33** (1995), 355–369.

[13] N. Schwartz, *Real Closed Rings — Examples and Applications*, Séminaire de Structures Algébriques Ordonnées, vol. 61, 1995-96 (Delon, Dickmann, Gondard eds), Paris VII-CNRS Logique, Prépublications, Paris, France, 1997.

[14] ______, *Epimorphic Extensions and Prüfer Extensions of Partially Ordered Rings*, Manuscripta Math. **102** (2000), 347–381.

[15] ______, *Convex Subrings of Partially Ordered Rings*, Math. Nachr. **283** (2010), 758–774.

(Recibido en noviembre de 2014. Aceptado en mayo de 2015)

Centro de Investigación en Matemática Pura y Aplicada
Escuela de Matemática, Universidad de Costa Rica,
11501
San José, Costa Rica
e-mail: `jorge.guier@ucr.ac.cr`