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On the limit cycles of quasihomogeneous polynomial systems

Sobre los ciclos límite de sistemas polinomiales cuasi-homogeneos

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Abstract. In this work, the nonexistence of limit cycles for classes of $p-q$-quasi-homogeneous polynomial planar systems of weighted degree l is established. Furthermore, we rule out the existence of limits cycles for certain perturbations of such planar systems. We present applications and examples in order to illustrate our results.

Key words and phrases. Integrating factors, Inverse integrating factors, Limit cycles, $p-q$-quasi-homogeneous systems.

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Resumen. En este trabajo, se establece la no existencia de ciclos límite para la clase de sistemas bidimensionales, polinomiales $p-q$-cuasi-homogeneos de grado ponderado l. Además, se descarta la existencia de ciclos límite para ciertas perturbaciones de tales sistemas. Finalmente, se presentan aplicaciones y ejemplos para ilustrar los resultados obtenidos.

Palabras y frases clave. Factores Integrantes, Factores integrantes inversos, ciclos límite, sistemas $p-q$-cuasi-homogeneos.

1. Introduction

The qualitative theory of differential equations deals with the local and global properties of solutions of autonomous differential systems. The principal aim of this theory is the geometrical description of solutions of these systems. In the planar case, the existence (or nonexistence) of limit cycles is an important property that is used to characterize differential systems. Given an open set $U \subseteq \mathbb{R}^2$, we consider a system given by

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2), \end{cases} \tag{1}$$

where $f_i : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, and $1 \leq i \leq 2$ are C^1 functions. Consider the vector field $F := f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$, then the system (1) can be rewritten in the form

$$\dot{x} = F(x), \qquad\qquad x := (x_1, x_2) \in U. \tag{2}$$

Inverse integrating factors (IIFs) are useful tools in the study of qualitative properties of differential systems. In particular, the relationship of IIFs with limit cycles, integrability, symmetries, center problems, bifurcations and other properties has been extensively studied (see [1, 6, 5, 7, 8]). We now recall the definition of an inverse integrating factor:

Definition 1.1. A C^1 function $\mathcal{V} : U \to \mathbb{R}$, is said to be an *inverse integrating factor (IIF)* of the system (1) if it is not locally null and satisfies the following partial differential equation:

$$f_1 \frac{\partial \mathcal{V}}{\partial x_1} + f_2 \frac{\partial \mathcal{V}}{\partial x_2} = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \mathcal{V}. \tag{3}$$

The name inverse integrating factor for the function $\mathcal{V}$ arises from the fact that $1/\mathcal{V}$ is an integrating factor for the vector field F, that is, $F/\mathcal{V}$ restricted to $U \backslash \mathcal{V}^{-1}(0)$ is divergence-free. Note that the vector field F is tangent to the curve $\mathcal{V}(x_1, x_2) = 0$, so the set $\mathcal{V}(x_1, x_2) = 0$ is formed by solutions of (1). An important class of trajectories of (1) are given by

Definition 1.2. A *limit cycle* of the system (1) is a periodic orbit $\gamma \in U$ for which there is at least one other solution tending towards γ when $t \to +\infty$ or $t \to -\infty$.

It is well known that for a polynomial vector field a limit cycle is a periodic orbit which has an annulus-like neighborhood free of other periodic solutions.

The aim of this paper is to study the link between limit cycles and IIFs to rule out the existence of limit cycles for certain classes of differential systems, namely, $p-q$-quasi-homogeneous systems. Furthermore, we prove nonexistence of limit cycles for certain perturbations of such planar systems. We present applications and examples in order to illustrate our results.

2. $p - q$-quasi-homogeneous systems

A useful relationship between limit cycles and inverse integrating factors is established in the following theorem (see [8]), that relates limit cycles with the zero level curve of an IIF.

Theorem 2.1. ([8], th. 9) *Let $\mathcal{V} : U \to \mathbb{R}$ be an inverse integrating factor of the system* (1). *If $\gamma \subset U$ is a limit cycle of* (1), *then γ is contained in the set* $\mathcal{V}^{-1}(0) := \{(x_1, x_2) \in U \mid \mathcal{V}(x_1, x_2) = 0\}$.

Thus, one of the most important applications of IIFs is the localization of limit cycles. We take advantage of this link to study the existence of limit cycles of certain systems whose inverse integrating factors can be explicitly obtained. In particular, we focus on systems whose IIF is given by quasi-homogeneous polynomials.

Definition 2.2. Let $p, q, k, l \in \mathbb{Z}^+$. A real function $f : \mathbb{R}^2 \to \mathbb{R}$ is called a $p - q$-quasi-homogeneous function of weighted degree k if $f(\alpha^p x_1, \alpha^q x_2) = \alpha^k f(x_1, x_2)$, $\forall \alpha \in \mathbb{R} \setminus \{0\}$. A vector field $F = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$ is called a $p - q$-quasi-homogeneous vector field of weighted degree l, if f_1 and f_2 are $p - q$-quasi-homogeneous functions of weighted degree $p + l - 1$ and $q + l - 1$, respectively. A $p - q$-quasi-homogeneous differential system of weighted degree l, is determined by a $p - q$-quasi-homogeneous vector field.

For our purposes, we focus most of our results on $p - q$-quasi-homogeneous polynomials. We note however, that any $p - q$-quasi-homogeneous differential system of degree l is invariant under the similarity transformation:

$$(x_1, x_2, t) \to (\alpha^p x_1, \alpha^q x_2, \alpha^{-l+1} t), \ \forall \alpha \in \mathbb{R} \setminus \{0\},$$

a fact that can be readily seen from the definition of a quasi-homogeneous system.

A nice property of quasi-homogenous differential systems is that there is an explicit formula for the IIF of the system. This fact is established in the following proposition:

Proposition 2.3. *Given a $p - q$-quasi-homogeneous vector field $F = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$, then $\mathcal{V} = q x_2 f_1 - p x_1 f_2$ is an IIF of the system.*

The previous result follows from the use of the generalized Euler's theorem [3] for quasi-homogeneous functions which states that given f_1 a $p - q$-quasi-homogeneous function, we can obtain the equality

$$p x_1 \frac{\partial f_1}{\partial x_1} + q x_2 \frac{\partial f_1}{\partial x_2} = (p + l - 1) f_1.$$

Similarly, it is possible to obtain an expression for a $p - q$ quasi-homogenous function f_2.

Definition 2.4. Let $\mathbb{R}[x_1, x_2]$ be the polynomial ring over $\mathbb{R}$ in two variables. Given $P \in \mathbb{R}[x_1, x_2]$, define its zero set by

$$V(P) := \{(x_1, x_2) \in \mathbb{R}^2 \mid P(x_1, x_2) = 0\}.$$

Definition 2.5. If $S \subset \mathbb{R}[x_1, x_2]$, then let $V(S)$ be the set of common zeros

$$V(S) = \cap_{P \in S} V(P).$$

The set $V(P)$ is known to be an algebraic curve, and the set $\cap_{P \in S} V(P)$ is called an algebraic set. These sets will be important in the following study of properties of the zero sets of IIFs of quasi-homogeneous polynomials.

Lemma 2.6. *If P is a non-zero $p{-}q$-quasi-homogeneous polynomial of weighted degree k, then its zero set $V(P)$ contains no subset homeomorphic to $\mathbb{S}^1$.*

Proof. Let P be a non-zero $p-q$-quasi-homogeneous polynomial of weighted degree k. Then it satisfies $P(\lambda^p x, \lambda^q y) = \lambda^k P(x,y)$.

Suppose that $V(P)$ contains a subset γ which is homeomorphic to $\mathbb{S}^1$, then we have that $P(x_0, y_0) = 0$ for all $(x_0, y_0) \in \gamma$. For each point $(x_0, y_0) \in \gamma$ there is a curve given by $C(\lambda, (x_0, y_0)) := \{(\lambda^p x_0, \lambda^q y_0) : \lambda \in \mathbb{R}\}$.

Evaluating P at $C(\lambda, (x_0, y_0))$, we obtain that

$$P(\lambda^p x_0, \lambda^q y_0) = \lambda^k P(x_0, y_0) = 0 \text{ for all } (x_0, y_0) \in \gamma.$$

Thus, for all $\lambda \in \mathbb{R}$ and any point $(x,y) \in \gamma$ we have that $C(\lambda, (x,y)) \in V(P)$.

Consider a line L not totally contained in $V(P)$ such that it intersects an infinite number of curves $C(\lambda, (x,y))$ in $V(P)$. Since the degrees of P and L are finite, using a theorem by Bezout (see [9]) we obtain that L can only intersect V(P) at a finite number of points, leading to a contradiction. Thus, $V(P)$ does not have subsets homeomorphic to $\mathbb{S}^1$. ☑

As a direct consequence of the previous Lemma, we obtain the following Theorem:

Theorem 2.7. *If a non-zero $p-q$-quasi-homogeneous polynomial of weighted degree k is an IIF of the system* (1)*, then it has no limit cycles.*

Proof. Let $\mathcal{V}$ be a $p-q$-quasi-homogeneous polynomial which is an IIF of the system (1). Suppose that (1) admits a limit cycle γ. By Theorem 2.1 we obtain that $\gamma \in V(\mathcal{V})$, contradicting Lemma 2.6. Therefore, the system (1) does not contain limit cycles. ☑

Using Theorem 2.7 it is possible to establish the non-existence of limit cycles for quasi-homogeneous polynomial vector fields, as follows:

Proposition 2.8. *Given a $p-q$-quasi-homogeneous polynomial vector field of weighted degree l given by $F = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2}$, then the system* (1) *has no limit cycles.*

Proof. By Proposition 2.3 the function $\mathcal{V} = qx_2f_1 - px_1f_2$ is an IIF of the vector field F. Since $\mathcal{V}$ is a $p-q$-quasi-homogeneous function of weighted degree $p+q+l-1$, the result follows from Theorem 2.7. ☑

3. Perturbed quasi-homogeneous systems

We now extend our results to perturbed quasi-homogeneous systems, where the perturbation is given by quasi-homogeneous polynomials.

Definition 3.1. A quasidegenerate infinite system is a planar polynomial system of the form

$$\begin{cases} \dot{x}_1 &= P(x_1,x_2) + px_1A(x_1,x_2), \\ \dot{x}_2 &= Q(x_1,x_2) + qx_2A(x_1,x_2), \end{cases} \tag{4}$$

where $P\frac{\partial}{\partial x_1} + Q\frac{\partial}{\partial x_2}$ is a $p-q$-quasi-homogeneous polynomial vector field of weighted degree l and A is given by a $p-q$-quasi-homogeneous polynomial of weighted degree α.

Theorem 3.2. ([4], Th. 8) *Assume that H is a $p-q$ quasi-homogeneous polynomial of weighted degree d such that H is a first integral of the $p-q$ quasi-homogeneous vector field $P\frac{\partial}{\partial x_1} + Q\frac{\partial}{\partial x_2}$. Then, the function*

$$\mathcal{V} := (px_1Q - qx_2P)H^{\frac{\alpha-l+1}{d}}(x_1,x_2),$$

is an IIF of (4).

Having determined an IIF for these perturbed systems, we can use Theorem 2.7 and Theorem 3.2 to extend the nonexistence of limit cycles for such perturbed systems, as follows:

Proposition 3.3. *Suppose the hypotheses of Theorem 3.2 are satisfied, if $H^{\frac{\alpha-l+1}{d}}$ is a $p-q$-quasi-homogeneous polynomial, then the system* (4) *admits no limit cycles.*

Proof. Since $H^{\frac{\alpha-l+1}{d}}(x_1,x_2)$ is a $p-q$-quasi-homogeneous polynomial, it follows that the IIF $\mathcal{V} := (px_1Q - qx_2P)H^{\frac{\alpha-l+1}{d}}(x_1,x_2)$ is a $p-q$ quasi-homogeneous polynomial. The result follows directly from Theorem 2.7. ☑

4. Examples and applications

We now explore examples and applications of quasi-homogeneous differential systems and pertinent perturbations of such systems. Using our previous results we can establish the nonexistence of limit cycles in these examples.

Example 4.1. The differential system given by

$$\begin{cases} \dot{x}_1 &= 2x_2^3 + x_1^2x_2^2 - x_1x_2^4 + 8x_2^6, \\ \dot{x}_2 &= -x_1^2x_2 + 3x_1x_2^3 + 7x_2^5, \end{cases} \tag{5}$$

is a 2−1-quasi-homogeneous polynomial system of weighted degree 5. By Proposition 2.8 the system (5) admits no limit cycles.

Example 4.2. Consider the system

$$\begin{cases} \dot{x}_1 &= x_1^4x_2 + x_1^2x_2^6 + x_2^{12}, \\ \dot{x}_2 &= -x_1^5 + 3x_1^3x_2^5 + 7x_1x_2^{10}. \end{cases} \tag{6}$$

This system is a $5-2$-quasi-homogeneous system of weighted degree 21. Using Proposition 2.8 we conclude that the system (6) admits no limit cycles.

Example 4.3. Consider $p = q = 1$. In this case, a $p-q$-quasi-homogeneous vector field reduces to a classical homogeneous system. Hence, we recover a well-known result:

Corollary 4.4. *If f_1 and f_2 are homogeneous polynomials of the same degree, then the system* (1) *has no limit cycles.*

Example 4.5. The differential system given by

$$\begin{cases} \dot{x}_1 &= -4x_1^4x_2^3 + 7x_1^3x_2^4 - x_1^2x_2^5 - 10x_2^7, \\ \dot{x}_2 &= 3x_1^7 - x_1^6x_2 + 8x_1^4x_2^3 + 7x_1x_2^6, \end{cases} \tag{7}$$

is a homogeneous polynomial system of degree 7. By Corollary 4.4 we obtain that the system (8) admits no limit cycles.

Example 4.6. In the case that q is odd; and p and l are even; the $p-q$-quasi-homogeneous vector field includes some types of time-reversible systems. In particular, we obtain the following result:

Corollary 4.7. *If a polynomial vector field* (1) *is invariant under the symmetry $(x_1, x_2, t) \to (x_1, -x_2, -t)$, then the system has no limit cycles.*

Example 4.8. The classical Lotka-Volterra system [2], is used to model interactions between two species, namely, predators (x_2) and preys (x_1). The system is given by

$$\begin{cases} \dot{x}_1 &= x_1(a - bx_2), \\ \dot{x}_2 &= x_2(cx_1 - d), \end{cases} \tag{8}$$

where a, b, c and d are parameters describing the interaction of the two species. An IIF of this system is given by $\mathcal{V} = x_1x_2$, which is a $p-q$-quasi-homogenous polynomial of degree $p+q$. Hence, by Theorem 2.7 we obtain that the system has no limit cycles.

Example 4.9. Let $H(x_1, x_2) = x_1^3x_2^3 + 2x_1x_2^9 - x_2^{12}$, then H is a $3-1$-quasi-homogeneous polynomial of weighted degree $d = 12$. Thus, the Hamiltonian system given by:

$$X_H = (\frac{\partial H}{\partial x_2}, -\frac{\partial H}{\partial x_1}) = (3x_1^3x_2^2 + 18x_1x_2^8 - 12x_2^{11}, -3x_1^2x_2^3 - 2x_2^9),$$

is a $3-1$-quasi-homogeneous polynomial vector field of weighted degree $l=9$. Consider $A(x_1,x_2)=x_1^6x_2^2+x_1^5x_2^5$. A is a $3-1$-quasi-homogeneous polynomial of weighted degree $\alpha=20$. We now consider the perturbed system composed by

$$\left\{\begin{array}{ll} \dot{x}_1 & =3x_1^3x_2^2+18x_1x_2^8-12x_2^{11}+3x_1(x_1^6x_2^2+x_1^5x_2^5), \\ \dot{x}_2 & =-3x_1^2x_2^3-2x_2^9+x_2(x_1^6x_2^2+x_1^5x_2^5), \end{array}\right. \tag{9}$$

For this quasidegenerate infinite system we have that $\frac{\alpha-l+1}{d}=1$. Using Proposition 3.3 we obtain that system (9) admits no limit cycles.

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