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Solutions of the hexagon equation for abelian anyons

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Abstract. We address the problem of determining the obstruction to existence of solutions of the hexagon equation for abelian fusion rules and the classification of prime abelian anyons.

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1. Introduction

Anyons are two-dimensional particles which in contrast to boson or fermions satisfy exotic statistics. The exchange of two identical anyons can in general be described by either abelian or non-abelian statistics. In the abelian case an exchange of two particles gives rise to a complex phase $e^{2\pi i\theta}$. Bosons and fermions correspond only to the phase changes $+1$ and -1 respectively. Particles with non-real phase change are considered anyons. In general, the statistics of anyons is described by unitary operators acting on a finite dimensional degenerate ground-state manifold, [12].

There has been increased interest in non-abelian anyons since they possess the ability to store, protect and manipulate quantum information [12, 13, 10, 22, 18]. In contrast, abelian anyons only seem good as quantum memory. Moreover, abelian anyons are interesting for two reasons. First, they have simpler physical realizations than non-abelian anyons; and second, gauging a finite group of topological symmetries of an abelian anyon theory, when it possible, leads to a new anyon theory that is in general is non-abelian, [4]. Moreover, all concrete known examples of non-abelian anyon theories with integer global dimension are constructed from a gauging of an abelian anyon theory.

Mathematically speaking, an abelian anyon theory is a modular pointed category ([5, 9]), and the latter comprise are the class of modular categories which are best understood. Abelian anyons correspond to triples (A, ω, c), where A is a finite abelian group and $(\omega, c) \in Z^3_{ab}(A, \mathrm{U}(1))$ is an abelian 3-cocycle. The set of modular categories up to gauge equivalence with a fixed abelian group A forms an abelian group denoted by $H^3_{ab}(A, \mathrm{U}(1))$ and called the *third abelian cohomology group* of A. The groups $H^3_{ab}(A, B)$ were defined and studied by Eilenberg and MacLane in [7, 8] for any pair of abelian groups [7, 8]. In this work, we address the problem of determining for an ordinary 3-cocycle $\omega \in Z^3(A, B)$ the obstruction to the existence of a map $c : A \times A \to B$ such that $(\omega, c) \in Z^3_{ab}(A, B)$. To that end, we construct a double complex associated to a finite abelian group and a map from the ordinary group cohomology to the total cohomology of the double complex. We find several exact sequences involving $H^3_{ab}(A, B)$ and provide an explicit method for the construction of all possible abelian 3-cocycles. We finish the note with a reformulation of an old result of Wall [21] and Durfee [6] on the classification of indecomposable symmetric forms on finite abelian groups in terms of classification of prime abelian anyons.

The paper is organized as follows. In Section 2 we recall the definitions of group cohomology and abelian group cohomology. Section 3 contains a brief introduction to fusion algebras and the pentagon and hexagon equation. In section 4 we present the main results of the paper. We recall a theorem of Eilenberg and MacLane about the isomorphism between $H^3_{ab}(A, B)$ and $\mathrm{Quad}(A, B)$ (the group of all quadratic forms from A to B). We also show that $\mathrm{Quad}(A, \mathbb{R}/\mathbb{Z})$ can be computed inductively from a decomposition of A as direct sum of cyclic groups. In this section we also define the obstruction for the existence of solutions of the hexagon equation. Section 5 contains the classification of prime abelian anyon theories.

2. Preliminaries

In this section we present some basic definitions of group cohomology and abelian group cohomology. A lot of this material can be found in [7] and [8].

We will denote by $\mathrm{U}(1)$ the group of complex numbers of modulus 1, which we will often write additively through the identification with $\mathbb{R}/\mathbb{Z}$.

Given an abelian group A we will denote by $S^2(A), \wedge^2 A$ and $A^{\otimes 2}$ the second symmetric power, second exterior power and second tensor power of A, respectively. Here, we see A as a $\mathbb{Z}$-module.

Given a group G we will denote by $\widehat{G}$ to the abelian group of all linear character of G, that is

$$\widehat{G} = \mathrm{Hom}(G, \mathrm{U}(1)) = \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z}).$$

2.1. Group cohomology

We will recall the usual cocycle description of group cohomology associated to the normalized bar resolution of $\mathbb{Z}$, see [7] for more details. Let G be a discrete group and let A be a $\mathbb{Z}[G]$-module. Let $C^0(G,A)=A$, and let

$$C^n(G,A)=\{f:\underbrace{G\times\cdots\times G}_{n-times}\to A \mid f(x_1,\ldots,x_n)=0,\ \text{if } x_i=1_G \text{ for some } i\},$$

for $n\geq 1$.

Consider the cochain complex

$$0\longrightarrow C^0(G,A)\xrightarrow{\delta_0}C^1(G,A)\xrightarrow{\delta_1}C^2(G,A)\cdots C^n(G,A)\xrightarrow{\delta_n}C^{n+1}(G,A)\cdots$$

where

$$\begin{aligned}\delta_n(f)(x_1,x_2,\ldots,x_{n+1}) &= x_1\cdot f(x_2,\ldots,x_{n+1})\\ &+\sum_{i=1}^{n}(-1)^i f(x_1,\ldots,x_{i-1},x_ix_{i+1},x_{i+2},\ldots,x_{n+1})\\ &+(-1)^{n+1}f(x_1,\ldots,x_n).\end{aligned}$$

We denote, $Z^n(G,A):=\ker(\delta_n)$ (n-cocycles), $B^n(G,A):=\operatorname{Im}(\delta_{n-1})$ (n-coboundaries) and

$$H^n(G,A):=Z^n(G,A)/B^n(G,A)\quad(n\geq 1),$$

the cohomology of G with coefficients in A.

2.2. Eilenberg-MacLane cohomology theory of abelian groups

Let A be an abelian group. A space X having only one nontrivial homotopy group $\pi_n(X)=A$ is called the Eilenberg-MacLane space $K(A,n)$. Such space can be constructed as a CW complex or using the *Dold-Kan correspondence* between chain complexes and simplicial abelian groups. If $A[n]$ is the chain complex which is A in dimension n and trivial elsewhere; the geometric realization of the corresponding simplicial abelian group is a $K(A,n)$ space.

The abelian cohomology theory of the abelian group M with coefficients in the abelian group N is defined as

$$H^n_{ab}(M,N):=\{\text{Homotopy classes}\quad K(M,2)\to K(N,n+1)\}$$

In [7, 8] Eilenberg and MacLane defined a chain complex associated to any abelian group M to compute the abelian cohomology groups of the space $K(M,2)$.

We use the following notations for X, Y any two groups:

- $X^p|Y^q = \{\mathbf{x}|\mathbf{y} = (x_1, \dots, x_p|y_1, \dots, y_q), x_i \in X, y_j \in Y\}, \;\; p, q \geq 0.$
- $\mathrm{Shuff}(p, q)$ the set of (p, q)-shuffles, i.e. an element in the symmetric group $\mathbb{S}_{p+q}$ such that $\lambda(i) < \lambda(j)$ whenever $1 \leq i < j \leq p$ or $p+1 \leq i < j \leq p+q$.
- Any $\pi \in \mathrm{Shuff}(p, q)$ defines a map

$$\pi : X^{p+q} \to X^{p+q} \tag{1}$$
$$(x_1, \dots, x_{p+q}) \mapsto (x_{\pi(1)} \dots, x_{\pi(p+q)}) \tag{2}$$

Let M and N be abelian groups. Define the abelian group $C^0_{ab}(M, N) = 0$ and for $n > 0$

$$C^n_{ab}(M, N) = \bigoplus_{p_1,\dots,p_r \geq 1 : r + \sum_{i=1}^r p_i = n+1} \mathrm{Maps}(M^{p_1}|\cdots|M^{p_r}, N),$$

where $\mathrm{Maps}(M^{p_1}|\cdots|M^{p_r}, N)$ denotes the abelian group of all maps from $M^{p_1}|\cdots|M^{p_r}$ to N.

The coboundary maps are defined as

$$\partial : C^n_{ab}(M, N) \to C^{n+1}_{ab}(M, N)$$

$$\begin{aligned}\partial(f)(\mathbf{x}^1|\mathbf{x}^2|\dots|\mathbf{x}^r) = & \sum_{\substack{1\leq i\leq r\\ 0\leq j\leq p_i}} (-1)^{\epsilon_{i-1}+j} f(\mathbf{x}^1|\dots|d_j\mathbf{x}^i|\cdots|\mathbf{x}^r) \\ & + \sum_{\substack{1\leq i\leq r-1\\ \pi\in\mathrm{Shuff}(p_i,p_{i+1})}} (-1)^{\epsilon_i+\epsilon(\pi)} f(\mathbf{x}^1|\dots|\pi(\mathbf{x}^i|\mathbf{x}^{i+1})|\cdots|\mathbf{x}^r)\end{aligned}$$

where

$$\begin{aligned} d_j : M^{p_i} &\to M^{p_i-1} \\ (x_1, \dots, x_{p_i}) &\mapsto (x_1, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_{p_i}) \end{aligned}$$

are the face operators; $\epsilon_i = p_1 + \cdots p_i + i$ and $\epsilon(\pi)$ is the sign of the shuffle π.

We denote, $Z^n_{ab}(M, N) := \ker(\partial_n)$ (called abelian n-cocycles), $B^n_{ab}(M, N) := \mathrm{Im}(\partial_{n-1})$ (called abelian n-coboundaries) and

$$H^n_{ab}(M, N) := Z^n_{ab}(M, N)/B^n_{ab}(M, N) \;\; (n \geq 1),$$

the abelian cohomology of M with coefficients in N.

Let us write the first cochains groups and their coboundaries.

- $C^0_{ab}(M, N) = 0,$

- $C^1_{ab}(M,N) = \mathrm{Maps}(M,N)$,
- $C^2_{ab}(M,N) = \mathrm{Maps}(M^2,N)$,
- $C^3_{ab}(M,N) = \mathrm{Maps}(M^3,N) \oplus \mathrm{Maps}(M|M,N)$
- $C^4_{ab}(M,N) = \mathrm{Maps}(M^4,N) \oplus \mathrm{Maps}(M^2|M,N) \oplus \mathrm{Maps}(M|M^2,N)$.

Thus

- Since $C^0_{ab}(M,N) = 0$, $H^1_{ab}(M,N) = Z^1_{ab}(M,N) = \mathrm{Hom}(M,N)$.
- For $f \in C^2_{ab}(M,N)$, we have

  $$\partial(f)(x,y,z) = f(y,z) - f(xy,z) + f(y,z) - f(x,yz), \;\; \partial(f)(x|y) = f(x,y) - f(y,x).$$

  Then $H^2_{ab}(M,N) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M,N)$ the group of abelian extensions of M by N.
- Finally, for $(\omega, c) \in C^3_{ab}(M,N)$ we have
  - (i) $\partial(\omega)(x,y,z,t) = \omega(y,z,t) - \omega(x+y,z,t) + \omega(x,y+z,t) - \omega(x,y,z+t) + \omega(x,y,z)$,
  - (ii) $\partial(c)(x|y,z) = c(x|z) - c(x|y+z) + c(x|y) + w(x,y,z) - \omega(y,x,z) + \omega(y,z,x)$,
  - (iii) $\partial(c)(x,y|z) = c(y|z) - c(x+y|z) + c(x|z) - \omega(x,y,z) + \omega(x,z,y) - \omega(z,x,y)$.

3. Fusion algebras

A fusion algebra is based on a finite set A (where elements will be called anyonic particles or simply particles). The elements in A will be denoted by $a, b, c, \ldots$.

For every particle a there exists a unique anti-particle, that we denote by $\overline{a}$. There is a unique trivial "vacuum" particle denoted by 1 (or sometimes 0).

The fusion algebra has *fusion rules*

$$a \times b = \sum_c N^c_{ab} c$$

where $N^c_{ab} \in \mathbb{Z}^{\geq 0}$ that count the number of ways the particles a and b fuse into c. The fusion rules obey the following relations

- associativity $(a \times b) \times c = a \times (b \times c)$,
- commutativity $a \times b = b \times a$,
- the vacuum is the identity for the fusion product, $a \times 1 = a$,

- the rule $a \mapsto \overline{a}$ defines an involution of the fusion rules, that is,

$$\overline{1} = 1, \quad \overline{\overline{a}} = a, \quad \overline{a} \times \overline{b} = \overline{a \times b},$$

where

$$\overline{a \times b} = \sum_c N_{ab}^c \overline{c}.$$

- The fusion of a with its antiparticle $\overline{a}$ contains the vacuum with multiplicity one, that is

$$N_{a\overline{a}}^1 = 1.$$

A fusion algebra is called *abelian* if

$$\sum_c N_{ab}^c = 1$$

for every a and b. This is if the fusion of two particles $a \times b = c$, is again one of the particles in A. If A is an abelian fusion algebra, then the fusion product defines a structure of abelian group on A and conversely every finite abelian group defines a set of abelian fusion rules.

If we have a fusion algebra on the set A with n particles, we can assign to each particle a the matrix N_a whose entries are exactly N_{ab}^c in the position (b, c). This is an $n \times n$ integer matrix that contains all the information about the fusion rules of a. It satisfies the equation

$$N_a N_b = \sum_c N_{ab}^c N_c.$$

3.1. The Pentagon equation for abelian anyons

Throughout this section, we will follow the notation of [20], slightly modified for our purposes. For further reading on these topics we direct the reader to [1, 2, 13].

Let A be a fusion algebra. Assign to each fusion product a vector space $\begin{bmatrix} \mathrm{c} \\ \mathrm{a,b} \end{bmatrix}$ of dimension $N_{a,b}^c$. If $N_{a,b}^c = 0$ then $\begin{bmatrix} \mathrm{c} \\ \mathrm{a,b} \end{bmatrix} = 0$. The vector spaces $\begin{bmatrix} \mathrm{c} \\ \mathrm{a,b} \end{bmatrix}$ are called the *fusion spaces* of A. The fusion space takes in account the ways in which the anyons a and b can fuse together to give c.

Now, consider the fusion of the particles a, b and c. The associativity of the fusion rules ensures that $(a \times b) \times c = a \times (b \times c)$, but with the fusion spaces there are two different objects that can do this. The first being

$$\bigoplus_{i \in A} \begin{bmatrix} \mathrm{i} \\ \mathrm{a,b} \end{bmatrix} \otimes \begin{bmatrix} \mathrm{d} \\ \mathrm{i,c} \end{bmatrix},$$

and the second being

$$\bigoplus_{i\in A}\begin{bmatrix}\mathrm{i}\\ \mathrm{b,c}\end{bmatrix}\otimes\begin{bmatrix}\mathrm{d}\\ \mathrm{a,i}\end{bmatrix}.$$

We would like a family of linear isomorphisms that takes in account the distinct ways of "associating" fusion spaces in this context, thus we have the following definition:

An F-matrix for a fusion algebra A is a family of linear isomorphisms

$$F\begin{bmatrix}\mathrm{d}\\ \mathrm{a,b,c}\end{bmatrix}:\bigoplus_{i\in A}\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}\otimes\begin{bmatrix}\mathrm{d}\\ \mathrm{i,c}\end{bmatrix}\longrightarrow\bigoplus_{j\in A}\begin{bmatrix}\mathrm{j}\\ \mathrm{b,c}\end{bmatrix}\otimes\begin{bmatrix}\mathrm{d}\\ \mathrm{a,j}\end{bmatrix}$$

which satisfies the pentagon equation:

$$\begin{array}{ccccc}
\bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}\begin{bmatrix}\mathrm{j}\\ \mathrm{i,c}\end{bmatrix}\begin{bmatrix}\mathrm{e}\\ \mathrm{j,d}\end{bmatrix} & \xrightarrow{F\begin{bmatrix}\mathrm{j}\\ \mathrm{a,b,c}\end{bmatrix}} & \bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{b,c}\end{bmatrix}\begin{bmatrix}\mathrm{j}\\ \mathrm{a,i}\end{bmatrix}\begin{bmatrix}\mathrm{e}\\ \mathrm{j,d}\end{bmatrix} & \xrightarrow{F\begin{bmatrix}\mathrm{e}\\ \mathrm{a,i,d}\end{bmatrix}} & \bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{b,c}\end{bmatrix}\begin{bmatrix}\mathrm{j}\\ \mathrm{i,d}\end{bmatrix}\begin{bmatrix}\mathrm{e}\\ \mathrm{a,j}\end{bmatrix}, \\
\Big\downarrow F\begin{bmatrix}\mathrm{e}\\ \mathrm{i,c,d}\end{bmatrix} & & & & \Big\downarrow F\begin{bmatrix}\mathrm{j}\\ \mathrm{b,c,d}\end{bmatrix} \\
\bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}\begin{bmatrix}\mathrm{j}\\ \mathrm{c,d}\end{bmatrix}\begin{bmatrix}\mathrm{e}\\ \mathrm{i,j}\end{bmatrix} & \xrightarrow{\tau} & \bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{c,d}\end{bmatrix}\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}\begin{bmatrix}\mathrm{e}\\ \mathrm{i,j}\end{bmatrix} & \xrightarrow{F\begin{bmatrix}\mathrm{e}\\ \mathrm{a,b,j}\end{bmatrix}} & \bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{c,d}\end{bmatrix}\begin{bmatrix}\mathrm{j}\\ \mathrm{b,i}\end{bmatrix}\begin{bmatrix}\mathrm{e}\\ \mathrm{a,j}\end{bmatrix},
\end{array}$$

or simply

$$\sum_{i,j\in A}F\begin{bmatrix}\mathrm{j}\\ \mathrm{b,c,d}\end{bmatrix}F\begin{bmatrix}\mathrm{e}\\ \mathrm{a,i,d}\end{bmatrix}F\begin{bmatrix}\mathrm{j}\\ \mathrm{a,b,c}\end{bmatrix}=\sum_{i,j\in A}F\begin{bmatrix}\mathrm{e}\\ \mathrm{i,c,d}\end{bmatrix}F\begin{bmatrix}\mathrm{e}\\ \mathrm{a,b,j}\end{bmatrix}. \tag{3}$$

In the diagram above,

$$\tau:\bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}\begin{bmatrix}\mathrm{j}\\ \mathrm{c,d}\end{bmatrix}\longrightarrow\bigoplus_{i,j}\begin{bmatrix}\mathrm{i}\\ \mathrm{c,d}\end{bmatrix}\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}$$

is the operator that swaps the components of $\begin{bmatrix}\mathrm{i}\\ \mathrm{a,b}\end{bmatrix}$ and $\begin{bmatrix}\mathrm{j}\\ \mathrm{c,d}\end{bmatrix}$. We also omit the tensor products and identity operators for simplicity.

We want that any transformation through the F-matrix starting and ending in the same spaces to be the same. Equation (3) ensures this.

Let us assume that A is an abelian fusion algebra. Then we must have that each fusion space is either one or zero dimensional and an F-matrix for A is determined by a family of scalars

$$\left\{\omega(a,b,c) := F\begin{bmatrix} \text{d} \\ \text{a,b,c} \end{bmatrix} \in \mathbb{C}^*\right\}_{a,b,c\in A}$$

such that

$$\omega(a_1a_2,a_3,a_4)\omega(a_1,a_2,a_3a_4) = \omega(a_1,a_2,a_3)\omega(a_1,a_2a_3,a_4)\omega(a_2,a_3,a_4), \quad (4)$$

for all $a_1, a_2, a_3, a_4 \in A$.

A function $\omega : A \times A \times A \to \mathrm{U}(1)$ satisfying equation (4) is just a standard 3-cocycle. Thus, the set of all solutions of the pentagon equation of an abelian fusion algebra is exactly $Z^3(A, \mathrm{U}(1))$.

A gauge transformation between two solution of pentagon equation $\omega, \omega' \in Z^3(A, \mathrm{U}(1))$ is determined by a family of non zero scalars $\{u(a,b)\}_{a,b\in A}$ such that

$$\omega'(a,b,c) = \frac{u(ab,c)u(a,b)}{u(a,bc)u(b,c)}\omega(a,b,c),$$

for all $a, b, c \in A$. Thus, the set of gauge equivalence classes of solution of the pentagon equation is the $H^3(A, \mathrm{U}(1))$.

3.2. The hexagon equation

In this section we will assume that A is an abelian group and $\omega \in Z^3(A, \mathrm{U}(1))$ is a 3-cocycle.

In the previous section, we extended the associativity of the fusion rules to the associativity of the fusion spaces through a family of linear operators called F-matrix. Now, we want to extend the commutativity as well.

In order to do this, we need a family of unitary operators

$$R^c_{a,b} : \begin{bmatrix} \text{c} \\ \text{a,b} \end{bmatrix} \to \begin{bmatrix} \text{c} \\ \text{b,a} \end{bmatrix}$$

that satisfy

$$R^a_{a,1} = \mathrm{Id} = R^a_{1,a}$$

and the hexagon equations

$$\sum_{i,j,k} R^i_{a,c} F\begin{bmatrix} \text{j} \\ \text{b,a,c} \end{bmatrix} R^k_{a,b} = \sum_{i,j,k} F\begin{bmatrix} \text{i} \\ \text{a,c,b} \end{bmatrix} R^j_{b,c} F\begin{bmatrix} \text{k} \\ \text{a,b,c} \end{bmatrix} \quad (5)$$

$$\sum_{i,j,k} (R^i_{a,c})^{-1} F\begin{bmatrix} \text{j} \\ \text{b,a,c} \end{bmatrix} (R^k_{a,b})^{-1} = \sum_{i,j,k} F\begin{bmatrix} \text{i} \\ \text{a,c,b} \end{bmatrix} (R^j_{b,c})^{-1} F\begin{bmatrix} \text{k} \\ \text{a,b,c} \end{bmatrix}. \quad (6)$$

We will call such family an *R-matrix*, or a *braiding*, for A. As before, these equations imply that any transformation within the R and the F-matrices are independent of the path.

In the case where A is an abelian theory with an associated 3-cocycle ω, a braiding is determined by a family of scalars $\{c_{a,b}\}_{a,b\in A}$ that satisfy the equations

$$\frac{\omega(b,a,c)}{\omega(a,b,c)\omega(b,c,a)} = \frac{\mathrm{c}(a,bc)}{\mathrm{c}(a,b)\mathrm{c}(a,c)}$$
$$\frac{\omega(a,b,c)\omega(c,a,b)}{\omega(a,c,b)} = \frac{\mathrm{c}(ab,c)}{\mathrm{c}(a,c)\mathrm{c}(b,c)}.$$

Thus, (ω, c) is an *abelian* 3-*cocycle*. The solutions of the hexagon up to gauge equivalence is the group $H^3_{ab}(A, \mathrm{U}(1))$.

4. Computing $H^3_{ab}(M,N)$

4.1. Quadratic forms and $H^3_{ab}(A,N)$

Let A and B be abelian groups. A quadratic form from A to B is a function $\gamma : A \to B$ such that

$$\gamma(a) = \gamma(-a) \tag{7}$$
$$\gamma(a+b+c) - \gamma(b+c) - \gamma(a+c) - \gamma(a+b) + \gamma(a) + \gamma(b) + \gamma(c) = 0, \tag{8}$$

for any $a, b, c \in A$. A map $\gamma : A \to \mathcal{B}$ such that $\gamma(a) = \gamma(-a)$ satisfies (8) if and only if the map

$$b_\gamma : A \times A \to B$$
$$(a_1, a_2) \mapsto \gamma(a_1 + a_2) - \gamma(a_1) - \gamma(a_2)$$

is a symmetric bilinear form. It follows by induction that $\gamma(na) = n^2\gamma(a)$ for any positive integer n.

We will denote by $\mathrm{Quad}(A, B)$, the group of all quadratic forms from A to B. Eilenberg and MacLane proved in [8, Theorem 26.1] that for any two abelian groups A, B, the map

$$\mathrm{Tr} : H^3_{ab}(A, B) \to \mathrm{Quad}(A, B) \tag{9}$$
$$(\omega, c) \mapsto [a \mapsto c(a, a)]$$

is a group isomorphism. If A is a finite abelian group, the group $\mathrm{Quad}(A, \mathbb{R}/\mathbb{Z})$ can be computed using the following results.

Proposition 4.1. *If n is odd, then* $\mathrm{Quad}(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z})$ *is a cyclic group of order n, with generator given by*

$$q_n : C_n \to \mathbb{R}/\mathbb{Z}$$
$$m \mapsto m^2/n.$$

If n is even $\mathrm{Quad}(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z})$ *is a cyclic group of order $2n$, with generator given by*

$$\begin{aligned} q_{2n} : C_n &\to \mathbb{R}/\mathbb{Z} \\ m &\mapsto m^2/2n. \end{aligned}$$

Proof. Let $\gamma : \mathbb{Z}_n \to \mathbb{R}/\mathbb{Z}$ be a quadratic form. Since $\gamma(m) = m^2\gamma(1)$, the quadratic form is completely determined by $\gamma(1) \in \mathbb{Q}/\mathbb{Z}$. Since $q(n) = 0$, $n^2q(1) = 0$, and since $q(1) = q(-1)$, $2nq(1) = 0$.

If n is odd. Then $nq(1) = 0$, so $q(1) \in \{1/n, 2/n, \dots, 0\} \subset \mathbb{Q}/\mathbb{Z}$ define all possible quadratic forms. If n is even, $q(1) \in \{1/2n, 2/2n, \dots, 0\} \subset \mathbb{Q}/\mathbb{Z}$ define the possible quadratic forms. ☑

***Remark* 4.2.** Let n be an even positive integer. An abelian 3-cocycle $(\omega, c) \in Z^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ representing the cohomology class of the quadratic form q_{2n} is given by

$$c(a,b) = \frac{ab}{2n}, \qquad \omega(a,b,c) = \begin{cases} \frac{a}{2}, & \text{if } b + c \geq n, \\ 0. & \text{other case.} \end{cases}$$

Proposition 4.3. *Let A and B be abelian group, then the map*

$$\begin{aligned} T : \mathrm{Hom}(A \otimes B, \mathbb{R}/\mathbb{Z}) \oplus \mathrm{Quad}(A, \mathbb{R}/\mathbb{Z}) \oplus \mathrm{Quad}(B, \mathbb{R}/\mathbb{Z}) &\to \mathrm{Quad}(A \oplus B, \mathbb{R}/\mathbb{Z}) \\ f \oplus \gamma_A \oplus \gamma_B &\to [(a,b) \mapsto f(a \oplus b) + \gamma_A(a) + \gamma_B(b)], \end{aligned}$$

is a group isomorphisms.

Proof. We will see that

$$\begin{aligned} W : \mathrm{Quad}(A \oplus B, \mathbb{R}/\mathbb{Z}) &\to \mathrm{Hom}(A \otimes B, \mathbb{R}/\mathbb{Z}) \oplus \mathrm{Quad}(A, \mathbb{R}/\mathbb{Z}) \oplus \mathrm{Quad}(B, \mathbb{R}/\mathbb{Z}) \\ \gamma &\mapsto \gamma_A + \gamma_B + b_\gamma|_{(A \oplus 0) \times (0) \oplus B}, \end{aligned}$$

is the inverse of T. In fact,

$$\begin{aligned} T \circ W(\gamma)(a \oplus b) &= \gamma(a) + \gamma(b) + (\gamma(a \otimes b) - \gamma(a) - \gamma(b)) \\ &= \gamma(a \oplus b), \end{aligned}$$

and

$$\begin{aligned} W \circ T(f \oplus \gamma_A \oplus \gamma_B)(a_1 \otimes b_1 \oplus a_2 \oplus b_2) &= b_{T((f \oplus \gamma_A \oplus \gamma_B)}(a_1 \otimes b_1) \oplus \gamma_A(a_2) \oplus \gamma_B(b_2) \\ &= b_{T((f \oplus 0 \oplus 0)}(a_1 \otimes b_1) \oplus \gamma_A(a_2) \oplus \gamma_B(b_2) \\ &= f(a_1 \otimes b_1) \oplus \gamma_A(a_2) \oplus \gamma_B(b_2). \end{aligned}$$

☑

Corollary 4.4. *If A is a finite abelian group, then*

$$|\operatorname{Quad}(A,\mathbb{R}/\mathbb{Z})| = |A/2A||S^2(A)|.$$

Proof. Recall that $S^2(A\oplus B) \cong S^2(A)\oplus S^2(B)\oplus A\otimes B$ for any pair of abelian groups. In particular,

$$|S^2(A\oplus B)| = |S^2(A)||S^2(B)||A\otimes B|. \tag{10}$$

If $A = B\oplus C$, by Proposition 4.3 we have

$$\begin{aligned}|\operatorname{Quad}(A,\mathbb{R}/\mathbb{Z})| &= |B/2B||S^2(B)|||C/2C||S^2(C)||B\otimes C| \\ &= (|B/2B||C/2C|)(|S^2(B)||S^2(B)||B\otimes C)|) \\ &= |A/2A||S^2(A)|.\end{aligned}$$

☑

4.2. A double complex for an abelian group.

To describe the obstruction to the existence of a solution of the hexagon equation of a 3-cocycle $\omega \in Z^3(A,U(1))$, in this section we will define a double complex associated to an abelian group.

Let A and N be abelian groups. We define a double complex by $D^{p,q}(A,N) = 0$ if p or q are zero and

$$D^{p,q}(A,N) := \operatorname{Maps}(A^p|A^q;N), \quad p,q>0$$

with horizontal and vertical differentials the standard differentials, that is,

$$\delta_h : D^{p,q}(A,N) = C^p(A,C^q(A,N)) \to D^{p+1,q}(A,N) = C^{p+1}(A,C^q(A,N))$$

and

$$\delta_v : D^{p,q}(A,N) = C^q(A,C^p(A,N)) \to D^{p,q+1}(A,N) = C^{q+1}(A,C^p(A,N))$$

defined by the equations

$$\begin{aligned}(\delta_h F)(g_1,...,g_{p+1}||k_1,...,k_q) &= F(g_2,...,g_{p+1}||k_1,...,k_q) \\ &\quad + \sum_{i=1}^{p}(-1)^i F(g_1,\dots,g_ig_{i+1},\dots,g_{p+1}||k_1,...,k_q) \\ &\quad + (-1)^{p+1}F(g_1,...,g_p||k_1,...,k_q) \\ (\delta_v F)(g_1,...,g_p||k_1,...,k_{q+1}) &= F(g_1,...,g_p||k_2,...,k_{q+1}) \\ &\quad + \sum_{j=1}^{q}(-1)^j F(g_1,...,g_p||k_1,...,k_jk_{j+1},...,k_{q+1}) \\ &\quad + (-1)^{q+1}F(g_1,...,g_p||k_1,...,k_q).\end{aligned}$$

For future reference it will be useful to describe the equations that define a 2-cocycle and the coboundary of a 1-cochains:

- $\mathrm{Tot}^0(D^{*,*}(A,N)) = \mathrm{Tot}^1(D^{*,*}(A,N)) = 0$,
- $\mathrm{Tot}^2(D^{*,*}(A,N)) = \mathrm{Maps}(A|A,N)$,
- $\mathrm{Tot}^3(D^{*,*}(A,N)) = \mathrm{Maps}(A|A^2,N) \oplus \mathrm{Maps}(A^2|A,N)$,

Thus,

- $H^2(\mathrm{Tot}^*(D^{*,*}(A,N))) = \mathrm{Hom}(A^{\otimes 2},N)$ the abelian group of all bicharacters from A to N.
- for $f \in \mathrm{Tot}^2(D^{*,*}(A,N))$,

$$\begin{aligned}\delta_h(f)(x,y||z) &= f(y||z) - f(x+y||z) + f(x||z)\\ \delta_v(f)(x||y,z) &= f(x||z) - f(x||y+z) + f(x||y)\end{aligned}$$

Let us describe the elements

$$(\alpha,\beta) \in Z^3(\mathrm{Tot}^*(D^{*,*}(A,N))),$$

$\alpha \in C^1(A, Z^2(A,N))$, that is $\alpha : A|A^2 \to N$ such that

$$\alpha(x;a,b) + \alpha(x;a+b,c) = \alpha(x;a,b+c) + \alpha(x;b,c)$$

for all $x,a,b,c \in A$,

$\beta \in C^1(A, Z^2(A,N))$, that is $\beta : A^2|A \to N$ such that

$$\beta(x+y,z;a) + \beta(x,y;a) = \beta(x,y+z,a) + \beta(y,z;a)$$

for all $x,y,z \in A, a \in A$, and $\delta_h(\alpha) = -\delta_v(\beta)$, that is,

$$\alpha(x;a,b) + \alpha(y;a,b) - \alpha(x+y;a,b) = \beta(x,y;a+b) - \beta(x,y;a) - \beta(x,y;b),$$

for all $x,y \in A, a,b \in B$.

4.3. Obstruction

Consider the group homomorphism

$$\tau_n : H^n(A,N) \to H^n(\mathrm{Tot}^*(D^{*,*}(A,N)))$$

induced by the cochain map

$$\begin{aligned}\tau : C^*(A,N) &\to C^n(\mathrm{Tot}^*(D^{*,*}(A,N)))\\ \alpha &\mapsto \oplus_{p=1}^{n-1}\alpha_p,\end{aligned}$$

where $\alpha_p \in \mathrm{Maps}(A^p|A^{n-p}, N)$ is defined by

$$\alpha_p(a_1,\dots,a_p|a_{p+1},\dots,a_n) = \sum_{\pi\in \mathrm{Shuff(p,n-p)}} (-1)^{\epsilon(\pi)}\alpha(a_{\lambda(1)},\dots,a_{\pi(n)}).$$

For every $n\in\mathbb{Z}^{\geq 2}$, we define the suspension homomorphism from

$$\begin{aligned} s_n : H^n_{ab}(A,N) &\to H^n(A,N)\\ \oplus_{p_1,\dots,p_r\geq 1: r+\sum_{i=1}^r p_i=n+1}\alpha_{p_1,\dots,p_r} &\mapsto \alpha_n. \end{aligned}$$

The group homomorphism

$$\begin{aligned} H^2(\mathrm{Tot}^*(D^{*,*}(A,N))) = \mathrm{Hom}(A^{\otimes 2},N) &\to Z^3_{ab}(A,N)\\ c &\mapsto (0,c), \end{aligned}$$

induces a group homomorphism $\iota : H^2(\mathrm{Tot}^*(D^{*,*}(A,N)) \to H^3_{ab}(A,N)$.

The following result shows that the shuffle homomorphism can be interpreted as the obstruction to the hexagon equation.

Theorem 4.5. *Let A and N be a abelian groups. Then, the sequence*

$$\begin{array}{l} 0 \longrightarrow H^2_{ab}(A,N) \xrightarrow{s_2} H^2(A,N) \xrightarrow{\tau_2} H^2(\mathrm{Tot}^*(D^{*,*}(A,N)) \\ \xrightarrow{\iota} H^3_{ab}(A,N) \xrightarrow{s_3} H^3(A,N) \xrightarrow{\tau_3} H^3(\mathrm{Tot}^*(D^{*,*}(A,N))). \end{array}$$

is exact.

Proof. The shuffle homomorphism $\tau_2 : H^2(A,N)\to H^2(\mathrm{Tot}^*(D^{*,*}(A,N))$ is given by $\tau(\alpha)(x,y) = \alpha(x,y)-\alpha(y,x)$, thus it is clear that the sequence is exact in $H^2(A,N)$.

An abelian 3-cocycle is in the kernel of the suspension map if it is cohomologous to an abelian 3-cocycle of the form $(0,c)$. But then $c\in\mathrm{Hom}(A^{\otimes 2},N) = H^2(\mathrm{Tot}^*(D^{*,*}(A,N))$, hence the sequence is exact in $H^3_{ab}(A,N)$. Finally, If $\omega\in Z^3(A,N)$, then $\tau(\omega) = (\alpha_\omega,\beta_\omega)$, where

$$\begin{aligned} \alpha_\omega(x|y,z) &= \omega(x,y,z)-\omega(y,x,z)+\omega(y,z,x)\\ \beta_\omega(x,y|z) &= \omega(x,y,z)-\omega(x,z,y)+\omega(z,x,y). \end{aligned}$$

Thus, $[(\alpha_\omega,\beta_\omega)] = 0$, if and only if there is $c : A\times A\to N$ such that

$$\delta_v(c) = \alpha_\omega, \quad -\delta_h(c) = \beta_\omega,$$

that is, $[\tau(\omega)] = 0$ if and only if there is $c : A\times A \to N$ such that $(\omega,c)\in Z^3_{ab}(A,N)$. Thus, the sequence is exact in $H^3(A,N)$. ☑

Corollary 4.6 (Total obstruction)**.** *A gauge class of a solution of the pentagon equation $\omega \in H^3(A, \mathbb{R}/\mathbb{Z})$ admits a solution of the hexagon equation if and only if $\tau(\omega) = 0$ in $H^3(\mathrm{Tot}^*(D^{*,*}(A, \mathbb{R}/\mathbb{Z})))$.*

☑

Proposition 4.7. *Let A be a finite abelian group.*

(1)
$$\ker(s_3) \cong \widehat{S^2(A)}.$$

(2) Under the isomorphism $\mathrm{Tr} : H^3_{ab}(A, \mathbb{R}/\mathbb{Z}) \to \mathrm{Quad}(A, \mathbb{R}/\mathbb{Z})$ (see (9)), $\ker(s_3)$ corresponds to the subgroup

$$\mathrm{Quad}_0(A, \mathbb{R}/\mathbb{Z}) = \{q \in \mathrm{Quad}(A, \mathbb{R}/\mathbb{Z}) : o(a)q(a) = 0, \forall a \in A\},$$

where $o(a)$ denotes the order of $a \in A$.

Proof. Since $\mathbb{R}/\mathbb{Z}$ is divisible and A is finite, the group $H^2_{ab}(A, \mathbb{R}/\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{R}/\mathbb{Z})$ is null. Thus, by Theorem 4.5 we have an exact sequence

$$0 \to H^2(A, \mathbb{R}/\mathbb{Z}) \xrightarrow{\tau} \mathrm{Hom}(A^{\otimes 2}, \mathbb{R}/\mathbb{Z}) \to \ker(s_3) \to 0.$$

The image of τ is $\mathrm{Hom}(\wedge^2 A, \mathbb{R}/\mathbb{Z})$, hence

$$\begin{aligned}\ker(s_3) &\cong \mathrm{Hom}(A^{\otimes 2}, \mathbb{R}/\mathbb{Z})/\mathrm{Hom}(\wedge^2 A, \mathbb{R}/\mathbb{Z})\\ &\cong \mathrm{Hom}(A^{\otimes 2}/\wedge^2 A, \mathbb{R}/\mathbb{Z})\\ &\cong \mathrm{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}),\end{aligned}$$

where the last isomorphism is defined using the exact sequence

$$0 \to \wedge^2 A \to A^{\otimes 2} \to S^2 A \to 0.$$

Now we will prove the second part. If A is cyclic the proposition follows by Remark 4.2. The general case follows from Proposition 4.3, since the image of $\mathrm{Hom}(A \otimes B, \mathbb{R}/\mathbb{Z})$ by T lies in $\mathrm{Quad}_0(A \oplus B, \mathbb{R}/\mathbb{Z})$. ☑

We will denote by $\mu_2 = \{1, -1\} \subset \mathrm{U}(1) \cong \mathbb{R}/\mathbb{Z}$.

Theorem 4.8. *Let A be an abelian finite group. The canonical projection $\pi : A \to A/2A$ induces an isomorphism between the images of the respective suspension maps of $H^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ and $H^3_{ab}(A/2A, \mathbb{R}/\mathbb{Z})$.*

Moreover, for an elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^{\oplus n}$,

$$\mathrm{Im}(s_3) \cong H^3(\mathbb{Z}/2\mathbb{Z}, \mu_2)^{\oplus n} = (\mathbb{Z}/2\mathbb{Z})^{\oplus n}.$$

***Proof*.** By Corollary 4.4, Proposition 4.7 and Theorem 4.5, we have that $|\operatorname{Im}(s_3)| = |A/2A|$. In particular the size of the image of the suspension maps of $H^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ and $H^3_{ab}(A/2A, \mathbb{R}/\mathbb{Z})$ are equal. Since π^* is an inyective map between the image of the suspension maps it is an isomorphisms.

Let $(\mathbb{Z}/2\mathbb{Z})^{\oplus n}$ be an elementary abelian 2-group. Recall that $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = H^3(\mathbb{Z}/2\mathbb{Z}, \mu_2) \cong \mathbb{Z}/2\mathbb{Z}$. Using the Remark 4.2 we have an injective group homomorphisms $H^3(\mathbb{Z}/2\mathbb{Z}, \mu_2)^{\oplus n} \to \operatorname{Im}(s_3)$, which is an isomorphism because both groups have the same order. ☑

Corollary 4.9. *For any abelian group we have an exact sequence*

$$0 \to S^2(\widehat{A}) \to H^3_{ab}(A, \mathbb{R}/\mathbb{Z}) \to A/2A \to 0. \tag{11}$$

☑

***Remark* 4.10.** A related result is established by Mason and Ng in [15, Lemma 6.2].

4.4. Explicit abelian 3-cocycles

Abelian 3-cocycles for the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ were study in detail in [3], for $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ in [11] and in full generality in [19].

Let A be a finite abelian group and $A = A_1 \oplus A_2$ the canonical decomposition, where A_1 has order a power of 2 and A_2 has odd order. Since $H^3_{ab}(A, \mathbb{R}/\mathbb{Z}) = H^3_{ab}(A_1, \mathbb{R}/\mathbb{Z}) \oplus H^3_{ab}(A_2, \mathbb{R}/\mathbb{Z})$, and $H^3_{ab}(A_2, \mathbb{R}/\mathbb{Z}) \cong S^2(\widehat{A_2})$, the problem of a general description can be divided in the case of group of odd order and the case of abelian 2-groups.

4.4.1. Case of A an odd abelian group

If A is an odd abelian group then map $A \to A, a \mapsto 2a$ is a group automorphism of A. Hence, given $q \in \operatorname{Quad}(A, \mathbb{R}/\mathbb{Z})$ the symmetric bilinear form $c := \frac{1}{2} b_q \in \operatorname{Hom}(A^{\otimes 2}, \mathbb{R}/\mathbb{Z})$, defines an abelian 3-cocycle $(0, c) \in Z^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ such that $\operatorname{Tr}(c) = q$.

4.4.2. Case of A an abelian 2-group

Let $A = \bigoplus_{i=1}^n \mathbb{Z}/2^{m_i}\mathbb{Z}$. Then by Corollary 4.9 and Proposition 4.7 we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^3_{ab}(A, \mathbb{R}/\mathbb{Z}) & \xrightarrow{s_3} & (\mathbb{Z}/2\mathbb{Z})^{\oplus n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\operatorname{Tr}} & & \downarrow \scriptstyle{=} & & \\
0 & \longrightarrow & \operatorname{Quad}_0(A, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \operatorname{Quad}(A, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\pi} & (\mathbb{Z}/2\mathbb{Z})^{\oplus n} & \longrightarrow & 0
\end{array} \tag{12}$$

where the horizontal sequences are exact and the vertical morphisms are isomorphisms.

Let $q \in \mathrm{Quad}_0(A, \mathbb{R}/\mathbb{Z})$, then $c \in \mathrm{Hom}(A^{\otimes 2}, \mathbb{R}/\mathbb{Z})$ defined by

$$c(\vec{x}, \vec{y}) = \sum_i x_i y_i q(\vec{e_i}) + \sum_{i<j} x_i y_j b_q(\vec{e_i}, \vec{e_j})$$

is a such that $(0, c) \in Z^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ represents q.

A set-theoretical section $j : (\mathbb{Z}/2\mathbb{Z})^{\oplus n} \to \mathrm{Quad}(A, \mathbb{R}/\mathbb{Z})$ of π in the exact sequence (12) is defined easily as follows

$$j(\vec{y})(\vec{x}) = \sum_i y_i x_i^2 / 2^{m_i+1}.$$

Abelian 3-cocycles representing the $j(\vec{e_j})'s$ are constructed as the pullback by the projection $\pi_j : \bigoplus_{i=1}^n \mathbb{Z}/2^{m_i} \to \mathbb{Z}/2^{m_j}$ of the abelian 3-cocycle $(w, c) \in Z^3(\mathbb{Z}/2^{m_j}, \mathbb{R}/\mathbb{Z})$ defined in Remark 4.2.

As a consequence of the previous discussion we have the following result.

Proposition 4.11. *Let A be an abelian group. Every abelian 3-cohomology class has a representative 3-cocycle $(\omega, c) \in Z^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ where $\omega(a, b, c) \in \{0, \frac{1}{2}\} \subset \mathbb{R}/\mathbb{Z}$.*

☑

***Remark* 4.12.** Proposition 4.11 implies that the cohomology class of the square power of an abelian 3-cocycle is zero. This result was established in [14, Lemma 4.4 (ii)].

4.5. Partial obstructions

Since the obstruction of the existence of a solution of the hexagon equation is an element in the total cohomology of a double complex, we can analyze the obstruction by partial obstructions as follows.

Proposition 4.13 (Partial obstruction 1)**.** *Let A and N be abelian groups and $\omega \in Z^3(A, N)$. If $\tau_3(\omega) = (\alpha_\omega, \beta_\omega)$, then the cohomology class of $\alpha_\omega(a|-,-) \in Z^2(A, N)$ only depends on the cohomology class of ω. If ω is in the image of the suspension map, then*

$$0 = [\alpha_\omega(a|-,-)] \in H^2(A, N)$$

for all $a \in A$.

***Proof*.** For $(\omega, c) \in H^3(A, N)$ we have that $\delta_v(c) = \alpha_\omega$, then $[\alpha_\omega(a|-,-)] = 0$ for all $a \in A$.

Let $u : A \times A \to N$, and

$$w'(a,b,c) = w(a,b,c) + u(a+b,c) + u(a,b) - u(a,b+c) - u(b,c).$$

Then

$$\alpha_{\omega'}(a|c,d) = \alpha_w(a|b,c) + l_a(b) + l_a(c) - l_a(b+c),$$

where $l_a(b) = u(a,b) - u(b,a)$. ☑

Assume that $[\alpha_\omega(a|-,-)] = 0$ for all $a \in A$. Thus, there exists $\eta \in C(A|A,N)$ such that $\delta_v(\eta) = \alpha_\omega$. We have that $(0, \delta_h(\eta)+\beta_\omega) \in Z^3(\mathrm{Tot}^*(D^{*,*}(A,N)))$, thus

$$\theta(\omega,\eta) := \delta_h(\eta) + \beta_\omega \in Z^2_{ab}(A, \mathrm{Hom}(A,N)).$$

In fact,

$$\begin{aligned}\delta_v(\delta_h(\eta) + \beta_\omega) &= \delta_v(\delta_h(\eta)) + \delta_v(\beta_\omega)\\ &= \delta_h(\delta_v(\eta)) + \delta_v(\beta_\omega)\\ &= \delta_h(\alpha_\omega) - \delta_h(\alpha_\omega) = 0,\end{aligned}$$

that is, $\theta(\omega,\eta)(a,b,x+y) = \theta(\omega,\eta)(a,b,x) + \theta(\omega,\eta)(a,b,y)$.

The cohomology of $\theta(\omega,\eta)$ does not depend on the choice of η. In fact, if $\eta \in C(A|A,N)$ such that $\delta_v(\eta) = \alpha_\omega$, then $\mu := \eta - \eta' \in C^1(A, \mathrm{Hom}(A, U(1)))$ and

$$\theta(\omega,\eta) - \theta(\omega,\eta') = \delta_h(\eta - \eta') = \delta_h(\mu).$$

Hence, we have defined a second obstruction

$$\theta(\omega) \in \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathrm{Hom}(A,N).$$

Corollary 4.14 (Obstruction 2)**.** *Let* $\omega \in Z^3(A,N)$*, such that* $0 = [\alpha_\omega(a|-,-)] \in Z^2(A,N)$ *for all* $a \in A$*. Then there exists* $c : A \times A \to N$ *such that* $(\omega, c) \in Z^3_{ab}(A,N)$ *if and only if* $\theta(\omega) = 0$*.*

Proof. If $(\omega,c) \in Z^3_{ab}(A,N)$, then $\tau(\omega) = 0$, that implies $\theta(\omega) = 0$. Now, let $\omega \in Z^3(A,N)$ and $\eta \in C(A|A,N)$ such that $\delta_v(\eta) = \alpha_\omega$, that is,

$$\theta(\omega,\eta) = \delta_h(\eta) + \beta_\omega \in Z^2_{ab}(A, \mathrm{Hom}(A,N)).$$

If $[\theta(\omega,\eta)] = 0$, there is $l : A \to \mathrm{Hom}(A,N)$ such that $\delta_h(l) = \delta_h(\eta) + \beta_\omega$. Thus $c := \eta - l$ is such that $(\omega,c) \in Z^3_{ab}(A,N)$, since

$$\delta_v(c) = \delta_v(\eta) - \delta_v(l) = \alpha_\omega$$

and

$$\begin{aligned}\delta_h(c) &= \delta_h(\eta) - (\delta_h(\eta) + \beta_\omega)\\ &= -\beta_\omega.\end{aligned}$$

☑

5. Abelian anyons

In this last section we will present the classification of all possible *prime* or *indecomposable* abelian anyon theories.

5.1. S and T matrices of abelian theories

By an abelian theory we will mean a triple (A, ω, c), where A is abelian group (or equivalently an abelian fusion rules) and $(\omega, c) \in Z^3(A, \mathbb{R}/\mathbb{Z})$ an abelian 3-cocycle (or equivalently a solution of the hexagon equation).

Let (A, ω, c) be an abelian theory. Recall that the associated quadratic form $q : A \to \mathbb{R}/\mathbb{Z}$ is defined by $q(a) = c(a, a)$. The *topological spin* of $a \in A$ is defined as the phase

$$\theta_a = e^{2\pi i q(a)},$$

thus, the topological spin is exactly the associated quadratic form and by the Eilenberg and MacLane theorem [8, Theorem 26.1] it determines up to gauge equivalence the abelian theory.

Recall that the symmetric bilinear form

$$b_q : A \times A \to \mathbb{R}/\mathbb{Z}$$

associated to the quadratic for $q : A \to \mathbb{R}/\mathbb{Z}$ is defined by

$$b_q(a, b) = q(a + b) - q(a) - q(b).$$

Since $q(a) = c(a, a)$, we also have that

$$b_q(a, b) = c(a, b) - c(b, a)$$

for all $a, b \in A$. The map

$$H^3_{ab}(A, \mathbb{R}/\mathbb{Z}) \xrightarrow{b} \mathrm{Hom}(S^2(A))$$

that associates the symmetric bilinear form b_q to an abelian 3-cocycle is a group homomorphism.

Definition 5.1. An anyonic abelian theory is an abelian theory (A, ω, c) such that one of the following equivalent conditions holds:

(1) The S-matrix $S_{ab} = |A|^{-1/2} e^{2\pi i b_q(a,b)}$ is non-singular.

(2) The symmetric bilinear form b_q is non-degenerated.

We will say that an abelian theory (A, ω, c) is *symmetric* if its b_q is trivial, or equivalently, if $-c(a, b) = c(b, a)$ for all $a, b \in A$. We will denote by $H^3_s(A, \mathbb{R}/\mathbb{Z}) \subset H^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ the subgroup of all equivalence class of symmetric abelian 3-cocycles.

The T-matrix of an abelian anyonic theory is the diagonal matrix of the topological spins, that is,

$$T_{ab} = \delta_{a,b} e^{2\pi i q(a)}.$$

Thus, for abelian anyons the T-matrix completely determines the theory. On the contrary, the S-matrix does not always determine the theory, however the following result result say that two abelian anyons with the same S-matrix only differ by a symmetric abelian 3-cocycle and their T-matrices by a linear character $\chi : A \to \{1, -1\}$.

Proposition 5.2. *Let A be an abelian group. Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^3_s(A, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^3_{ab}(A, \mathbb{R}/\mathbb{Z}) & \xrightarrow{b} & \operatorname{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \operatorname{Hom}(A, \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}) & \longrightarrow & \operatorname{Quad}(A, \mathbb{R}/\mathbb{Z}) & \xrightarrow{b} & \operatorname{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

commutes, the vertical morphisms are isomorphisms and the horizontal sequences are exact.

Proof. Clearly the kernel of $b : \operatorname{Quad}(A, \mathbb{R}/\mathbb{Z}) \to \operatorname{Hom}(S^2(A), \mathbb{R}/\mathbb{Z})$ is $\operatorname{Hom}(A, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$. Thus, by Corollary 4.4 the sequence

$$0 \to \operatorname{Hom}(A, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \to \operatorname{Quad}(A, \mathbb{R}/\mathbb{Z}) \to \operatorname{Hom}(S^2(A), \mathbb{R}/\mathbb{Z}) \to 0,$$

is exact. ☑

***Remark* 5.3.** A related result was established in [15, Lemma 6.2 (ii), (iii)].

5.2. Prime abelian anyons

If (A, ω, c) and (A', ω', c') are abelian anyons theory, their direct sum is defined as the anyon theory $(A \oplus A', \omega \times \omega', c \times c')$, where $\omega \times \omega'((a, a'), (b, b'), (c, c')) = \omega(a, b, c)\omega'(a', b', c')$ and similarly for $c \times c'$.

We will say that an anyon theory (A, ω, c) is *prime* if for any non trivial subgroup $B \subset A$, the restriction of the associated bilinear form b_q is degenerated.

Two abelian theories (A, ω, c) and (A, ω, c) are called equivalents if there is a group isomorphism $f : A \to A'$ such that $(f^*(\omega), f^*(c)), (\omega, c) \in Z^3_{ab}(A, \mathbb{R}/\mathbb{Z})$ are cohomologous, or equivalently if $q'(f(a)) = q(a)$ for all $a \in A$, where q and q' are the quadratic forms associated.

By [17, Theorem 4.4] any abelian anyon theory is a direct sum of prime abelian anyon theory. Thus, the classification of abelian anyons is reduced to the classification of prime abelian anyons.

The Legendre symbol is a function of $a \in \mathbb{Z}^{>0}$ and a prime number p defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Following the notation of [16], we establish the classification of prime abelian anyons, that follows from results of Wall [21] and Durfee [6] about the classification of indecomposable non-degenerated quadratic forms on abelian groups.

Theorem 5.4. *The following is the list of all equivalence classes of prime abelian anyons theories:*

(i) If $p \neq 2$ and $\epsilon = \pm 1$, $\omega^{\epsilon}_{p,k}$ denotes the abelian anyon with fusion rules given by $\mathbb{Z}/p^k\mathbb{Z}$ and abelian 3-cocycle $(0, c)$, where $c(x,y) = \frac{uxy}{p^k}$, for some $u \in \mathbb{Z}^{>0}$ with $(p,u) = 1$ and $\left(\frac{2u}{p}\right) = \epsilon$.

(ii) If $\epsilon \in (\mathbb{Z}/8\mathbb{Z})^{\times}$, $\omega^{\epsilon}_{2,k}$ denotes the abelian anyon with fusion rules given by $\mathbb{Z}/2^k\mathbb{Z}$ and abelian 3-cocycle

$$c(x,y) = \frac{uxy}{2^{k+1}}, \qquad \omega(x,y,z) = \begin{cases} \frac{x}{2}, & \text{if } y+z \geq 2^k, \\ 0. & \text{otherwise.} \end{cases}$$

for some $u \in \mathbb{Z}^{>0}$ with $u \equiv \epsilon \pmod 8$. The abelian anyons $w^{1}_{2,k}$ and $w^{-1}_{2,k}$ are defined for all $k \geq 1$ and $w^{5}_{2,k}$ and $w^{-5}_{2,k}$ for all $k \geq 2$.

(iii) E_k denoted the abelian anyon with fusion rules given by $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ and abelian 3-cocycle $(0,c)$, where $c \in \mathrm{Hom}(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is defined by

$$c(\vec{e}_i, \vec{e}_j) = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{2^k}, & \text{if } i \neq j. \end{cases}$$

(iv) F_k denoted the abelian anyon with fusion rules given by $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ and abelian 3-cocycle $(0,c)$, where $c \in \mathrm{Hom}(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is defined by

$$c(\vec{e}_i, \vec{e}_j) = \begin{cases} \frac{1}{2^{k-1}}, & \text{if } i = j, \\ \frac{1}{2^k}, & \text{if } i \neq j. \end{cases}$$

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