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Operator-valued Fourier multipliers on toroidal Besov spaces

Multiplicadores de Fourier operador-valuados sobre espacios de Besov toroidales

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Abstract. We prove in this paper that a sequence $M : \mathbb{Z}^n \to \mathcal{L}(E)$ of bounded variation is a Fourier multiplier on the Besov space $B^s_{p,q}(\mathbb{T}^n, E)$ for $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$ and E a Banach space, if and only if E is a UMD-space. This extends the Theorem 4.2 in [3] to the $n-$dimensional case. As illustration of the applicability of this results we study the solvability of two abstract Cauchy problems with periodic boundary conditions.

Key words and phrases. Fourier multipliers, operator-valued symbols, UMD-spaces, toroidal Besov spaces.

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Resumen. En el presente artículo se prueba que una sucesión $M : \mathbb{Z}^n \to \mathcal{L}(E)$ de variación acotada, es un multiplicador de Fourier sobre el espacio de Besov $B^s_{p,q}(\mathbb{T}^n, E)$ para $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$ y E un espacio de Banach, si y solo si, E es un espacio UMD. Este resultado extiende el Teorema 4.2 en [3] al caso n-dimensional. Como ilustración de la aplicabilidad de este resultado, se estudia la solubilidad de dos problemas de Cauchy abstractos con condiciones de frontera periódicas.

Palabras y frases clave. Multiplicadores de Fourier, símbolos operador-valuados, espacios UMD, espacios de Besov toroidales.

1. Introduction

We are interested in obtaining a Fourier multiplier theorem on the toroidal or periodic Besov spaces $B^s_{p,q}(\mathbb{T}^n, E)$ for $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$ and E a $UMD-$space, with discrete symbols satisfying a bounded variation condition similar to the introduced in [11] and [10]. To reach this goal we give an extension of Theorem 4.2 in [3] and then, as an example of the applicability of the theory, we consider the following two Cauchy problems:

$$\begin{cases} \partial_t u(t,x) + A(t)u(t,x) = f(t,x), & t \in (0,T], x \in \mathbb{T}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{T}^n, \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t u(t,x) + A_\omega u(t,x) = f(t,x), & t \in [0,2\pi], x \in \mathbb{T}^n, \\ u(0,x) = u(2\pi,x), & x \in \mathbb{T}^n, \end{cases} \tag{2}$$

where $T > 0$ and $A(t)$ in (1) is a family of uniformly normal elliptic differential operators given by

$$A(t) := \sum_{|\alpha| \leq m} a_\alpha(t) D^\alpha. \tag{3}$$

Here $m \in \mathbb{N}$, $a_\alpha \in C_b([0,\infty), \mathcal{L}(E))$ for $|\alpha| \leq m$, E is a UMD-space and $D_j := -i\partial_j$. For the Problem (2), $A_\omega := \omega + A$, where A is as in (3) but with constant coefficients and $\omega \geq \omega_0$ with ω_0 appropiated. We stress that in [2] the autors use the evolution equation (1) with $n = 3$ and $A(t) = -\Delta$ to investigate the concentration of a pool of soluble polymers in a small cubical section $[0,2\pi]^3$ ($\cong \mathbb{T}^3$) of a biological cell (see also [7]). We remark that the results apply also in the case that the in (3) defined operator $A(t)$ is replaced by a Fourier-multiplier operator $A(t) = \mathcal{F}^{-1}_{\mathbb{T}^n}\big(a(t,\cdot)\mathcal{F}_{\mathbb{T}^n}\big)$, where $\{a(t,\cdot) : \mathbb{Z}^n \to \mathcal{L}(E); t \geq 0\}$ is a family of bounded variation symbols satisfying a certain uniform boundedness condition (see (64)).

For E a real (or complex) Banach space, $1 \leq p < \infty$ and $n \in \mathbb{N}$ let $L^p(\mathbb{R}^n, E)$ and $L^p(\mathbb{T}^n, E)$ be the usual Bochner space of p-integrable E-valued functions on $\mathbb{R}^n$ and on the n-dimensional torus $\mathbb{T}^n$ respectively. Now, we say that a function $M : \mathbb{Z}^n \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the Banach space of bounded linear operators $T : E \to E$ endowed with the usual operator norm, is a Fourier multiplier on $L^p(\mathbb{T}^n, E)$ if for each $f \in L^p(\mathbb{T}^n, E)$ there exists $g \in L^p(\mathbb{T}^n, E)$ such that

$$\hat{g}(k) = M(k)\hat{f}(k) \text{ for all } k \in \mathbb{Z}^n, \tag{4}$$

where $^\wedge$ denotes the Fourier transform. In the same way, we say that M is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}^n, E)$ if for each $f \in B^s_{p,q}(\mathbb{T}^n, E)$ there exists $g \in B^s_{p,q}(\mathbb{T}^n, E)$ such that (4) holds. We recall that E is called a UMD-space, if M, defined by $M(k) := I_E$ for $k \geq 0$, and by $M(k) = 0$ otherwise, is a Fourier-multiplier on $L^p(\mathbb{T}^n, E)$ for some $1 < p < \infty$.

In contrast to extensive theory on $E-$valued distributions in general, and Fourier multiplier theorems on $L^p(\mathbb{R}^n, E)$ and $B^s_{p,q}(\mathbb{R}^n, E)$ (and its aplications to partial differential equations) in particular, the contribution in literature to $E-$valued periodic distributions is rather sparse. For example, in [6], [5], [7], [10] and [11], the classical Fourier multiplier theorems of Marcinkiewicz and Mikhlin are extended to vector-valued functions and operator-valued multipliers on $\mathbb{Z}^n$ satisfying certain $\mathcal{R}$-boundedness condition. More specifically, the authors of those works established Fourier multiplier theorems on $L^p(\mathbb{T}^n, E)$ if $1 < p < \infty$, E is a UMD-space and, instead uniform boundedness, a $\mathcal{R}$-boundedness condition holds. This $\mathcal{R}$-boundedness condition is similar to our uniform boundedness condition (52). The firth results about the vector-valued periodic Besov spaces $B^s_{p,q}(\mathbb{T}^n, E)$ and Fourier multiplier theorems on these spaces appeared in [3] but with $n = 1$. There, in Theorem 4.2, the autors proved that each sequence $M : \mathbb{Z} \to \mathcal{L}(E)$ satisfaying the variational Marcinkiewicz condition is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}, E)$ if and only if $1 < p < \infty$ and E is a UMD-space. The corresponding result of this theorem for Besov spaces on the real line has been established by Bu and Kim in [5]. The variational Marcinkiewicz condition, giving in [3] is equivalent to the bounded variation condition (52) in case $n = 1$.

In this paper we obtain (Theorem 5.7) an analogous result to the assertion of Theorem 4.2 in [3] for the toroidal Besov space $B^s_{p,q}(\mathbb{T}^n, E)$. Indeed we prove that, given $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$, each fuction $M : \mathbb{Z}^n \to \mathcal{L}(E)$ which satisfies (52) is a Fourier multiplier in $B^s_{p,q}(\mathbb{T}^n, E)$ iff E is a $UMD-$space. Of course, the proof of the implication where the UMD property of E is the thesis, is similar to the case $n = 1$ in [3]. The hard part of this work was to show the other direction of the equivalence. With this, we establish results of existence and uniqueness of solution for the problems (1) and (2), since we prove that the sequences

$$M_{t,\lambda}(k) := \lambda\left(\lambda + a(t,k)\right)^{-1}, \quad k \in \mathbb{Z}^n,$$

are of bounded variation, where $a(t, \cdot)$ is the symbol of $A(t)$.

The plan of the paper is as follows: After some preliminary definitions and remarks in Section 2, we develop in Section 3 some fundamental elements on Besov spaces. In particular it is proved in Lemma 3.1 the existence of a resolution of the unity (very useful in the following sections) and in Theorem 3.7 the independence of norms on the resolution of unity in the space $B^s_{p,q}(\mathbb{T}^n, E)$. In Section 4 we define discrete Fourier multipliers on $L^p(\mathbb{T}, E)$ and $B^s_{p,q}(\mathbb{T}^n, E)$, the UMD-spaces and show in Corollary 4.8 an elementary result to characterize UMD-spaces. We prove in Section 5 the main result of the present paper, the Theorem 5.7. As an example, we prove in Section 6 (Corollary 6.5) the existence and uniqueness of solution for the problems (1) and (2) in certain periodic Besov spaces.

In the next three sections we explain in detail definitions and preliminary results for this work in order to do more clear the study of the periodic Besov spaces $B^s_{p,q}(\mathbb{T}^n, E)$ and the main result.

2. Functions and distributions on $\mathbb{T}^n$ and $\mathbb{Z}^n$

In this section we will present some notations, function spaces on the torus $\mathbb{T}^n$ and on the lattice $\mathbb{Z}^n$, as well as spaces of periodic and tempered distributions. Furthermore we will give some results, which are proven in similar way to the one-dimensional case discussed in [3] (see also [4] and [7]).

Throughout this paper $n \in \mathbb{N}$ is fixed, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\{\delta_j : j = 1, \ldots, n\}$ is the standard basis of $\mathbb{R}^n$, $\langle x \rangle := (1+|x|)^{1/2}$ for $x \in \mathbb{R}^n$, where $|x|$ is the euclidean norm of x, $B_r(a)$ and $\bar{B}_r(a)$ denote the open ball and closed ball (respectively) of radius $r > 0$ centered at a point $a \in \mathbb{R}^n$. E denotes an arbitary Banach space with norm $\|\cdot\|_E$. If X and Y are local convex spaces, then $\mathcal{L}(X, Y)$ denotes the space of all linear and continuous applications from X into Y. As usual $\mathcal{L}(X) := \mathcal{L}(X, X)$. For $\alpha, \beta \in \mathbb{Z}^n$ the writing $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for each $i = 1, \ldots, n$, and $[\alpha, \beta] := \{k \in \mathbb{Z}^n : \alpha \leq k \leq \beta\}$. In the following $đx := (2\pi)^{-n}dx$, where dx is the Lebesgue measure. Furthermore $C^m_c(\mathbb{R}^n, E)$, for $m \in \mathbb{N}_0 \cup \{\infty\}$, denote as usual the set of all m-times continuously differentible functions $\varphi : \mathbb{R}^n \to E$ with compact support and we write $C_c := C^0_c$.

Definition 2.1 (The spaces $C^\infty(\mathbb{T}^n, E)$)**.** We denote with $C^m(\mathbb{T}^n, E)$, $m \in \mathbb{N}_0$, the space of all 2π-periodic (in each component), E-valued and m-times continuously differentiable functions defined in $\mathbb{R}^n$. The space of test functions is the space $C^\infty(\mathbb{T}^n, E) := \bigcap_{m \in \mathbb{N}_0} C^m(\mathbb{T}^n, E)$.

The topology of $C^\infty(\mathbb{T}^n, E)$ is induced by the contable family of seminorms $\{q_k;\, k \in \mathbb{N}_0\}$ given by

$$q_k(\varphi) := \max_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq k}} \sup_{x \in [0,2\pi]^n} \|\partial^\alpha \varphi(x)\|, \quad \varphi \in C^\infty(\mathbb{T}^n, E). \tag{5}$$

It can be shown that $(C^\infty(\mathbb{T}^n, E), \{q_k;\, k \in \mathbb{N}_0\})$ is a Frechet space.

Definition 2.2 (The space of periodic distributions $\mathcal{D}'(\mathbb{T}^n, E)$)**.** The space $\mathcal{D}'(\mathbb{T}^n, E) := \mathcal{L}(C^\infty(\mathbb{T}^n), E)$ is called the space of E-valued periodic (or toroidal) distributions. The value of $u \in \mathcal{D}'(\mathbb{T}^n, E)$ on a test function $\varphi \in C^\infty(\mathbb{T}^n)$ will be denoted by $u(\varphi)$ or $\langle u, \varphi \rangle$.

The topology of $\mathcal{D}'(\mathbb{T}^n, E)$ is the weak-*-topology, i.e. a sequence $(u_k)_{k \in \mathbb{N}}$ in $\mathcal{D}'(\mathbb{T}^n, E)$ converges to $u \in \mathcal{D}'(\mathbb{T}^n, E)$ iff,

$$\langle u_k, \varphi \rangle \underset{k \to \infty}{\longrightarrow} \langle u, \varphi \rangle \quad \text{in } E, \quad \forall\, \varphi \in C^\infty(\mathbb{T}^n).$$

Indeed the topology of $\mathcal{D}'(\mathbb{T}^n, E)$ is induced by the family of seminorms $\{q'_\varphi;\, \varphi \in C^\infty(\mathbb{T}^n)\}$ where

$$q'_\varphi(u) := \|u(\varphi)\|, \quad u \in \mathcal{D}'(\mathbb{T}^n, E), \quad \varphi \in C^\infty(\mathbb{T}^n).$$

For example, for any $\psi \in C^\infty(\mathbb{T}^n, E)$, the map

$$C^\infty(\mathbb{T}^n) \ni \varphi \mapsto \int\limits_{[0,2\pi]^n} \varphi(x)\psi(x)\, đx,$$

defines a $E-$ valued periodic distribution, which we call again ψ.

Definition 2.3. We denote with $L^p(\mathbb{T}^n, E)$, $1 \le p \le \infty$, the space of all strongly measurable 2π-periodic (in each component) functions $f : \mathbb{R}^n \to E$, such that $\|f\|_{L^p(\mathbb{T}^n,E)} < \infty$, where

$$\|f\|_{L^p(\mathbb{T}^n,E)} := \left(\int_{[0,2\pi]^n} \|f(x)\|_E^p \, đx \right)^{1/p}, \qquad 1 \le p < \infty,$$

and with the usual definition for $p = \infty$.

As the continuous case it holds

$$L^p(\mathbb{T}^n, E) \hookrightarrow \mathcal{D}'(\mathbb{T}^n, E), \qquad \forall\, 1 \le p \le \infty.$$

Definition 2.4. The space $\mathcal{S}(\mathbb{Z}^n, E)$ consists of all functions $\varphi : \mathbb{Z}^n \longrightarrow E$ for which the following holds: For each $M \in \mathbb{R}$ there exists a constant $C_{\varphi,M}$ such that

$$\|\varphi(\xi)\|_E \le C_{\varphi,M} \langle \xi \rangle^{-M}, \qquad \text{for all } \xi \in \mathbb{Z}^n. \tag{6}$$

The elements of $\mathcal{S}(\mathbb{Z}^n, E)$ are called $E-$valued rapidly decreasing functions on $\mathbb{Z}^n$. As usual $\mathcal{S}(\mathbb{Z}^n) := \mathcal{S}(\mathbb{Z}^n, \mathbb{C})$.

The topology in $\mathcal{S}(\mathbb{Z}^n, E)$ is given by the contable family of seminorms $\{p_k : k \in \mathbb{N}_0\}$ defined by

$$p_k(\varphi) := \sup_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^k \|\varphi(\xi)\|_E, \qquad \text{for } \varphi \in \mathcal{S}(\mathbb{Z}^n, E). \tag{7}$$

Then a sequence $(\varphi_l)_{l \in \mathbb{N}}$ in $\mathcal{S}(\mathbb{Z}^n, E)$ converges to a function $\varphi \in \mathcal{S}(\mathbb{Z}^n, E)$ iff

$$p_k(\varphi_l - \varphi) \underset{l \to \infty}{\longrightarrow} 0 \text{ for all } k \in \mathbb{N}_0.$$

The space of $E-$valued tempered distributions on $\mathbb{Z}^n$ will be denoted by $\mathcal{S}'(\mathbb{Z}^n, E)$ and consists of all linear and continuous mappings from $\mathcal{S}(\mathbb{Z}^n)$ into E. This distributions space is also endowed with the weak-*-topology.

Example 2.5. Let $\phi \in C_c(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{Z}^n, E)$. Then the mapping $\phi f : \mathcal{S}(\mathbb{Z}^n) \longrightarrow E$ defining by $(\phi f)(\varphi) := f(\phi\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{Z}^n)$ belongs to $\mathcal{S}'(\mathbb{Z}^n, E)$.

Definition 2.6. *a*) For a function $f \in C^\infty(\mathbb{T}^n, E)$ we define

$$(\mathcal{F}_{\mathbb{T}^n} f)(\xi) := \int_{\mathbb{T}^n} e^{-ix\cdot\xi} f(x) đx = \int_{[0,2\pi]^n} e^{-ix\cdot\xi} f(x) đx, \qquad \xi \in \mathbb{Z}^n. \quad (8)$$

We call $\mathcal{F}_{\mathbb{T}^n} f$ the toroidal or periodic Fourier transform of f.

b) For $g \in \mathcal{S}(\mathbb{Z}^n, E)$ we define

$$\left(\mathcal{F}_{\mathbb{T}^n}^{-1} g\right)(x) := \sum_{\xi\in\mathbb{Z}^n} e^{ix\cdot\xi} g(\xi), \quad x \in \mathbb{T}^n. \quad (9)$$

We call $\mathcal{F}_{\mathbb{T}^n}^{-1} g$ the inverse periodic Fourier transform of g.

c) Let $u \in \mathcal{D}'(\mathbb{T}^n, E)$. The periodic Fourier transform of u is defined by

$$(\mathcal{F}_{\mathbb{T}^n} u)(\varphi) := u\left([\mathcal{F}_{\mathbb{T}^n}^{-1}\varphi](-\cdot)\right), \qquad \varphi \in \mathcal{S}(\mathbb{Z}^n). \quad (10)$$

d) For $v \in \mathcal{S}'(\mathbb{Z}^n, E)$ we define the inverse *periodic Fourier transform of* v by

$$\left(\mathcal{F}_{\mathbb{T}^n}^{-1} v\right)(\psi) := v\left([\mathcal{F}_{\mathbb{T}^n}\psi](-\cdot)\right), \qquad \psi \in C^\infty(\mathbb{T}^n). \quad (11)$$

Proposition 2.7. *The following mappings are linear and continuous: a*) $C^\infty(\mathbb{T}^n, E) \ni f \mapsto \mathcal{F}_{\mathbb{T}^n} f \in \mathcal{S}(\mathbb{Z}^n, E)$, *b*) $\mathcal{S}(\mathbb{Z}^n, E) \ni g \mapsto \mathcal{F}_{\mathbb{T}^n}^{-1} g \in C^\infty(\mathbb{T}^n, E)$, *c*) $\mathcal{D}'(\mathbb{T}^n, E) \ni u \mapsto \mathcal{F}_{\mathbb{T}^n} u \in \mathcal{S}'(\mathbb{Z}^n, E)$ *and d*) $\mathcal{S}'(\mathbb{Z}^n, E) \ni v \mapsto \mathcal{F}_{\mathbb{T}^n}^{-1} v \in \mathcal{D}'(\mathbb{T}^n, E)$.

Definition 2.8. We say that a function $u : \mathbb{Z}^n \longrightarrow E$ grows at most polynomially at infinity if there exist constans $M \in \mathbb{R}$ and $C \geq 0$ (both depending on u) such that

$$\|u(\xi)\|_E \leq C\langle\xi\rangle^M, \qquad \text{for all } \xi \in \mathbb{Z}^n. \quad (12)$$

The space of all functions $u : \mathbb{Z}^n \longrightarrow E$ with at most polynomial growth at infinity will be denoted by $\mathcal{O}(\mathbb{Z}^n, E)$.

Note that if $u \in \mathcal{S}(\mathbb{Z}^n, E)$, then $u \in \mathcal{O}(\mathbb{Z}^n, E)$. We can also identify the space $\mathcal{O}(\mathbb{Z}^n, E)$ with the space of all sequences $(a_k)_{k\in\mathbb{Z}^n}$ in E, for which there are constants C and M such that $\|a_k\|_E \leq C\langle k\rangle^M$ for all $k \in \mathbb{Z}^n$.

Example 2.9. Let $u \in \mathcal{S}'(\mathbb{Z}^n, E)$. The function defined by $\bar{u}(\xi) := u(\psi_\xi)$, $\xi \in \mathbb{Z}^n$ belongs to $\mathcal{O}(\mathbb{Z}^n, E)$, where $\psi_\xi \in \mathcal{S}(\mathbb{Z}^n)$ is defined by

$$\psi_\xi(k) := \delta_{\xi,k} := \begin{cases} 1, & \text{if } \xi = k, \\ 0, & \text{if } \xi \neq k. \end{cases} \quad (13)$$

Proposition 2.10. *The map* $\mathcal{O}(\mathbb{Z}^n, E) \ni u \mapsto \Lambda_u \in \mathcal{S}'(\mathbb{Z}^n, E)$, *where*

$$\Lambda_u(\varphi) := \sum_{\xi \in \mathbb{Z}^n} \varphi(\xi) u(\xi), \qquad \forall \varphi \in \mathcal{S}(\mathbb{Z}^n), \tag{14}$$

is bijective.

***Remark* 2.11.** Due to the last proposition, one can identify $u \in \mathcal{S}'(\mathbb{Z}^n, E)$ with $\Lambda_{\bar{u}}$, and so

$$u(\varphi) = \Lambda_{\bar{u}}(\varphi) = \sum_{\xi \in \mathbb{Z}^n} \varphi(\xi) u(\psi_\xi), \qquad \forall \varphi \in \mathcal{S}(\mathbb{Z}^n). \tag{15}$$

Definition 2.12. For $\phi \in C^\infty(\mathbb{T}^n)$, $u \in \mathcal{D}'(\mathbb{T}^n)$ and $e \in E$, the tensor products $\phi \otimes e$ and $u \otimes e$ are defined by

$$(\phi \otimes e)(x) := \phi(x) e, \quad x \in [0, 2\pi]^n,$$
$$(u \otimes e)(\varphi) := u(\varphi) e, \quad \varphi \in C^\infty(\mathbb{T}^n).$$

It is straightforward to prove that $\phi \otimes e \in C^\infty(\mathbb{T}^n, E)$ and $u \otimes e \in \mathcal{D}'(\mathbb{T}^n, E)$.

Proposition 2.13. *Let* $(a_k)_{k \in \mathbb{Z}^n} \subset E$ *be a sequence with at most polynomial growth at infinity (i.e.* $[k \mapsto a_k] \in \mathcal{O}(\mathbb{Z}^n, E)$*), then the mapping* $g : C^\infty(\mathbb{T}^n) \to E$*, defined by*

$$g(\varphi) := \sum_{k \in \mathbb{Z}^n} \left(\mathcal{F}_{\mathbb{T}^n} \varphi\right)(k) a_k, \qquad \varphi \in C^\infty(\mathbb{T}^n), \tag{16}$$

belongs to $\mathcal{D}'(\mathbb{T}^n, E)$*. Furthermore, for all* $g \in \mathcal{D}'(\mathbb{T}^n, E)$ *it holds*

$$g(\varphi) = \sum_{k \in \mathbb{Z}^n} \left(\mathcal{F}_{\mathbb{T}^n} \varphi\right)(k) g(e_k) \qquad (\varphi \in C^\infty(\mathbb{T}^n)), \tag{17}$$

where $e_k(x) := e^{ik \cdot x}$ *for all* $x \in \mathbb{R}^n$*.*

It follows from the last proposition that for all $g \in \mathcal{D}'(\mathbb{T}^n, E)$,

$$g = \sum_{k \in \mathbb{Z}^n} e_k \otimes \hat{g}(k) \quad \text{in } \mathcal{D}'(\mathbb{T}^n, E), \tag{18}$$

where $\hat{g}(k) := g(e_{-k}),\ k \in \mathbb{Z}^n$, are call the*Fourier coefficients* of g. To see this fact note that $e_k(\varphi) = \int\limits_{\mathbb{T}^n} e^{ix \cdot k} \varphi(x)\, đx = \left(\mathcal{F}_{\mathbb{T}^n} \varphi\right)(-k)$ for all $\varphi \in C^\infty(\mathbb{T}^n)$.

We finish this section with some results, which we will need for the following one. Before, note that

$$e_k(e_{-\xi}) = \int\limits_{[0,2\pi]^n} e^{i(k-\xi) \cdot x}\, đx = \delta_{\xi,k}, \qquad \text{for all } \xi, k \in \mathbb{Z}^n.$$

Lemma 2.14. If $(a_k)_{k\in\mathbb{Z}^n} \in \mathcal{O}(\mathbb{Z}^n, E)$, then $\sum_{k\in\mathbb{Z}^n} e_k \otimes a_k \in \mathcal{D}'(\mathbb{T}^n, E)$. Furthermore, if $(b_k)_{k\in\mathbb{Z}^n} \in \mathcal{O}(\mathbb{Z}^n, E)$,

$$\sum_{k\in\mathbb{Z}^n} e_k \otimes a_k = \sum_{k\in\mathbb{Z}^n} e_k \otimes b_k \text{ in } \mathcal{D}'(\mathbb{T}^n, E) \Longleftrightarrow a_k = b_k \text{ for each } k \in \mathbb{Z}^n. \quad (19)$$

Proof. Due to $e_{-k}(\varphi) = (\mathcal{F}_{\mathbb{T}^n}\varphi)(k)$ for all $\varphi \in C^\infty(\mathbb{T}^n)$, it follows from Proposition 2.13 that the mapping

$$\varphi \mapsto \sum_{k\in\mathbb{Z}^n} (\mathcal{F}_{\mathbb{T}^n}\varphi)(k) a_{-k} = \sum_{k\in\mathbb{Z}^n} e_k(\varphi) a_k$$

belongs to $\mathcal{D}'(\mathbb{T}^n, E)$, i.e. there exists some $g \in \mathcal{D}'(\mathbb{T}^n, E)$ such that

$$g = \sum_{k\in\mathbb{Z}^n} e_k \otimes a_k.$$

Now, if $\sum_{k\in\mathbb{Z}^n} e_k \otimes a_k = \sum_{k\in\mathbb{Z}^n} e_k \otimes b_k$, then it holds for each $\xi \in \mathbb{Z}^n$ that

$$a_\xi = \Big(\sum_{k\in\mathbb{Z}^n} e_k \otimes a_k\Big)(e_{-\xi}) = \Big(\sum_{k\in\mathbb{Z}^n} e_k \otimes b_k\Big)(e_{-\xi}) = b_\xi.$$

The reciprocal is trivial. ☑

As a direct consequence of the last lemma and the equalities (17) and (18) we have that:

Theorem 2.15. *Let* $f, g \in \mathcal{D}'(\mathbb{T}^n, E)$ *and* $(a_k)_{k\in\mathbb{Z}^n} \in \mathcal{O}(\mathbb{Z}^n, E)$.

a) $f = g \Longleftrightarrow \hat{f}(k) = \hat{g}(k)$ *for all* $k \in \mathbb{Z}^n$.

b) $f = \sum_{k\in\mathbb{Z}^n} e_k \otimes a_k \Longleftrightarrow \hat{f}(k) = a_k$ *for all* $k \in \mathbb{Z}^n$.

3. The periodic Besov spaces $B^s_{p,q}(\mathbb{T}^n, E)$

A sequence $\phi := (\phi_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ is called a *resolution of unity* , denoted $(\phi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$, if it satisfies the following three conditions:

(1) $\operatorname{supp}(\phi_0) \subset \Omega_0 := \overline{B}_2(0)$ and

$$\operatorname{supp}(\phi_j) \subset \Omega_j := \left\{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\right\}, \quad j \in \mathbb{N}. \quad (20)$$

(2) $\sum_{j\ge 0} \phi_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n$.

(3) For each $\alpha \in \mathbb{N}_0^n$ there exists a constant $C_\alpha > 0$ such that

$$|(\partial^\alpha \phi_j)(\xi)| \le C_\alpha 2^{-j|\alpha|} \chi_{\Omega_j}(\xi), \text{ for all } \xi \in \mathbb{R}^n \text{ and } j \in \mathbb{N}_0,$$

where χ_{Ω_j} denotes the characteristic function on Ω_j.

The set $\Phi(\mathbb{R}^n)$ is not empty as it is shown in the following generalization of Lemma 4.1 in [3].

Lemma 3.1. *There exists a sequence* $(\phi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ *such that*

a) $\phi_j \geq 0$ *for all* $j \in \mathbb{N}_0$,

b) $\operatorname{supp}(\phi_j) \subsetneq \Omega_j$ *for all* $j \in \mathbb{N}_0$,

c) $\phi_j(\xi) = 1$ *if* $|\xi| \in [7\cdot 2^{j-3}, 3\cdot 2^{j-1}]$ *and* $j \geq 3$,

d) $|\xi| \in [7\cdot 2^{j-3}, 3\cdot 2^{j-1}]$ *and* $j \geq 3$ *implies* $\xi \notin \operatorname{supp}(\phi_{j-1}) \cap \operatorname{supp}(\phi_{j+1})$.

Proof. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $0 \leq \varphi_0 \leq 1$, $\operatorname{supp}(\varphi_0) \subsetneq \overline{B}_2(0)$ and $\varphi_0(\xi) = 1$, if $|\xi| \leq \frac{13}{8}$. Let φ be another function in $\mathcal{S}(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1$, $\operatorname{supp}(\varphi) \subsetneq \left\{x \in \mathbb{R}^n : \frac{3}{2} \leq |x| \leq \frac{7}{2}\right\}$ and $\varphi(\xi) = 1$, if $\frac{13}{8} \leq |\xi| \leq \frac{13}{4}$. Now, for $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, define

$$\varphi_j(\xi) := \varphi\left(\frac{\xi}{2^{j-1}}\right) \qquad \text{and} \qquad \Psi(\xi) := \sum_{k=0}^{\infty} \varphi_k(\xi).$$

Then it is clear that for $j = 1, 2, \ldots$ it holds

$$\operatorname{supp}(\varphi_j) \subset K_j := \left\{\xi \in \mathbb{R}^n : 3\cdot 2^{j-2} \leq |\xi| \leq 7\cdot 2^{j-2}\right\}, \tag{21}$$

$$\varphi_j(\xi) = 1, \text{ if } |\xi| \in [13\cdot 2^{j-4}, 13\cdot 2^{j-3}]. \tag{22}$$

Due to $K_j \cap K_{j+2} = \varnothing$ and $\varphi_j(13\cdot 2^{j-3}) = 1 = \varphi_{j+1}(13\cdot 2^{j-3})$ for each $j \in \mathbb{N}_0$, we have that

$$\operatorname{supp}(\varphi_j) \cap \operatorname{supp}(\varphi_{j+1}) \neq \varnothing \quad \text{and} \quad \operatorname{supp}(\varphi_j) \cap \operatorname{supp}(\varphi_{j+2}) = \varnothing, \tag{23}$$

for all $j \in \mathbb{N}_0$.

Assertion: For each $\xi \in \mathbb{R}^n$, there exists some $j \in \mathbb{N}_0$ such that

$$\Psi(\xi) = \sum_{\ell=-1}^{1} \varphi_{j+\ell}(\xi) \geq 1, \tag{24}$$

where $\varphi_{-1} := 0$. In fact: Let $\xi \in \mathbb{R}^n$. Because $\mathbb{R}_0^+ = [0, 13/8] \cup \bigcup_{j=1}^{\infty} [13\cdot 2^{j-4}, 13\cdot 2^{j-3}]$, then $|\xi| \leq 13/8$ or $13\cdot 2^{j-4} \leq |\xi| \leq 13\cdot 2^{j-3}$ for some $j \in \mathbb{N}$. From this, (22) and (23) it follows clearly (24).

Now, define for $j = 0, 1, 2, \ldots$

$$\phi_j(\xi) := \frac{\varphi_j(\xi)}{\Psi(\xi)}, \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{25}$$

By direct calculation we get that $(\phi)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ and it satisfies $a) - d)$. ☑

In the continuous case (in $\mathbb{R}^n$) it can be shown that if $(\phi_j) \in \Phi(\mathbb{R}^n)$, then there exists a constant $C_n > 0$ such that

$$\left\|\mathcal{F}_{\mathbb{R}^n}^{-1}\phi_j\right\|_{L^1(\mathbb{R}^n)} \leq C_n, \quad \text{for all } j \in \mathbb{N}_0, \tag{26}$$

where $\mathcal{F}_{\mathbb{R}^n}^{-1}\phi_j$ is the inverse Fourier transform (in $\mathbb{R}^n$) of ϕ_j.

Theorem 3.2. *Let $\phi \in C_c(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{T}^n, E)$. Then*

$$\mathcal{F}_{\mathbb{T}^n}^{-1}(\phi\mathcal{F}_{\mathbb{T}^n} f) = \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi(k)\hat{f}(k). \tag{27}$$

Proof. From Example 2.5 and Remark 2.11 it follows for $\phi \in C_c(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{T}^n, E)$ that

$$\begin{aligned}\left[\mathcal{F}_{\mathbb{T}^n}^{-1}(\phi\mathcal{F}_{\mathbb{T}^n} f)\right]\widehat{}(k) &= \left(\mathcal{F}_{\mathbb{T}^n}^{-1}(\phi\mathcal{F}_{\mathbb{T}^n} f)\right)(e_{-k}) = (\phi\mathcal{F}_{\mathbb{T}^n} f)((\mathcal{F}_{\mathbb{T}^n} e_{-k})(-\cdot)) \\ &= \sum_{\xi\in\mathbb{Z}^n} \phi(\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\psi_\xi)(\mathcal{F}_{\mathbb{T}^n} e_{-k})(-\xi) \\ &= \sum_{\xi\in\mathbb{Z}^n} \phi(\xi)(\mathcal{F}_{\mathbb{T}^n} f)(\psi_\xi)\delta_{\xi,k} = \phi(k)(\mathcal{F}_{\mathbb{T}^n} f)(\psi_k) \\ &= \phi(k) f\left([\mathcal{F}_{\mathbb{T}^n}^{-1}\psi_k](-\cdot)\right) = \phi(k) f(e_{-k}) = \phi(k)\hat{f}(k).\end{aligned}$$

That is,

$$\left[\mathcal{F}_{\mathbb{T}^n}^{-1}(\phi\mathcal{F}_{\mathbb{T}^n} f)\right]\widehat{}(k) = \phi(k)\hat{f}(k) \quad \text{for all } k \in \mathbb{Z}^n, \tag{28}$$

and therefore (27) holds, due to (18). ☑

In similar way to the proof of Proposition 2.2 in [3], we obtain the following result.

Proposition 3.3. *For each $\phi \in C_c^\infty(\mathbb{R}^n)$ and $1 \leq p \leq \infty$, it holds*

$$\left\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi(k)\hat{f}(k) \right\|_{L^p(\mathbb{T}^n,E)} \leq \left\|\mathcal{F}_{\mathbb{R}^n}^{-1}\phi\right\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{T}^n,E)}\,, \tag{29}$$

for all $f \in C^\infty(\mathbb{T}^n, E)$.

Definition 3.4. A function $f : \mathbb{R}^n \longrightarrow E$ is called an *$E-$trigonometric polynomial*, if there exist $\alpha, \beta \in \mathbb{Z}^n$ with $\alpha \leq \beta$ and $(x_k)_{k\in[\alpha,\beta]} \subset E$ such that

$$f = \sum_{k\in[\alpha,\beta]} e_k \otimes x_k, \tag{30}$$

where $e_k(x) := e^{ik\cdot x}$ for all $x \in \mathbb{R}^n$.

We can write the $E-$ trigonometric polynomial f in (30) as

$$f = \sum_{k\in\mathbb{Z}^n} e_k \otimes x_k,$$

with $x_k := 0$ for $k \notin [\alpha, \beta]$, or in the form

$$f = \sum_{k\in[-N,N]^n} e_k \otimes x_k,$$

for some $N \in \mathbb{N}$.

We denote the class of all $E-$valued trigonometric polynomials on $\mathbb{T}^n$ by $\mathcal{T}(\mathbb{T}^n, E)$. It is clear that $\mathcal{T}(\mathbb{T}^n, E) \subset C^\infty(\mathbb{T}^n, E)$.

***Remark* 3.5.** Due to Theorem 2.15 *b)* if $f = \sum_{k\in[\alpha,\beta]} e_k \otimes x_k \in \mathcal{T}(\mathbb{T}^n, X)$, then

$$\hat{f}(k) = \begin{cases} x_k, & \text{if } k \in [\alpha, \beta], \\ 0, & \text{otherwise,} \end{cases}$$

when f is seen as a distribution in $\mathcal{D}'(\mathbb{T}^n, E)$.

Definition 3.6. Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and $\phi := (\phi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$. We define the $E-$valued and $n-$dimensional toroidal or periodic *Besov space by*

$$B^{s,\phi}_{p,q}(\mathbb{T}^n, E) := \left\{ f \in \mathcal{D}'(\mathbb{T}^n, E) : \|f\|_{B^{s,\phi}_{p,q}} := \|f\|_{B^{s,\phi}_{p,q}(\mathbb{T}^n,E)} < \infty \right\},$$

where

$$\|f\|_{B^{s,\phi}_{p,q}} := \begin{cases} \left(\sum_{j\ge 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)} \right)^{1/q}, & \text{if } 1 \le q < \infty, \\ \sup_{j\in\mathbb{N}_0} 2^{sj} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{f}(k) \Big\|_{L^p(\mathbb{T}^n,E)}, & \text{if } q = \infty, \end{cases} \tag{31}$$

Note that for $f \in \mathcal{D}'(\mathbb{T}^n, E)$, $(\phi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ and $j \ge 0$, $\mathcal{F}^{-1}_{\mathbb{T}^n}(\phi_j \mathcal{F}_{\mathbb{T}^n} f) \overset{(27)}{=} \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{f}(k)$ is a trigonometric polynomial. Furthermore $B^{s,\phi}_{p,q}(\mathbb{T}^n, E)$ is a Banach space with the norm defined in (31).

In the following theorem we prove that the $E-$valued and $n-$dimensional periodic *Besov spaces are independent on* $\phi \in \Phi(\mathbb{R}^n)$.

Theorem 3.7. *Let* $\phi := (\phi_j)_{j\in\mathbb{N}_0}$, $\varphi := (\varphi_j)_{j\in\mathbb{N}_0}$ *in* $\Phi(\mathbb{R}^n)$. *Then the norms* $\|\cdot\|_{B^{s,\phi}_{p,q}(\mathbb{T}^n,E)}$ *and* $\|\cdot\|_{B^{s,\varphi}_{p,q}(\mathbb{T}^n,E)}$ *are equivalent.*

Proof. We must prove that there are constants $c, C > 0$ such that

$$c\,\|f\|_{B^{s,\varphi}_{p,q}(\mathbb{T}^n,E)} \leq \|f\|_{B^{s,\phi}_{p,q}(\mathbb{T}^n,E)} \leq C\,\|f\|_{B^{s,\varphi}_{p,q}(\mathbb{T}^n,E)}\,, \tag{32}$$

for all $f \in B^{s,\varphi}_{p,q}(\mathbb{T}^n, X)$. But, due to the transitivity of $\leq$, it suffices to show (32) for ϕ as in Lemma 3.1. We will show this for $1 \leq p < \infty$, the case $p = \infty$ is proved in similar way. Let $1 \leq p < \infty$, $\varphi \in \Phi(\mathbb{R}^n)$ and ϕ as in Lemma 3.1. Since $\operatorname{supp}(\phi_j) \cap \operatorname{supp}(\phi_{j+2}) = \varnothing$ for all $j \in \mathbb{N}_0$,

$$\phi_j(x) = \phi_j(x) \sum_{l=-1}^{1} \varphi_{j+l}(x), \qquad \forall x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}_0, \tag{33}$$

where $\phi_{-1} = \varphi_{-1} := 0$. From $(|a| + |b|)^p \leq c_p(|a|^p + |b|^p)$ for all $a, b \in \mathbb{C}$, Proposition 3.3 and (26) it follows that

$$\begin{aligned}
\|f\|^q_{B^{s,\phi}_{p,q}} &\overset{(33)}{=} \sum_{j\geq 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k) \sum_{l=-1}^{1} \varphi_{j+l}(k)\hat{f}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)} \\
&\leq c_q \sum_{l=-1}^{1} \sum_{j\geq 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\varphi_{j+l}(k)\hat{f}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)} \\
&\overset{(28)}{=} c_q \sum_{l=-1}^{1} \sum_{j\geq 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k) \left(\mathcal{F}^{-1}_{\mathbb{T}^n} (\varphi_{j+l}\mathcal{F}_{\mathbb{T}^n} f)\right)^{\wedge}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)} \\
&\leq c_q \sum_{l=-1}^{1} \sum_{j\geq 0} 2^{jsq} \left\|\mathcal{F}^{-1}_{\mathbb{R}^n}\phi_j\right\|^q_{L^1(\mathbb{R}^n)} \left\|\mathcal{F}^{-1}_{\mathbb{T}^n} (\varphi_{j+l}\mathcal{F}_{\mathbb{T}^n} f)\right\|^q_{L^p(\mathbb{T}^n,E)} \\
&\leq c_q c_n \sum_{l=-1}^{1} \sum_{j\geq 0} 2^{jsq} \left\|\mathcal{F}^{-1}_{\mathbb{T}^n} (\varphi_{j+l}\mathcal{F}_{\mathbb{T}^n} f)\right\|^q_{L^p(\mathbb{T}^n,E)} \\
&= c_q c_n \sum_{l=-1}^{1} \sum_{j\geq 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \varphi_{j+l}(k)\hat{f}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)}.
\end{aligned} \tag{34}$$

Now, because

$$\begin{aligned}
&\sum_{j\geq 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \varphi_{j\pm 1}(k)\hat{f}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)} \\
&\qquad\qquad \leq 2^{\mp jsq} \sum_{j\geq 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \varphi_j(k)\hat{f}(k) \Big\|^q_{L^p(\mathbb{T}^n,E)},
\end{aligned} \tag{35}$$

then

$$\|f\|^q_{B^{s,\phi}_{p,q}(\mathbb{T}^n,E)} \leq c_q c_n (2\pi)^{-n} (1 + 2^{sq} + 2^{-sq})\, \|f\|^q_{B^{s,\varphi}_{p,q}(\mathbb{T}^n,E)}\,. \tag{36}$$

Exchanging the roles of ϕ and φ in the expressions (33) - (36), we have that (32) follows from (36) with ϕ as in Lemma 3.1. ☑

Due to the last theorem we will write $B^s_{p,q}(\mathbb{T}^n, E)$ instead $B^{s,\phi}_{p,q}(\mathbb{T}^n, E)$. From now on, $B^s_{p,q}(\mathbb{T}^n, E)$ will be considered with the resolution of the unity of Lemma 3.1.

***Remark* 3.8.** Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $x \in E$ fixed. Note that the function $f : \mathbb{T}^n \to E$, defined by $f := e_0 \otimes x$, with $e_0(y) = 1$ for all $y \in \mathbb{R}^n$, satisfies (due to Remark 3.5)

$$\begin{aligned}\sum_{j\ge 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{f}(k)\Big\|^q_{L^p(\mathbb{T}^n,E)} &= \sum_{j\ge 0} 2^{sjq} \left|\phi_j(0)\right|^q \|x\|^q_E \\ &= \left|\phi_0(0)\right|^q \|x\|^q_E =: C_0^q \|x\|^q_E\,, \qquad (37)\end{aligned}$$

if $q < \infty$. Similar result holds for $q = \infty$. Because of $(\phi_j)_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$, one obtains with this idea that $\mathcal{T}(\mathbb{T}^n, E) \subset B^s_{p,q}(\mathbb{T}^n, E)$.

4. Discrete Fourier multipliers

Definition 4.1. A function $M : \mathbb{Z}^n \to \mathcal{L}(E, F)$ is called a *discrete operator-valued* ($B^s_{p,q}-$)*Fourier multiplier* from $B^s_{p,q}(\mathbb{T}^n, E)$ to $B^s_{p,q}(\mathbb{T}^n, F)$ if for each $f \in B^s_{p,q}(\mathbb{T}^n, E)$ there exists $g \in B^s_{p,q}(\mathbb{T}^n, F)$ such that $\hat{g}(k) = M(k)\hat{f}(k)$ for all $k \in \mathbb{Z}^n$. If $E = F$, we will say that M is a discrete Fourier multiplier on $B^s_{p,q}(\mathbb{T}^n, E)$.

Theorem 4.2. *Let $M : \mathbb{Z}^n \longrightarrow \mathcal{L}(E, F)$ be a function. Then the following assertion are equivalent:*

a) M is a discrete $B^s_{p,q}-$Fourier multiplier.

b) There exists a constant $C > 0$ such that

$$\Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes M(k)\hat{f}(k)\Big\|_{B^s_{p,q}(\mathbb{T}^n,F)} \le C \|f\|_{B^s_{p,q}(\mathbb{T}^n,E)}\,, \qquad (38)$$

for all $f \in B^s_{p,q}(\mathbb{T}^n, E)$.

Proof. $a) \Rightarrow b)$] Let $M : \mathbb{Z}^n \to \mathcal{L}(E, F)$ be a discrete $B^s_{p,q}-$Fourier multiplier. For $f = \sum_{k\in\mathbb{Z}^n} e_k \otimes \hat{f}(k) \in B^s_{p,q}(\mathbb{T}^n, E)$, define

$$S_M(f) := \sum_{k\in\mathbb{Z}} e_k \otimes M(k)\hat{f}(k). \qquad (39)$$

Due to the hypothesis there exists a $g \in B^s_{p,q}(\mathbb{T}^n, F)$ such that $\hat{g}(k) = M(k)\hat{f}(k)$ far all $k \in \mathbb{Z}^n$. Therefore, due to (18) we have

$$S_M(f) = \sum_{k\in\mathbb{Z}} e_k \otimes \hat{g}(k) = g,$$

i.e. S_M is a well defined application from $B^s_{p,q}(\mathbb{T}^n, X)$ into $B^s_{p,q}(\mathbb{T}^n, F)$. Now, we will prove that S_M is a closed linear operator. Let $(f_m)_{m\in\mathbb{N}_0} = \left(\sum_{k\in\mathbb{Z}^n} e_k \otimes \hat{f}_m(k)\right)_{m\in\mathbb{N}_0} \subset B^s_{p,q}(\mathbb{T}^n, E)$ such that

$$f_m \xrightarrow[m\to\infty]{} f \qquad \text{and} \qquad S_M f_m \xrightarrow[m\to\infty]{} h$$

in $B^s_{p,q}(\mathbb{T}^n, E)$ and $B^s_{p,q}(\mathbb{T}^n, F)$, respectively. Since

$$2^{sjq}\Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)(f_m - f)\hat{}(k)\Big\|_{L^p(\mathbb{T}^n,E)} \le \|f_m - f\|_{B^s_{p,q}(\mathbb{T}^n,E)} \xrightarrow[m\to\infty]{} 0$$

in $\mathbb{C}$, then

$$\sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)(f_m - f)\hat{}(k) \xrightarrow[m\to\infty]{} 0 \quad \text{in } L^p(\mathbb{T}^n, E),$$

for each $j \in \mathbb{N}_0$. Because of $L^p(\mathbb{T}^n, E) \hookrightarrow \mathcal{D}'(\mathbb{T}^n, E)$, it holds for each $l \in \mathbb{Z}^n$ that

$$\phi_j(l)(f_m - f)\hat{}(l) = \sum_{k\in\mathbb{Z}^n} e_k(e_{-l})\phi_j(k)(f_m - f)\hat{}(k) \xrightarrow[m\to\infty]{} 0 \quad \text{in } E.$$

Then

$$\phi_j(l)\hat{f}_m(l) \xrightarrow[m\to\infty]{} \phi_j(l)\hat{f}(l) \quad \text{in } E,\ \forall\, l \in \mathbb{Z}^n \text{ and } j \in \mathbb{N}_0. \tag{40}$$

In the same way one obtains that

$$\phi_j(l)M(l)\hat{f}_m(l) \xrightarrow[m\to\infty]{} \hat{h}(l) \quad \text{in } F,\ \forall\, l \in \mathbb{Z}^n \text{ and } j \in \mathbb{N}_0. \tag{41}$$

Because $M(k) \in \mathcal{L}(E, F)$, it follows from (40) that for each $k \in \mathbb{Z}^n$

$$\phi_j(k)M(k)\hat{f}_m(k) \xrightarrow[m\to\infty]{} \phi_j(k)M(k)\hat{f}(k) \quad \text{in } F.$$

Therefore

$$\sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)M(k)\hat{f}_m(k) \xrightarrow[m\to\infty]{F} \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)M(k)\hat{f}(k),$$

because these sums are finite. In the same way it follows from (41) that

$$\sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)M(k)\hat{f}_m(k) \xrightarrow[m\to\infty]{F} \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{h}(k).$$

Then

$$\sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)M(k)\hat{f}(k) = \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{h}(k), \quad \text{for } j \in \mathbb{N}_0$$

and thus

$$\begin{aligned}
&\|S_M f - h\|^q_{B^s_{p,q}(\mathbb{T}^n,F)} \\
&= \sum_{j\geq 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k) \left(S_M f - h\right)^\wedge (k) \Big\|^q_{L^p(\mathbb{T}^n;F)} \\
&= \sum_{j\geq 0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k) \widehat{S_M f}(k) - \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k) \hat{h}(k) \Big\|^q_{L^p(\mathbb{T}^n;F)} \\
&= 0,
\end{aligned}$$

i.e. $S_M f = h$, and hence S_M is a closed linear operator. Thus, by the closed graph theorem, S_M is bounded and consequently (38) holds.

$b) \Rightarrow a)$] Suppose that (38) holds for each $f \in B^s_{p,q}(\mathbb{T}^n, E)$. From this and (37) there exists a constant $c > 0$ such that

$$\|M(k)\|_{\mathcal{L}(E,F)} \leq c \quad \text{ for all } k \in \mathbb{Z}^n.$$

Let $f \in B^s_{p,q}(\mathbb{T}^n, E)$. Because $f \in \mathcal{D}'(\mathbb{T}^n, E)$, there exist constants $d > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}
\big\|M(k)\hat{f}(k)\big\|_F \leq cd\, q_N(e_{-k}) &= cd \max_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\leq N}} \sup_{x\in[0,2\pi]^n} |(-i)^{|\alpha|} k^\alpha e^{-ikx}| \\
&\leq cd|k|^N \leq C\langle k\rangle^N \qquad \forall k \in \mathbb{Z}^n.
\end{aligned}$$

Therefore $(M(k)\hat{f}(k))_{k\in\mathbb{Z}^n} \in \mathcal{O}(\mathbb{Z}^n, F)$. Thus $g := \sum_{k\in\mathbb{Z}^n} e_k \otimes M(k)\hat{f}(k) \in \mathcal{D}'(\mathbb{T}^n, F)$, due to Proposition 2.13, and thereby $\hat{g}(k) = M(k)\hat{f}(k)$, due to Theorem 2.15. Furthermore, $g \in B^s_{p,q}(\mathbb{T}^n, F)$ because of (38). Consequently $M : \mathbb{Z}^n \to \mathcal{L}(E, F)$ is a $B^s_{p,q}$−Fourier multiplier. ☑

***Remark* 4.3.** i) In the proof of Theorem 4.2 it was shown that $M : \mathbb{Z}^n \to \mathcal{L}(E)$ is a uniformly bounded function, if M is a discrete $B^s_{p,q}$−Fourier multiplier.

ii) If $M : \mathbb{Z}^n \to \mathcal{L}(E)$ is a uniformly bounded function, then the operator $S_M : \mathcal{D}'(\mathbb{T}^n, E) \longrightarrow \mathcal{D}'(\mathbb{T}^n, E)$ defined by

$$S_M f := \sum_{k\in\mathbb{Z}^n} e_k \otimes M(k)\hat{f}(k) \tag{42}$$

is well defined and

$$(S_M f)\widehat{\ }(k) = M(k)\hat{f}(k) \quad \text{for all } k \in \mathbb{Z}^n,$$

as shown in the proof of $b) \Rightarrow a)$ in the previous theorem.

A definition of L^p-Fourier multiplier, equivalent to the definition given in the introduction, is the following (see [7], Lemma 3.10):

Definition 4.4. Let $1 \leq p < \infty$. A uniformly bounded function $M : \mathbb{Z}^n \to \mathcal{L}(E)$ is called a discrete L^p- Fourier multiplier, if there exists a constant $C > 0$ such that

$$\|S_M f\|_{L^p(\mathbb{T}^n,E)} \leq C \|f\|_{L^p(\mathbb{T}^n,E)}, \qquad \forall\, f \in \mathcal{T}(\mathbb{T}^n, E), \tag{43}$$

where S_M is defined by (42). In this case $S_M \in \mathcal{L}(L^p(\mathbb{T}^n, E))$, due to the density of $\mathcal{T}(\mathbb{T}^n, X)$ in $L^p(\mathbb{T}^n, E)$ (see [7], Proposition 2.4.). If M a discrete L^p- Fourier multiplier, we will write $M \in \widetilde{\mathcal{M}}_p(E)$ and $\|M\|_p := \|M\|_{\widetilde{\mathcal{M}}_p}$ denotes the smallest constant C such that (43) holds.

Theorem 0.1 in [11] motivates the following definition of $UMD-$spaces.

Definition 4.5. E is called a UMD$-$space, if the map $R : \mathbb{Z}^n \to \mathcal{L}(E)$ defined by

$$R(k) := \begin{cases} I_E, & \text{if } k \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{44}$$

is a discrete operator-valued L^p-Fourier multiplier for some (or equivalently, for all) $p \in (1, \infty)$, where I_E is the identity operator in E. We call S_R the operator-valued $n-$dimensional Riesz proyection.

***Remark* 4.6.** It is easy to prove that R is a discrete L^p-Fourier multiplier if and only if, the map $N : \mathbb{Z}^n \to \mathcal{L}(E)$ defined by

$$N(k) := \begin{cases} I_E, & \text{if } k \leq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{45}$$

is also a discrete L^p-Fourier multiplier.

Theorem 4.7. *Let* $M,\ M_l \in \widetilde{\mathcal{M}}_p(E),\ l = 1, 2$*, then:*

a) $M_1 + M_2 \in \widetilde{\mathcal{M}}_p(E)$ *with* $S_{M_1+M_2} = S_{M_1} + S_{M_2}$.

b) $M_1 \cdot M_2 \in \widetilde{\mathcal{M}}_p(E)$ *with* $S_{M_1 \cdot M_2} = S_{M_1} \circ S_{M_2}$, *where* $M_1 \cdot M_2 : \mathbb{Z}^k \to \mathcal{L}(E)$ *is given by* $(M_1 \cdot M_2)(k) := M_1(k) \circ M_2(k)$ *for* $k \in \mathbb{Z}^n$.

c) For each $\alpha \in \mathbb{Z}^n$ *fixed, the application* $M_\alpha : \mathbb{Z}^n \to \mathcal{L}(E)$ *defined by*

$$M_\alpha(k) := M(k - \alpha) \quad \text{for all } k \in \mathbb{Z}^n, \tag{46}$$

is a discrete L_p-*Fourier multiplier with* $\|M_\alpha\|_p = \|M\|_p$.

Proof. The proof of a) and b) follow directly from the definition. For the proof of c) let $\alpha \in \mathbb{Z}^n$ fixed and $f = \sum_{k\in\mathbb{Z}^n} e_k \otimes x_k$ in $\mathcal{T}(\mathbb{T}^n, E)$. Then

$$\begin{aligned}
&\|S_{M_\alpha} f\|^p_{L^p(\mathbb{T}^n,E)} \\
&= \int_{\mathbb{T}^n} \Big\| \sum_{k\in\mathbb{Z}^n} e^{ik\cdot x} M(k-\alpha)x_k \Big\|^p_E \,đx = \int_{\mathbb{T}^n} \Big\| \sum_{\xi\in\mathbb{Z}^n} e^{i(\xi+\alpha)\cdot x} M(\xi) x_{\xi+\alpha} \Big\|^p_E \,đx \\
&\overset{y_\xi := x_{\xi+\alpha}}{=} \int_{\mathbb{T}^n} \Big\| \sum_{\xi\in\mathbb{Z}^n} e^{i\xi\cdot x} M(\xi) y_\xi \Big\|^p_E \,đx = \Big\| S_M\big(\sum_{\xi\in\mathbb{Z}^n} e_\xi \otimes y_\xi \big) \Big\|^p_{L^p(\mathbb{T}^n,E)} \\
&\le \|M\|^p_p \Big\| \sum_{\xi\in\mathbb{Z}^n} e_\xi \otimes y_\xi \Big\|^p_{L^p(\mathbb{T}^n,E)} = \|M\|^p_p \Big\| \sum_{\xi+\alpha\in\mathbb{Z}^n} e_{\xi+\alpha} \otimes x_{\xi+\alpha} \Big\|^p_{L^p(\mathbb{T}^n,E)} \\
&= \|M\|^p_p \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes x_k \Big\|^p_{L^p(\mathbb{T}^n,E)}.
\end{aligned}$$

From this follows that $M_\alpha \in \widetilde{\mathcal{M}}_p(E)$ with $\|M_\alpha\|_p \le \|M\|_p$. In the same way one proves that $\|M\|_p \le \|M_\alpha\|_p$. ☑

Corollary 4.8. E is a UMD-space if and only if for each $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that for

$$f = \sum_{k\in[-K,K]^n} e_k \otimes x_k \;\in\; \mathcal{T}(\mathbb{T}^n, E) \quad (K \in \mathbb{N}_0)$$

there exists some $\beta \in \mathbb{Z}^n$ which satisfies $\beta_j \ge K$ (for all $j = 1, \ldots, n$) and

$$\Big\| \sum_{k\in[0,\beta]} e_k \otimes x_k \Big\|_{L^p(\mathbb{T}^n,E)} \le C_p \,\|f\|_{L^p(\mathbb{T}^n,E)}. \tag{47}$$

Proof. $\Rightarrow$] Let E be a UMD$-$space, $1 < p < \infty$ and

$$f = \sum_{k\in[-K,K]^n} e_k \otimes x_k \;\in\; \mathcal{T}(\mathbb{T}^n, E) \quad (K \in \mathbb{N}_0).$$

Then R and N, defined as in (44) and (45), respectively, are L^p-discrete Fourier multipliers. Due to Theorem 4.7 c), R_α and N_β are also L^p-discrete Fourier multipliers for all $\alpha, \beta \in \mathbb{Z}^n$. We set $x_k := \mathbf{0}$ for $k \notin [-K,K]^n$. Then for all $\alpha, \beta \in \mathbb{Z}^n$ with $\alpha \le \beta$ it holds

$$\begin{aligned}
&\Big\| \sum_{k\in[\alpha,\beta]} e_k \otimes x_k \Big\|_{L^p(\mathbb{T}^n,E)} \\
&\quad = \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes N_\beta(k) R_\alpha(k) x_k \Big\|_{L^p(\mathbb{T}^n,E)} = \Big\| S_{N_\beta\cdot R_\alpha} \sum_{k\in\mathbb{Z}^n} e_k \otimes x_k \Big\|_{L^p(\mathbb{T}^n,E)}
\end{aligned}$$

$$\leq \|N_\beta \cdot R_\alpha\|_p \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes x_k \Big\|_{L^p(\mathbb{T}^n,E)} \leq \|N\|_p \|R\|_p \|f\|_{L^p(\mathbb{T}^n,E)}, \tag{48}$$

due to Theorem 4.7.

$\Leftarrow$] Suppose that for $1 < p < \infty$ there exists $C_p > 0$ such that for each $f = \sum_{k\in[-K,K]^n} e_k \otimes x_k \in \mathcal{T}(\mathbb{T}^n, E)$ we can find $\beta \in \mathbb{Z}^n$ with $\beta_j \geq K$ for all $j = 1, \dots, n$ and such that (47) holds. Then

$$\begin{aligned}\|S_R f\|_{L^p(\mathbb{T}^n,E)} &= \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes R(k)x_k \Big\|_{L^p(\mathbb{T}^n,E)} = \Big\| \sum_{k\in[0,K]^n} e_k \otimes x_k \Big\|_{L^p(\mathbb{T}^n,E)} \\ &= \Big\| \sum_{k\in[0,\beta]} e_k \otimes x_k \Big\|_{L^p(\mathbb{T}^n,E)} \overset{(47)}{\leq} C_p \|f\|_{L^p(\mathbb{T}^n,E)},\end{aligned}$$

for all $f = \sum_{k\in[-K,K]^n} e_k \otimes x_k \in \mathcal{T}(\mathbb{T}^n, E)$, and thus the operator vector-valued $n-$dimensional Riesz proyection is bounded in $L^p(\mathbb{T}^n, E)$. Therefore E is a UMD$-$space. ☑

5. Multipliers of bounded variation; main result

Definition 5.1. Let $G \subset \mathbb{Z}^n$. For a function $M : \mathbb{Z}^n \to \mathcal{L}(E,F)$ let the *restriction* of M to G be defined by

$$M_G(k) := \begin{cases} M(k), & \text{if } k \in G, \\ 0, & \text{if } k \notin G. \end{cases}$$

In particular $M_{\mathbb{Z}^n} = M$. Let $\alpha, \beta \in (\mathbb{Z} \cup \{-\infty, \infty\})^n$ with $\alpha \leq \beta$. For the standard basis of $\mathbb{R}^n$ $\{\delta_j : j = 1, \dots, n\}$, the difference operators Δ^{δ_j} are defined by

$$\Delta^{\delta_j} M_{[\alpha,\beta]}(x) := \begin{cases} M_{[\alpha,\beta]}(x) - M_{[\alpha,\beta]}(x-\delta_j), & \text{if } x_j \neq \alpha_j, \\ 0, & \text{if } x_j = \alpha_j. \end{cases}$$

Moreover, let $\Delta^0 M_{[\alpha,\beta]} := M_{[\alpha,\beta]}$,

$$\Delta^\gamma M_{[\alpha,\beta]} := \Delta^{\gamma_1\delta_1} \cdots \Delta^{\gamma_n\delta_n} M_{[\alpha,\beta]}, \qquad \text{for } \gamma = (\gamma_1, \dots, \gamma_n) \in \{1,0\}^n,$$

and let the *variation* of M on $[\alpha,\beta]$ be defined by

$$\operatorname*{Var}_{[\alpha,\beta]} M_{[\alpha,\beta]} := \sum_{\xi\in[\alpha,\beta]} \left\| \Delta^{\gamma_\xi} M_{[\alpha,\beta]}(\xi) \right\|, \tag{49}$$

where $\gamma_\xi = (\gamma_{\xi_1}, \dots, \gamma_{\xi_n})$ with

$$\gamma_{\xi_j} := \begin{cases} 1, & \text{if } \xi_j \neq \alpha_j, \\ 0, & \text{if } \xi_j = \alpha_j. \end{cases} \tag{50}$$

Note that if $\alpha, \beta \in \mathbb{Z}^n$ with $\alpha \leq \beta$ and $M : \mathbb{Z}^n \to \mathcal{L}(E, F)$ is a function such that $M_{[\alpha,\beta]} = 0$ in $\mathbb{Z}^n$, then $\Delta^{\delta_j} M_{[\alpha,\beta]}(k) = 0$ for all $k \in \mathbb{Z}^n$ and $j = 1, \ldots, n$, and in consequence $\operatorname*{Var}_{[\alpha,\beta]} M_{[\alpha,\beta]} = 0$.

***Remark* 5.2.** Using properties of telescopic sums it can be seen that for each $\beta \in \mathbb{Z}^n$ it holds

$$M(\beta) = \sum_{\xi \in [\alpha,\beta]} \Delta^{\gamma_\xi} M_{[\alpha,\beta]}(\xi) \tag{51}$$

for all $\alpha \in \mathbb{Z}^n$ with $\alpha \leq \beta$.

Definition 5.3. The coarse decomposition of $\mathbb{Z}^n$ is defined by: $D_0 := \{0\}$ and for $d \in \mathbb{N}$,

$$D_d := \big\{k \in \mathbb{Z}^n : |k_1|, \ldots, |k_{l-1}| < 2^{r+1},\ 2^r \leq |k_l| < 2^{r+1},\ |k_{l+1}|, \ldots, |k_n| < 2^r\big\},$$

where $d = nr + l$ with $r \in \mathbb{N}_0$ and $l \in \{1, 2, \ldots, n\}$. For $d \in \mathbb{N}$, $D_d = D_{d^+} \cup D_{d^-}$ where $D_{d^\pm} := \{k \in D_d \,:\, \pm k_l > 0\}$. Furthermore

$$\begin{aligned}&\operatorname*{Var}_{D_d} M := \operatorname*{Var}_{D_{d^+}} M_{D_{d^+}} + \operatorname*{Var}_{D_{d^-}} M_{D_{d^-}} \quad (d \in \mathbb{N}) \quad \text{and}\\ &\operatorname*{Var}_{D_0} M := \operatorname*{Var}_{D_0} M_{D_0}.\end{aligned}$$

Note that for $d \in \mathbb{N}$ and $D_{d^\pm}$ as in the above definition, $D_{d^\pm} = [\alpha_{d^\pm}, \beta_{d^\pm}]$ for some $\alpha_{d^\pm}, \beta_{d^\pm} \in \mathbb{Z}^n$ (for example, $\alpha_{d^+} = \big(-2^{r+1}, \ldots, -2^{r+1}, 2^r, -2^r, \ldots, -2^r\big)$, where 2^r is in the $l-$th position). Therefore $\operatorname*{Var}_{D_{d^\pm}} M_{D_{d^\pm}}$ make sense.

Now, the variational Marcinkiewicz condition, given in [3] by

$$\sup_{k \in \mathbb{Z}} \|M_k\| + \sup_{j \geq 0} \sum_{2^j \leq |k| \leq 2^{j+1}} \|M_{k+1} - M_k\| < \infty,$$

will be generalised by the following definition.

Definition 5.4. Let $M : \mathbb{Z}^n \to \mathcal{L}(E, F)$ be uniformly bounded. M is called a function of bounded variation with respect to the coarse decomposition of $\mathbb{Z}^n$, if there exists a positive constant C such that

$$\sup_{d \in \mathbb{N}_0} \operatorname*{Var}_{D_d} M < C. \tag{52}$$

Lemma 5.5. *Let $j \in \mathbb{N}_0$. Then the function $M := M_j : \mathbb{Z}^n \to \mathcal{L}(E)$ defined by*

$$M(k) := \begin{cases} I_E, & \text{if } k = k_1\delta_1 \text{ with } k_1 \in \big[7 \cdot 2^{j-3}, 2^j\big] \text{ and } j \geq 3,\\ 0, & \text{otherwise}\end{cases} \tag{53}$$

is of bounded variation with respect to the coarse decomposition of $\mathbb{Z}^n$.

Proof. By definition $M : \mathbb{Z}^n \to \mathcal{L}(E)$ satisfies $\|M(k)x\| \leq \|x\|$ for all $k \in \mathbb{Z}^n$ and $x \in E$. Therefore $\{M(k) : k \in \mathbb{Z}^n\} \subset \mathcal{L}(E)$ is uniformly bounded with

$$\|M(k)\|_{\mathcal{L}(E)} \leq 1 \quad \text{for all } k \in \mathbb{Z}^n. \tag{54}$$

We will show that M satisfies (52). In fact, $\operatorname*{Var}_{D_0} M = 0$ because $M_{D_0} = 0$. Now, we fix $d \in \mathbb{N}$ with $d = nj + l$, $j \in \mathbb{N}_0$ and $l \in \{1, \dots, n\}$. Due to (53), we have:

i) If $j < 3$, then $M_{D_d} = 0$ and therefore $\operatorname*{Var}_{D_d} M = 0$.

ii) From $j \geq 3$ and $l \in \{2, \dots, n\}$ it follows $M_{D_d} = 0$ because if $k \in D_d$, then $k_l \neq 0$ and hence $k \neq k_1\delta_1$ which yields $M_{D_d}(k) = 0$. Moreover, by definition $M_{D_d}(k) = 0$ if $k \notin D_d$. Therefore $\operatorname*{Var}_{D_d} M = 0$.

iii) If $j \geq 3$ and $l = 1$, then for each $k \in \mathbb{Z}^n$ it holds that

$$M_{D_d}(k) = \begin{cases} I_E, & \text{if } k = 2^j\delta_1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $M_{D_{d^-}} = M_{D_{d^+}\setminus\{2^j\delta_1\}} = 0$, and then

$$\operatorname*{Var}_{D_d} M = \sum_{k \in D_{d^+}} \left\| \Delta^{\gamma_k} M_{D_{d^+}}(k) \right\|_{\mathcal{L}(E)} = \left\| \Delta^{\gamma_{2^j\delta_1}} M(2^j\delta_1) \right\|_{\mathcal{L}(E)}.$$

Since $D_{d^+} = [\alpha_{d^+}, \beta_{d^+}]$ with $\alpha_{d^+} = (2^j, -2^j, \dots, -2^j)$, we have

$$\begin{aligned}
\operatorname*{Var}_{D_d} M &= \left\| \Delta^{\gamma_{2^j\delta_1}} M_{D_{d^+}}(2^j\delta_1) \right\|_{\mathcal{L}(E)} = \left\| \Delta^{0}\Delta^{\delta_2} \cdots \Delta^{\delta_n} M_{D_{d^+}}(2^j\delta_1) \right\|_{\mathcal{L}(E)} \\
&= \left\| \Delta^{\delta_2} \cdots \Delta^{\delta_n} M_{D_{d^+}}(2^j\delta_1) \right\|_{\mathcal{L}(E)} \\
&= \left\| \Delta^{\delta_2} \cdots \Delta^{\delta_{n-1}} \big(M_{D_{d^+}}(2^j\delta_1) - \underbrace{M_{D_{d^+}}(2^j\delta_1 - \delta_n)}_{=0} \big) \right\|_{\mathcal{L}(E)} \\
&= \left\| \Delta^{\delta_2} \cdots \Delta^{\delta_{n-1}} M_{D_{d^+}}(2^j\delta_1) \right\|_{\mathcal{L}(E)} = \cdots = \left\| M(2^j\delta_1) \right\|_{\mathcal{L}(E)} \leq 1,
\end{aligned}$$

due to (54).

In consequence $M : \mathbb{Z}^n \to \mathcal{L}(E)$ defined by (53) is of bounded variation with respect to the coarse decomposition of $\mathbb{Z}^n$. ☑

Lemma 5.6. *Let $(\phi_j)_{j\geq 0}$ be as in Lemma 3.1.*

a) For $j \geq 1$ it holds

$$\operatorname{supp}(\phi_j) \cap \mathbb{Z}^n \subset \bigcup_{d=n(j-m-1)+1}^{nj+n} D_d \;=:\; D_j^n, \tag{55}$$

where $D_{-d} := D_0$ for $d \in \mathbb{N}$ and m is the smallest non-negative integer satisfying $\sqrt{n} \leq 2^m$.

b) $\operatorname{supp}(\phi_0) \cap \mathbb{Z}^n \subset \bigcup_{d=0}^{n} D_d$.

Proof. a) Let $k \in \mathbb{Z}^n \cap \operatorname{supp}(\phi_j)$ with $j \geq 1$, then $2^{j-1} < |k| < 2^{j+1}$. If $k \notin D_j^n$, $k \in D_d$ with $d = nr + l$ for some $l \in \{1, \dots, n\}$ and some $r \geq j + 1$ or $r \leq j - m - 2$. In the first case it holds

$$|k| \geq |k_l| \geq 2^r \geq 2^{j+1},$$

which contradicts that $|k| < 2^{j+1}$. Now, we consider the second case, i.e. $k \in D_d$ with $d = nr + l$ for some $l \in \{1, \dots, n\}$ and some $r \leq j - m - 2$. If $r \geq 0$, then $|k_s| < 2^{r+1} \leq 2^{j-m-1}$ for all $s \in \{1, \dots, n\}$, and thus

$$|k| \leq \sqrt{n}|k|_\infty \leq \sqrt{n}2^{j-m-1} \leq 2^{j-1},$$

which now is in contradiction with $|k| > 2^{j-1}$. The same happens when $r < 0$ since $D_{-d} = D_0$. In consequence $k \in D_d$ with $d = nr + l$ for some $l \in \{1, \dots, n\}$ and some $r \in \{j - m - 1, \dots, j\}$.

b) If $0 \neq k = (k_1, \dots, k_n) \in \operatorname{supp}(\phi_0) \cap \mathbb{Z}^n$, then $|k_s| < 2$ for all $s \in \{1, 2, \dots, n\}$ and therefore $k \in D_l$ for some $l \in \{1, 2, \dots, n\}$, since otherwise there would be some $r \in \mathbb{N}$ and $s \in \{1, 2, \dots, n\}$ such that $|k_s| \geq 2^r$. Then we have that

$$[\operatorname{supp}(\phi_0) \cap \mathbb{Z}^n] \backslash \{0\} \subset \bigcup_{d=1}^{n} D_d.$$

From this follows b), due to $0 \in D_0$. ☑

Now, we will prove the main result of this paper. But before note that

$$\sum_{k\in[\alpha,\beta]} \sum_{l\in[\alpha,k]} a_l b_k = \sum_{k\in[\alpha,\beta]} a_k \sum_{l\in[k,\beta]} b_l, \tag{56}$$

for all $\alpha, \beta \in \mathbb{Z}^n$ with $\alpha \leq \beta$.

Theorem 5.7. *Let $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Each function $M : \mathbb{Z}^n \to \mathcal{L}(E)$ of bounded variation with respect to the coarse decomposition of $\mathbb{Z}^n$ is a Fourier multiplier on $B^s_{p,q}(\mathbb{T}^n, E)$ if and only if E is a UMD-space.*

Proof. $\Leftarrow$] Let E be a UMD-space. Suppose that $M : \mathbb{Z}^n \to \mathcal{L}(E)$ satisfies (52), $f \in B^s_{p,q}(\mathbb{T}^n, E)$ and let $(\phi_j)_{j\geq 0}$ be as in Lemma 3.1. Due to Lemma 5.6 we obtain that for $j \geq 1$ and $x \in \mathbb{T}^n$ fixed it holds

$$\Big\| \sum_{k\in\mathbb{Z}^n} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_E = \Big\| \sum_{k\in\operatorname{supp}(\phi_j)\cap\mathbb{Z}^n} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_E$$

$$
\begin{aligned}
&\leq \sum_{d=n(j-m-1)+1}^{n(j+1)} \Big\| \sum_{k\in D_d} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_E \\
&\leq \sum_{d=n(j-m-1)+1}^{n(j+1)} \bigg(\Big\| \sum_{k\in D_{d+}} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_E \\
&\qquad\qquad + \Big\| \sum_{k\in D_{d-}} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_E \bigg).
\end{aligned} \tag{57}
$$

Now we consider the sum over $D_{d^+} := [\alpha_{d^+}, \beta_{d^+}]$.

$$
\begin{aligned}
&\Big\| \sum_{k\in D_{d+}} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_E \\
&\overset{(51)}{=} \Big\| \sum_{k\in D_{d+}} \sum_{\xi\in[\alpha_{d+},k]} \Delta^{\gamma_\xi} M_{[\alpha_{d+},k]}(\xi) e^{ik\cdot x}\phi_j(k)\hat{f}(k)\Big\|_E \\
&= \Big\| \sum_{k\in D_{d+}} \sum_{\xi\in[\alpha_{d+},k]} \Delta^{\gamma_\xi} M_{[\alpha_{d+},\beta_{d+}]}(\xi) e^{ik\cdot x}\phi_j(k)\hat{f}(k)\Big\|_E \\
&\overset{(56)}{=} \Big\| \sum_{k\in D_{d+}} \Delta^{\gamma_k} M_{[\alpha_{d+},\beta_{d+}]}(k) \sum_{\xi\in[k,\beta_{d+}]} e^{i\xi\cdot x}\phi_j(\xi)\hat{f}(\xi)\Big\|_E \\
&\leq \sup_{k\in D_{d+}} \sum_{k\in D_{d+}} \Big\| \Delta^{\gamma_k} M_{[\alpha_{d+},\beta_{d+}]}(k)\Big\|_{\mathcal{L}(E)} \Big\| \sum_{\xi\in[k,\beta_{d+}]} e^{i\xi\cdot x}\phi_j(\xi)\hat{f}(\xi)\Big\|_E \\
&\overset{(52)}{\leq} C \sup_{k\in D_{d+}} \Big\| \sum_{\xi\in[k,\beta_{d+}]} e^{i\xi\cdot x}\phi_j(\xi)\hat{f}(\xi)\Big\|_E.
\end{aligned}
$$

We get the same estimate for the sum over D_{d^-} with a similar procedure. Then, from (48) and (57) it follows that

$$
\begin{aligned}
&\Big\| \sum_{k\in\mathbb{Z}^n} e^{ik\cdot x}\phi_j(k)M(k)\hat{f}(k)\Big\|_{L^p(\mathbb{T}^n,E)} \\
&\qquad\qquad \leq 2K_p n(m+2) \Big\| \sum_{k\in\mathbb{Z}^n} e^{ik\cdot x}\phi_j(k)\hat{f}(k)\Big\|_{L^p(\mathbb{T}^n,E)}.
\end{aligned}
$$

Analogously, using Lemma 5.6 *b*) we obtain

$$
\Big\| \sum_{k\in\mathbb{Z}^n} e^{ik\cdot x}\phi_0(k)M(k)\hat{f}(k)\Big\|_{L^p(\mathbb{T}^n,E)} \leq 2K_p n \Big\| \sum_{k\in\mathbb{Z}^n} e^{ik\cdot x}\phi_0(k)\hat{f}(k)\Big\|_{L^p(\mathbb{T}^n,E)}.
$$

Thus, there exists a constant $C > 0$ such that

$$
\Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)M(k)\hat{f}(k)\Big\|_{L^p(\mathbb{T}^n.E)} \leq C \Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes \phi_j(k)\hat{f}(k)\Big\|_{L^p(\mathbb{T}^n.E)}
$$

for all $f \in B^s_{p,q}(\mathbb{T}^n, E)$ and $j \in \mathbb{N}_0$, and therefore

$$\Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes M(k)\hat{f}(k) \Big\|_{B^s_{p,q}(\mathbb{T}^n,E)} \leq C \, \|f\|_{B^s_{p,q}(\mathbb{T}^n,E)} .$$

Thus, Theorem 4.2 implies that M is a $B^s_{p,q}(\mathbb{T}^n, E)$-Fourier multiplier.

$\Rightarrow$] Now, we suppose that each function $M : \mathbb{Z}^n \to \mathcal{L}(E)$ satisfying (52) is a $B^s_{p,q}(\mathbb{T}^n, E)$-Fourier multiplier. Let $(\phi_\ell)_{\ell\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be as in Lemma 3.1 and fix $j \in \mathbb{N}$ with $j \geq 3$. For this j let $M : \mathbb{Z}^n \to \mathcal{L}(E)$ be the function given in Lemma 5.5. Moreover, let us consider an arbitrary sequence $(x_k)_{k\in\mathbb{Z}^n}$ in E and the E-valued trigonometric polynomial

$$h := \sum_{\substack{k=k_1\delta_1, \\ 7\cdot 2^{j-3}\leq k_1\leq 3\cdot 2^{j-1}}} e_k \otimes x_k.$$

This h can be written as

$$h = \sum_{\substack{k=k_1\delta_1, \\ 7\cdot 2^{j-3}\leq k_1\leq 3\cdot 2^{j-1}}} e_k \otimes \hat{h}(k),$$

where $\hat{h}(k) = 0$ for $k \notin \{k_1\delta_1 : 7 \cdot 2^{j-3} \leq k_1 \leq 3 \cdot 2^{j-1}\}$ and $\hat{h}(k) = x_k$ else, due to Remark 3.5. By Lemma 3.1, $\phi_j(x) = 1$ for all $x \in \mathbb{R}^n$ with $7 \cdot 2^{j-3} \leq |x| \leq 3 \cdot 2^{j-1}$ and $\phi_l(x) = 0$ for all $x \in \mathbb{R}^n$ with $7 \cdot 2^{j-3} \leq |x| \leq 3 \cdot 2^{j-1}$ and $l \neq j$. Thus

$$\begin{aligned}
&\Big\| \sum_{k\in\mathbb{Z}^n} e_k \otimes M(k)\hat{h}(k) \Big\|^q_{B^s_{p,q}(\mathbb{T}^n,E)} \\
&= \sum_{l\geq 0} 2^{qsl} \Big\| \sum_{\substack{k=k_1\delta_1, \\ 7\cdot 2^{j-3}\leq k_1\leq 3\cdot 2^{j-1}}} e_k \otimes \phi_l(k)M(k)\hat{h}(k) \Big\|^q_{L^p(\mathbb{T}^n.E)} \\
&\overset{(53)}{=} \sum_{l\geq 0} 2^{qsl} \Big\| \sum_{\substack{k=k_1\delta_1, \\ 7\cdot 2^{j-3}\leq k_1\leq 2^j}} e_k \otimes \phi_l(k)\hat{h}(k) \Big\|^q_{L^p(\mathbb{T}^n.E)} \\
&= 2^{qsj} \Big\| \sum_{\substack{k=k_1\delta_1 \\ 7\cdot 2^{j-3}\leq k_1\leq 2^j}} e_k \otimes \hat{h}(k) \Big\|^q_{L^p(\mathbb{T}^n.E)} .
\end{aligned} \tag{58}$$

Similarly we obtain that

$$\|h\|^q_{B^s_{p,q}(\mathbb{T}^n,E)} = \sum_{l\geq 0} 2^{qsl} \Big\| \sum_{\substack{k=k_1\delta_1, \\ 7\cdot 2^{j-3}\leq k_1\leq 3\cdot 2^{j-1}}} e_k \otimes \phi_l(k)\hat{h}(k) \Big\|^q_{L^p(\mathbb{T}^n.E)}$$

$$= 2^{qsj} \Big\| \sum_{\substack{k=k_1\delta_1,\\ 7\cdot 2^{j-3}\le k_1\le 3\cdot 2^{j-1}}} e_k\otimes \hat{h}(k)\Big\|^q_{L_p(\mathbb{T}^n,E)}. \tag{59}$$

From Theorem 4.2, (58) and (59) it follows that

$$\Big\| \sum_{\substack{k=k_1\delta_1,\\ 7\cdot 2^{j-3}\le k_1\le 2^j}} e_k\otimes x_k\Big\|_{L^p(\mathbb{T}^n.E)} \le C\Big\| \sum_{\substack{k=k_1\delta_1,\\ 7\cdot 2^{j-3}\le k_1\le 3\cdot 2^{j-1}}} e_k\otimes x_k\Big\|_{L^p(\mathbb{T}^n.E)}, \tag{60}$$

where C do not depend on j. Note that we can write (60) as

$$\Big\| \sum_{\ell=7\cdot 2^{j-3}}^{2^j} e_\ell\otimes x_\ell\Big\|_{L^p(\mathbb{T},E)} \le C_n\Big\| \sum_{\ell=7\cdot 2^{j-3}}^{3\cdot 2^{j-1}} e_\ell\otimes x_\ell\Big\|_{L^p(\mathbb{T},E)}$$

for all $(x_\ell)_{\ell\in\mathbb{N}_0}\subset E$ and therefore

$$\Big\| \sum_{k\in[0,2^{j-3}]} e_k\otimes x_k\Big\|_{L^p(\mathbb{T},E)} \le C_n\Big\| \sum_{k\in[-2^{j-3},2^{j-1}]} e_k\otimes x_k\Big\|_{L^p(\mathbb{T},E)} \tag{61}$$

for all $(x_k)_{k\in\mathbb{N}_0}\subset E$ and $j\ge 3$.

Now, let $f=\sum_{k\in[-N,N]} e_k\otimes x_k\in\mathcal{T}(\mathbb{T},E)$ and set $x_k=0$ for $k\notin[-N,N]$. There exists some $j_N\ge 3$ such that $N\le 2^{j_N-3}$ and

$$\begin{aligned}\Big\| \sum_{k\in[0,2^{j_N-3}]} e_k\otimes x_k\Big\|_{L^p(\mathbb{T}.E)} &\overset{(61)}{\le} C_n\Big\| \sum_{k\in[-2^{j_N-3},2^{j_N-1}]} e_k\otimes x_k\Big\|_{L^p(\mathbb{T},.E)}\\ &= C_n\,\|f\|_{L^p(\mathbb{T},E)}.\end{aligned}$$

Therefore E is a UMD-space due to Corollary 4.8. ☑

***Remark* 5.8.** In the proof of Theorem 5.7 we have proved that if E is a $UMD-$space, $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$ and M is of bounded variation with respect to the coarse decomposition of $\mathbb{Z}^n$, then there exists $C>0$ such that

$$\|S_M\|_{\mathcal{L}\left(B^s_{p,q}(\mathbb{T}^n,E)\right)} \le C\sup_{d\in\mathbb{N}_0}\operatorname*{Var}_{D_d} M.$$

***Remark* 5.9.** Let $M:\mathbb{Z}^n\longrightarrow\mathcal{L}(E,F)$ be uniformly bounded.

a) As a particular case of the proof of Theorem 3.24 a) in [7], it holds that M is of bounded variation with $\sup_{d\in\mathbb{N}_0}\operatorname*{Var}_{D_d} M\le 2^{3n+1}$, if the set

$$\left\{|k|^{|\gamma_k|}\,\Delta^{\gamma_k}M_{D_d}(k) : d\in\mathbb{N}_0 \text{ and } k\in D_d\right\}$$

is uniformly bounded.

b) It is easy to see that

$$\left\{ |k|^{|\gamma_k|} \Delta^{\gamma_k} M_{D_d}(k) \,:\, d \in \mathbb{N}_0 \text{ and } k \in D_d \right\} \\ \subset \left\{ |k|^{|\gamma|} \Delta^{\gamma} M(k) : k \in \mathbb{Z}^n \text{ and } \gamma \in \{0,1\}^n \right\}.$$

6. Periodic boundary valued problems

In this section we will study the existence and uniqueness of solution for the problems (1) and (2). Note that $A(t)$ given in (3) is a (formal) lineal differential operator with $\mathcal{L}(E)$ −valued coefficients, where $a^0 : [0,\infty) \times \mathbb{Z}^n \to \mathcal{L}(E)$,

$$a^0(t,k) := \sum_{|\alpha|=m} a_\alpha(t)\, k^\alpha \tag{62}$$

is called its *principal symbol*.

For $\theta \in [0,\pi]$, set $\sum_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| \le \theta\} \cup \{0\}$. Given $\kappa \ge 1$ and $\theta \in [0,\pi)$, the operator A is called (uniformly) (κ,θ) −elliptic if $\sum_\theta \subset \rho\left(-a^0(t,k)\right)$ and

$$\left\| \left[\lambda I + a^0(t,k)\right]^{-1} \right\|_{\mathcal{L}(E)} \le \frac{\kappa}{1+|\lambda|} \quad \text{for all } \lambda \in \sum\nolimits_\theta \tag{63}$$

and $(t,k) \in [0,\infty) \times \mathbb{Z}^n$ with $|k| = 1$. It is called θ−elliptic, if it is (κ,θ) −elliptic for some $\kappa \ge 1$, and normally-elliptic if it is $\frac{\pi}{2}$−elliptic.

***Remark* 6.1.** Similarly to the continuous case (see [1], Remarks 3.1) we have:

a) Condition (63) is equivalent to

$$\left\| \left[\lambda I + a^0(t,k)\right]^{-1} \right\|_{\mathcal{L}(E)} \le \frac{\kappa}{|k|^m + |\lambda|}$$

for all $\lambda \in \sum_\theta$ and $(t,\xi) \in [0,\infty) \times \mathbb{Z}^n$ with $k \neq 0$.

b) The order m is even whenever A is normally elliptic.

***Remark* 6.2.** Let A be uniformly (κ,θ) −elliptic, $a(t,k) := \sum_{|\alpha|\le m} a_\alpha(t)\, k^\alpha$ and $b := a - a^0$. Due to

$$\lambda I + a(t,k) = \left[I + b(t,k)\left(\lambda I + a^0(t,k)\right)^{-1}\right]\left(\lambda I + a^0(t,k)\right),$$

by a Neumann series argument, there exists some $\omega_0 > 0$ such that

$$\left\| \left[\lambda I + a(t,k)\right]^{-1} \right\|_{\mathcal{L}(E)} \le \frac{2\kappa}{|k|^m + |\lambda|} \tag{64}$$

for all $\lambda \in \omega_0 + \sum_\theta$ and $(t,k) \in [0,\infty) \times \mathbb{Z}^n$.

Proposition 6.3. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, E a $UMD-$space, A an uniformly (κ, θ)-elliptic differential operator satisfaying $\sum_{|\alpha| \leq m} \|a_\alpha\|_\infty \leq C$, and let*

$$\mathcal{A} := \mathcal{A}_{B^s_{p,q}} : B^{s+m}_{p,q}(\mathbb{T}^n, E) \to B^s_{p,q}(\mathbb{T}^n, E), \quad u \longmapsto Au,$$

be the $B^s_{p,q}-$realization of A. Then there exist $C_\kappa \geq 1$ and $\omega_0 > 0$ such that $\omega_0 + \sum_\theta \subset \rho\left(-\mathcal{A}\left(t\right)\right)$ and

$$\left\| \left(\lambda I + \mathcal{A}\left(t\right)\right)^{-1} \right\|_{\mathcal{L}\left(B^s_{p,q}(\mathbb{T}^n, E)\right)} \leq \frac{C_\kappa}{1 + |\lambda|} \tag{65}$$

for all $\lambda \in \omega_0 + \sum_\theta$ and $t \geq 0$. In particular, each $\mathcal{A}\left(t\right)$ generates an analytic semigroup on $B^s_{p,q}(\mathbb{T}^n, E)$, if A is uniformly normally elliptic.

For the proof of this proposition we will use the following notations and lemma, whose proof can be found in [7].

Giving $\alpha \in \mathbb{N}^n_0 \backslash \{0\}$, let

$$\mathcal{Z}_\alpha := \left\{ \mathcal{W} = \left(w^1, ..., w^r\right) : 1 \leq r \leq |\alpha|, 0 < w^j \leq \alpha, \sum_{j=1}^r w^j = \alpha \right\}$$

denote the set of all additive decompositions of α into $r = r_{\mathcal{W}}$ multi-indices. For the sake of consistence we set $\mathcal{Z}_0 := \{\varnothing\}$ and $r_\varnothing := 0$. For $\mathcal{W} = \left(w^1, ..., w^r\right) \in \mathcal{Z}_\alpha$ let w^j_* be defined by

$$w^j_* := \sum_{l=j+1}^r w^l.$$

Lemma 6.4 ([7], Lemma 7.1c)**.** *Let $S : \mathbb{Z}^n \to \mathcal{L}\left(E, F\right)$ be a function such that the inverse $\left(S^{-1}\right)(k) := \left(S(k)\right)^{-1}$ exists for all $k \in \mathbb{Z}^n$. Then for $\alpha \in \mathbb{N}^n_0$, we have*

$$\Delta^\alpha \left(S^{-1}\right)(k) = \sum_{\mathcal{W} \in \mathcal{Z}_\alpha} (-1)^{r_{\mathcal{W}}} \left(S^{-1}\right)(k - \alpha) \prod_{j=1}^{r_{\mathcal{W}}} \Big(\big(\Delta^{w^j} S\big) S^{-1} \Big)(k - w^j_*)$$

for $k \in \mathbb{Z}^n$.

Proof of Proposition 6.3. Let a be as in Remark 6.2 and $\gamma \in \{0,1\}^n$. For $\lambda \in \omega_0 + \sum_\theta$ and $t \geq 0$, we define $M_{\lambda,t}\left(k\right) := \lambda \left(\lambda + a\left(t, \cdot\right)\right)^{-1}(k)$, $k \in \mathbb{Z}^n$. Using Lemma 6.4, the triangular inequality, the fact that $\Delta^{w^j}\left(k - w^j_*\right)^\alpha$ is a polynomial in $k - w^j_*$ of degree not greater than $|\alpha| - |w^j|$ and (64), we obtain for all $k \in \mathbb{Z}^n$ that

$$|k|^{|\gamma|} \left\| \Delta^\gamma M_{\lambda,t}\left(k\right) \right\|$$

$$
\begin{aligned}
&\leq |\lambda|\,|k|^{|\gamma|} \sum_{\mathcal{W}\in\mathcal{Z}_\gamma} \left\| \left(\lambda + a\left(t,k-\gamma\right)\right)^{-1} \right\| \\
&\qquad\qquad \cdot \prod_{j=1}^{r_{\mathcal{W}}} \left(\left\| \Delta^{w^j} a\left(t, k - w_*^j\right) \right\| \left\| \left(\lambda + a\left(t, k - w_*^j\right)\right)^{-1} \right\| \right) \\
&\leq C\,|\lambda|\,|k|^{|\gamma|} \sum_{\mathcal{W}\in\mathcal{Z}_\gamma} \left\| \left(\lambda + a\left(t,k-\gamma\right)\right)^{-1} \right\| \\
&\qquad\qquad \cdot \prod_{j=1}^{r_{\mathcal{W}}} \Big(\sum_{|\alpha|\leq m} \left|\Delta^{w^j} \left(k - w_*^j\right)^{\alpha}\right| \left\| \left(\lambda + a\left(t, k - w_*^j\right)\right)^{-1} \right\| \Big) \\
&\leq C 2\kappa\,|k|^{|\gamma|} \sum_{\mathcal{W}\in\mathcal{Z}_\gamma} \frac{|\lambda|}{|k-\gamma|^m + |\lambda|} \\
&\qquad\qquad \cdot \prod_{j=1}^{r_{\mathcal{W}}} \Big(\sum_{|\alpha|\leq m} \sum_{\text{finite}} c_{\alpha,w^j} |k - w_*^j|^{m-|w^j|} \frac{2\kappa}{|k - w_*^j|^m + |\lambda|} \Big) \\
&\leq C_\kappa\,|k|^{|\gamma|} \sum_{\mathcal{W}\in\mathcal{Z}_\gamma} C_{\mathcal{W}} \frac{|\lambda|}{|k-\gamma|^m + |\lambda|} \prod_{j=1}^{r_{\mathcal{W}}} \Big(|k - w_*^j|^{-|w^j|} \frac{|k - w_*^j|^m}{|k - w_*^j|^m + |\lambda|} \Big) \\
&\leq C_\kappa\,|k|^{|\gamma|} \sum_{\mathcal{W}\in\mathcal{Z}_\gamma} C_{\mathcal{W}} \prod_{j=1}^{r_{\mathcal{W}}} |k - w_*^j|^{-|w^j|} \leq \widehat{C}_\kappa,
\end{aligned}
$$

where $\widehat{C}_\kappa$ is a constant which do not depend on λ and t, and $\|\cdot\|$ abbreviates $\|\cdot\|_{\mathcal{L}(E)}$. It follows that $M_{\lambda,t}$ is of bounded variation due to Remark 5.9. Thus Theorem 5.7 implies that $M_{\lambda,t}$ is a discrete Fourier multiplier on $B^s_{p,q}(\mathbb{T}^n, E)$ and (65) holds due to Remark 5.8. ☑

Corollary 6.5. *Let $0 < \rho < 1$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, E a UMD-space and A a uniformly normally elliptic differential operator satisfaying*

$$
(t \mapsto a_\alpha(t)) \in C^\rho\left([0,T], \mathcal{L}(E)\right) \tag{66}
$$

for all $|\alpha| \leq m$.

a) If $f \in C^\rho\left([0,T], B^s_{p,q}(\mathbb{T}^n, E)\right)$, then the problem (1) has a unique classical solution

$$
u \in C^\rho\left((0,T], B^{m+s}_{p,q}(\mathbb{T}^n, E)\right) \cap C^{1+\rho}\left((0,T], B^s_{p,q}(\mathbb{T}^n, E)\right).
$$

b) If $s_1 \in \mathbb{R}$, $1 \leq p_1, q_1 \leq \infty$ and ω_0 as in Proposition 6.3, then for each $f \in B^{s_1}_{p_1,q_1}\left(\mathbb{T}, B^s_{p,q}(\mathbb{T}^n, E)\right)$ and $\omega \geq \omega_0$ there exists a unique $u \in B^{1+s_1}_{p_1,q_1}\left(\mathbb{T}, B^s_{p,q}(\mathbb{T}^n, E)\right)$ such that $u(t) + A_\omega u(t) = f(t)$ for almost all $t \in [0, 2\pi]$. In this sense u is the unique solution for the problem (2).

Proof. a) This is a consequence of Proposition 6.3, (66), Theorems 1.2 and 1.3 in [9] and Sätzes 4.11 and 4.12 in [8].

b) This follows from Proposition 6.3 and Theorem 5.1 in [3].

☑

***Remark* 6.6.** The results of this section are still valid if the operator $A(t)$ defined in (3) is replaced by a Fourier-multiplier operator $A(t) = \mathcal{F}_{\mathbb{T}^n}^{-1}\big(a(t,\cdot)\mathcal{F}_{\mathbb{T}^n}\big)$, where $\{a(t,\cdot) : \mathbb{Z}^n \to \mathcal{L}(E)\,;\, t \geq 0\}$ is a family of bounded variation symbols satisfying the condition (64) uniformly.

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