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On the space-time admitting some geometric structures on energy-momentum tensors

Sobre el espacio-tiempo admitiendo algunas estructuras geométricas en tensores de energía-momento

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Abstract. This paper presents a study of a general relativistic perfect fluid space-time admitting various types of curvature restrictions on energy-momentum tensors and brings out the conditions for which fluids of the space–time are sometimes phantom barrier and some other times quintessence barrier. The existence of a space–time where fluids behave as phantom barrier is ensured by an example.

Key words and phrases. General relativistic perfect fluid space-time, Einstein's field equation, energy-momentum tensor, semi-symmetric energy-momentum tensor.

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Resumen. Este artículo presenta un estudio del tiempo-espacio fluido perfecto relativista general admitiendo varios tipos de restricciones de curvatura en los tensores de energía-momento y saca a relucir las condiciones para las cuales los fluidos del espacio-tiempo son a veces barrera fantasma y otras veces barrera de quintaesencia. La existencia de un espacio–tiempo donde los líquidos se comportan como barrera fantasma es garantizado por un ejemplo.

Palabras y frases clave. Espacio-tiempo fluido general relativista perfecto, campo de Einstein, tensor energía-momento, tensor semi-simétrico de energía-momento.

1. Introduction

Recently, in tune with Yano and Sawaki [16], Baishya and Roy Chowdhury [4] introduced and studied quasi-conformal curvature tensors in the frame of N(k,μ)-manifolds. The generalized quasi-conformal curvature tensor is defined for n dimensional manifolds as

$$\begin{aligned}\mathcal{W}(X,Y)Z =& \frac{n-2}{n}\left[1+(n-1)(a-b)-\{1+(n-1)(a+b)\}c\right]C(X,Y)Z\\ &+[1-b+(n-1)a]E(X,Y)Z+(n-1)(b-a)P(X,Y)Z\\ &+\frac{n-2}{n}(c-1)\{1+(n-1)(a+b)\}\hat{C}(X,Y)Z \qquad (1)\end{aligned}$$

for all $X,Y,Z \in \chi(M)$, the set of all vector fields of the manifold M, where scalars a,b,c are real constants. The beauty of such curvature tensors lies on the fact that it has the flavour of

(i) Riemannian curvature tensors R if the scalar triple $(a,b,c) \equiv (0,0,0)$,

(ii) Conformal curvature tensors C [9] if $(a,b,c) \equiv (-\frac{1}{n-2},-\frac{1}{n-2},1)$,

(iii) Conharmonic curvature tensors $\hat{C}$ [10] if $(a,b,c) \equiv (-\frac{1}{n-2},-\frac{1}{n-2},0)$,

(iv) Concircular curvature tensors E [15, p. 84] if $(a,b,c) \equiv (0,0,1)$,

(v) Projective curvature tensors P [15, p. 84] if $(a,b,c) \equiv (-\frac{1}{n-1},0,0)$,

(vi) m-projective curvature tensors H [12] if $(a,b,c) \equiv (-\frac{1}{2n-2},-\frac{1}{2n-2},0)$.

Note that (1) can also be written as

$$\begin{aligned}\mathcal{W}(X,Y)Z =& R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y]\\ &+ b[g(Y,Z)QX - g(X,Z)QY]\\ &- \frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(Y,Z)X - g(X,Z)Y]. \qquad (2)\end{aligned}$$

The space-time under various curvature restrictions is a subject of vast literature, e.g., [1, 5, 7, 8] and the references there in.

In analogy with [14], an energy-momentum tensor T of type $(0,2)$ is said to be semi-symmetric type if

$$\mathcal{W}(X,Y)\cdot T = 0 \qquad (3)$$

holds where $\mathcal{W}(X,Y)$ acts on T as a derivation.

The paper is structured as follows. Section 2 is concerned with general relativistic perfect fluid space-time (briefly *GRPFS)* obeying Einstein's equation with $\mathcal{W}(X,Y)\cdot T = 0$. It is observed that a fluid of such space-time always behaves as phantom barrier for each of the restrictions $E(X,Y)\cdot T=0$ and $H(X,Y)\cdot T=0$ whereas the same behaves either as a phantom barrier or quintessence barrier for each of the restrictions $R(X,Y)\cdot T=0$ and $\hat{C}(X,Y)\cdot T=0$. A detailed study of *GRPFS* obeying Einstein's equation admitting $((X\wedge_S Y)\cdot\mathcal{W})=0$ and $(X\wedge_T Y)=0$, where the endomorphism is defined as $(X\wedge_B Y)Z=B(Y,Z)X-B(X,Z)Y$, has been carried out in Section 3 and 4 respectively, with similar types of results as in Section 2. Finally, we give an example in Section 5 of a fluid whose character is phantom barrier.

2. *GRPFS* with semi-symmetric type energy momentum tensors

Einstein's equation can be written as

$$S=kT+\frac{r}{2}g, \tag{4}$$

where k is the gravitational constant and r is the scalar curvature. Let (M^4,g) be a *GRPFS* with (3). Now (3) implies that

$$T(\mathcal{W}(X,Y)U,V)+T(U,\mathcal{W}(X,Y)V)=0. \tag{5}$$

In view of (4) and (5), we have

$$\begin{aligned}((\mathcal{W}(X,Y)\cdot S)U,V)&=\frac{r}{2}((\mathcal{W}(X,Y)\cdot g)U,V)\\&=0.\end{aligned}$$

In consequence of the above, we have the following:

Proposition 2.1. *A general relativistic space-time with a semi-symmetric type energy-momentum tensor is Ricci semi-symmetric type and vice-versa.*

By [5, Theorem 1, p 1029], we can easily bring out the following:

Proposition 2.2. *A general relativistic space-time with a covariant constant energy-momentum tensor is Ricci semi-symmetric type.*

By virtue of a result of Aikawa and Matsuyama [2] if a tensor field L is recurrent or birecurrent, then $R(X,Y)\cdot L=0$. Hence we have the following:

Theorem 2.3. *A general relativistic space-time with a recurrent or birecurrent energy-momentum tensor is always Ricci semi-symmetric type.*

Next we consider a perfect fluid space-time whose energy-momentum tensor is semi-symmetric type. An energy-momentum tensor is said to describe a perfect fluid [11] if

$$T(X,Y)=(\sigma+\rho)A(X)A(Y)+\rho g(X,Y), \tag{6}$$

where σ is the energy density, ρ is the isotropic pressure of the fluid and A is a non-zero 1-form such that $g(X,\theta) = A(X)$ for all X, θ being the velocity vector field of the fluid which is a time-like vector that is,

$$g(\theta,\theta) = A(\theta) = -1.$$

By virtue of (4) and (6), we get

$$S(X,Y) = \frac{k(\sigma-\rho)}{2} g(X,Y) + k(\sigma+\rho)A(X)A(Y). \tag{7}$$

In view of (2) and (7), we have

$$\begin{aligned}\mathcal{W}(X,Y)Z =& R(X,Y)Z + k\left[\frac{a+b}{2} - \frac{c(\sigma-3\rho)}{4}\left(\frac{1}{3}\right.\right.\\ &\left.\left.+a+b\right)\right][g(Y,Z)X - g(X,Z)Y]\\ &+ ak(\sigma+\rho)A(Z)[A(Y)X - A(X)Y]\\ &+ bk(\sigma+\rho)[g(Y,Z)A(X) - g(X,Z)A(Y)]\theta.\end{aligned} \tag{8}$$

As consequences of (5) and (6), it follows that

$$\begin{aligned}0 =& (\sigma+\rho)[A(\mathcal{W}(X,Y)U)A(V) + A(\mathcal{W}(X,Y)V)A(U)]\\ &+ \rho[g(\mathcal{W}(X,Y)U,V) + g(\mathcal{W}(X,Y)V,U)]\end{aligned} \tag{9}$$

which yields

$$-\sigma A(\mathcal{W}(X,Y)U) + \rho g(\mathcal{W}(X,Y)\theta,U) = 0 \tag{10}$$

for $V = \theta$ which in turn on contraction gives

$$\begin{aligned}0 = k&\left[\frac{\sigma+3\rho}{2}\{\sigma(1-a+3b) + \rho(1-b+3a)\}\right.\\ &\left.-(\sigma-3\rho)\left\{(a\sigma+\rho b) - \frac{c(\sigma+\rho)(1+3a+3b)}{4}\right\}\right].\end{aligned} \tag{11}$$

From (11), one can easily bring out the following table by substituting the triple (a,b,c) by $(0,0,0)$, $(-\frac{1}{2},-\frac{1}{2},1)$, etc.

Curvature restrictions	Relations between σ & ρ
$R(X,Y)\cdot T = 0$	$\sigma+\rho=0$ or $\sigma+3\rho=0$
$C(X,Y)\cdot T = 0$	σ, ρ are independent
$\hat{C}(X,Y)\cdot T = 0$	$\sigma+\rho=0$ or $\sigma-3\rho=0$
$E(X,Y)\cdot T = 0$	$\sigma+\rho=0$
$P(X,Y)\cdot T = 0$	$\sigma+\rho=0$ or $\sigma=0$
$H(X,Y)\cdot T = 0$	$\sigma+\rho=0$

Now $\sigma + \rho = 0$ means that the fluid behaves as a cosmological constant [13]. This is also termed as phantom barrier [6]. In cosmology, a choice $\sigma = -\rho$ leads to a rapid expansion of the space-time which is now termed as inflation. Also $\sigma + 3\rho = 0$ or $(\sigma - 3\rho = 0)$ is known as the quintessence barrier. Here the strong energy condition begins to be violated. The present observations indicate that our universe is in quintessence era [3]. Thus from the above discussion we can state the following:

Theorem 2.4. *Let (M^4, g) be a GRPFS obeying Einstein's equation admitting $C(X,Y) \cdot T = 0$. Then the density of the matter and pressure are independent.*

Theorem 2.5. *The behavior of fluids in GRPFS obeying Einstein's equation is always phantom barrier for each of the restrictions $E(X,Y) \cdot T = 0$ and $H(X,Y) \cdot T = 0$.*

Theorem 2.6. *The behavior of a fluid in GRPFS obeying Einstein's equation is either phantom barrier or quintessence barrier for each of the restrictions $R(X,Y) \cdot T = 0$ or $\hat{C}(X,Y) \cdot T = 0$.*

3. GRPFS satisfying $((X \wedge_S Y) \cdot \mathcal{W})(Z,U)V = 0$

Let us consider a perfect fluid space-time satisfying $((X \wedge_S Y) \cdot \mathcal{W})(Z,U)V = 0$,

$$\begin{aligned} \text{i. e., } 0 = &(X \wedge_S Y)\mathcal{W}(Z,U)V + \mathcal{W}((X \wedge_S Y)Z,U)V \\ &+ \mathcal{W}(Z,(X \wedge_S Y)U)V + \mathcal{W}(Z,U)(X \wedge_S Y)V, \end{aligned} \tag{12}$$

where the endomorphism $(X \wedge_S Y)Z$ is defined as

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{13}$$

In view of (13), (12) becomes

$$\begin{aligned} 0 = &S(Y, \mathcal{W}(Z,U)V)X - S(X, \mathcal{W}(Z,U)V)Y \\ &+ S(Y,Z)\mathcal{W}(X,U)V - S(X,Z)\mathcal{W}(Y,U)V \\ &+ S(Y,U)\mathcal{W}(Z,X)V - S(X,U)\mathcal{W}(Z,Y)V \\ &+ S(Y,V)\mathcal{W}(Z,U)X - S(X,V)\mathcal{W}(Z,U)Y. \end{aligned} \tag{14}$$

Replacing Y and V by θ, we get

$$0 = (\sigma - \rho)g(X, \mathcal{W}(Z,U)\theta) - (\sigma + 3\rho)A(\mathcal{W}(Z,U)X), \tag{15}$$

which on contraction yields

$$\begin{aligned} 0 = &\frac{k(\sigma + 3\rho)}{2}[\{(\sigma + 3\rho)(1 - a + 3b) + (\sigma - \rho)(1 - b + 3a)\}] \\ &- (\sigma - 3\rho)[b(\sigma - \rho) + a(\sigma + 3\rho)] \\ &+ \frac{c}{2}(\sigma + \rho)(\sigma - 3\rho)(1 + 3a + 3b). \end{aligned} \tag{16}$$

Consequently, from (16) one can easily bring out the following:

Curvature restrictions	Relations between σ & ρ
$((X \wedge_S Y) \cdot R)(Z,U)V = 0$	$\sigma + \rho = 0$ or $\sigma + 3\rho = 0$
$((X \wedge_S Y) \cdot C)(Z,U)V = 0$	σ, ρ are independent
$((X \wedge_S Y) \cdot \hat{C})(Z,U)V = 0$	$\sigma + \rho = 0$ or $\sigma - 3\rho = 0$
$((X \wedge_S Y) \cdot E)(Z,U)V = 0$	$\sigma + \rho = 0$
$((X \wedge_S Y) \cdot P)(Z,U)V = 0$	$\sigma + \rho = 0$ or $\sigma + 3\rho = 0$
$((X \wedge_S Y) \cdot H)(Z,U)V = 0$	$\sigma + \rho = 0$

Theorem 3.1. *The behavior of fluids in GRPFS obeying Einstein's equation is always phantom barrier for each of the restrictions* $(X \wedge_S Y) \cdot E = 0$ *and* $(X \wedge_S Y) \cdot H = 0$.

Theorem 3.2. *The behavior of fluids in GRPFS obeying Einstein's equation is either phantom barrier or quintessence barrier for each of the restrictions* $(X \wedge_S Y) \cdot R = 0$, $(X \wedge_S Y) \cdot \hat{C} = 0$ *and* $(X \wedge_S Y \cdot P) = 0$.

4. The Perfect fluid space-time satisfying $((X \wedge_T Y) \cdot \mathcal{W}) = 0$

Let us consider the perfect fluid space-time satisfying $((X \wedge_T Y) \cdot \mathcal{W}) = 0$,

$$\begin{aligned} \text{i. e., } 0 = {} & (X \wedge_T Y)\mathcal{W}(Z,U)V + \mathcal{W}((X \wedge_T Y)Z,U)V \\ & + \mathcal{W}(Z,(X \wedge_T Y)U)V + \mathcal{W}(Z,U)(X \wedge_T Y)V, \end{aligned} \tag{17}$$

where the endomorphism $(X \wedge_T Y)Z$ is defined as

$$(X \wedge_T Y)Z = T(Y,Z)X - T(X,Z)Y. \tag{18}$$

In consequence of (18), (17) becomes

$$\begin{aligned} 0 = {} & T(Y,\mathcal{W}(Z,U)V)X - T(X,\mathcal{W}(Z,U)V)Y \\ & + T(Y,Z)\mathcal{W}(X,U)V - T(X,Z)\mathcal{W}(Y,U)V \\ & + T(Y,U)\mathcal{W}(Z,X)V - T(X,U)\mathcal{W}(Z,Y)V \\ & + T(Y,V)\mathcal{W}(Z,U)X - T(X,V)\mathcal{W}(Z,U)Y. \end{aligned} \tag{19}$$

In view of (6) and (19), we have

$$\begin{aligned} 0 = {} & (\sigma + \rho)[A(\mathcal{W}(X,U)V)A(Y)A(Z) - A(\mathcal{W}(Y,U)V)A(X)A(Z) \\ & + A(\mathcal{W}(Z,X)V)A(Y)A(U) - A(\mathcal{W}(Z,Y)V)A(X)A(U) \\ & + A(\mathcal{W}(Z,U)X)A(Y)A(V) - A(\mathcal{W}(Z,U)Y)A(X)A(V)] \\ & + \rho[\bar{\mathcal{W}}(Z,U,V,Y)A(X) - \bar{\mathcal{W}}(Z,U,V,X)A(Y) \\ & + g(Y,Z)A(\mathcal{W}(X,U)V) - g(X,Z)A(\mathcal{W}(Y,U)V) \\ & + g(Y,U)A(\mathcal{W}(Z,X)V) - g(X,U)A(\mathcal{W}(Z,Y)V) \\ & + g(Y,V)A(\mathcal{W}(Z,U)X) - g(X,V)A(\mathcal{W}(Z,U)Y)]. \end{aligned} \tag{20}$$

Replacing Y and V by θ in (20), we get

$$\sigma A(\mathcal{W}(Z,U)X) + \rho g(\mathcal{W}(Z,U)\theta, X) = 0 \tag{21}$$

which on contraction gives

$$0 = -\frac{k(\sigma+3\rho)}{2}\left[\{\sigma(1-a+3b) - \rho(1-b+3a)\}\right] + k(\sigma-3\rho)[(a\sigma - b\rho) - \frac{c}{4}(\sigma-\rho)(1+3a+3b)]. \tag{22}$$

From the above one can easily bring out the following:

Curvature Restrictions	Relations between σ & ρ
$((X \wedge_T Y)\cdot R)(Z,U)V = 0$	$\sigma - \rho = 0$ or $\sigma + 3\rho = 0$
$((X \wedge_T Y)\cdot C)(Z,U)V = 0$	σ, ρ are independent
$((X \wedge_T Y)\cdot \hat{C})(Z,U)V = 0$	$\sigma - \rho = 0$ or $\sigma - 3\rho = 0$
$((X \wedge_T Y)\cdot E)(Z,U)V = 0$	$\sigma - \rho = 0$ or $\sigma + \rho = 0$
$((X \wedge_T Y)\cdot P)(Z,U)V = 0$	$\sigma + \rho = 0$ or $\sigma = 0$
$((X \wedge_T Y)\cdot H)(Z,U)V = 0$	$\sigma - \rho = 0$ or $\sigma + \rho = 0$

Theorem 4.1. *The behavior of fluids in GRPFS obeying Einstein's equation is always phantom barrier for each of the restrictions $(X \wedge_T Y)\cdot E = 0$ (for $\sigma \neq \rho$), $(X \wedge_T Y)\cdot H = 0$ for $\sigma \neq \rho$ and $(X \wedge_T Y)\cdot P = 0$ for $\sigma \neq 0$.*

Theorem 4.2. *The behavior of fluids in GRPFS obeying Einstein's equation is always quintessence barrier for each of the restrictions $(X \wedge_T Y)\cdot R = 0$ for $\sigma \neq \rho$ and $(X \wedge_S Y)\cdot \hat{C} = 0$* for *$\sigma \neq \rho$.*

5. An example of a fluid whose character is phantom barrier

Example 5.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2e^{x^1})[(dx^1)^2 + (dx^3)^2 + (dx^2)^2 - (dx^4)^2],$$

$(i,j = 1,2,3,4)$. The only non-vanishing components of the Christoffel symbols and the Ricci tensors (up to symmetry) are

$$\Gamma^1_{11} = \Gamma^1_{44} = \Gamma^2_{12} = \Gamma^3_{13} = -\Gamma^4_{14} = -\Gamma^1_{22} = -\Gamma^1_{33} = \frac{e^{x^1}}{1+2e^{x^1}},$$

$$R_{1221} = R_{1331} = -R_{1441} = \frac{e^{x^1}}{1+2e^{x^1}},$$

$$R_{2442} = R_{3443} = -R_{2332} = -\frac{e^{2x^1}}{1+2e^{x^1}},$$

$$\frac{1}{3}(1+2e^{x^1})R_{11} = R_{22} = R_{33} = -R_{44} = \frac{e^{x^1}}{1+2e^{x^1}}.$$

The scalar curvature r of the resulting space $(\mathbb{R}^4, g)$ is $r = -\frac{6e^{x^1}(1+e^{x^1})}{(1+2e^{x^1})^3}$.

Now using the above results, we may have

$$
\begin{aligned}
E_{1221} &= E_{1331} = -E_{1441} = \frac{e^{x^1}(3+e^{x^1})}{2+4e^{x^1}},\\
E_{2442} &= E_{3443} = -E_{2332} = -\frac{e^{x^1}(1+3e^{x^1})}{(2+4e^{x^1})},\\
\hat{C}_{1221} &= \hat{C}_{1331} = -\hat{C}_{1441} = \frac{e^{x^1}(3+e^{x^1})}{2+4e^{x^1}},\\
\hat{C}_{2332} &= -\hat{C}_{2442} = -\hat{C}_{3443} = \frac{e^{x^1}(1+3e^{x^1})}{2+4e^{x^1}},\\
P_{1212} &= P_{1313} = -P_{1414} = -\frac{2e^{x^1}}{1+2e^{x^1}},\\
P_{1221} &= P_{1331} = -P_{1441} = \frac{2e^{x^1}(2+e^{x^1})}{3+6e^{x^1}},\\
P_{2323} &= -P_{2424} = -P_{3434} = -\frac{e^{x^1}(1+5e^{x^1})}{3+6e^{x^1}},\\
P_{2332} &= -P_{2442} = -P_{3443} = \frac{e^{x^1}(1+5e^{x^1})}{3+6e^{x^1}},\\
H_{1221} &= H_{1331} = -H_{1441} = \frac{e^{x^1}(5+e^{x^1})}{3+6e^{x^1}},\\
H_{2332} &= -H_{2442} = -H_{3443} = \frac{e^{x^1}(5+e^{x^1})}{3+6e^{x^1}}.
\end{aligned}
$$

Assuming the associate vector field θ in the direction of x^4, we have

$$T_{11} = \rho = T_{22} = T_{33},\ T_{44} = \sigma.$$

As consequences of the above relations, we can easily bring out the following:

$$(R\circ T)_{1414} = -\frac{(\sigma+\rho)e^{x^1}}{(1+2e^{x^1})^2},$$

$$(R\circ T)_{2424} = (R\circ T)_{3434} = -\frac{(\sigma+\rho)e^{2x^1}}{(1+2e^{x^1})^2},$$

$$(E\circ T)_{1414} = -\frac{(\sigma+\rho)e^{x^1}(3+e^{x^1})}{2(1+2e^{x^1})^2},$$

$$(E\circ T)_{2424} = (E\circ T)_{3434} = -\frac{(\sigma+\rho)e^{x^1}(1+3e^{x^1})}{2(1+2e^{x^1})^2},$$

$$(\hat{C}\circ T)_{1414} = -\frac{(\sigma+\rho)e^{x^1}(3+e^{x^1})}{(1+2e^{x^1})^2},$$

$$(\hat{C}\circ T)_{2424} = (\hat{C}\circ T)_{3434} = -\frac{(\sigma+\rho)e^{x^1}(1+3e^{x^1})}{2(1+2e^{x^1})^2},$$

$$(P\circ T)_{1414} = -\frac{2e^{x^1}(\sigma+\rho)}{(1+2e^{x^1})^2},\ (P\circ T)_{1441} = \frac{2e^{x^1}(\sigma+\rho)(2+e^{x^1})}{3(1+2e^{x^1})^2},$$

$$\begin{aligned}(P\circ T)_{2424} &= -(P\circ T)_{2442} = (P\circ T)_{3434} = -(P\circ T)_{3443}\\ &= -\frac{(1+5e^{x^1})(\sigma+\rho)e^{x^1}}{3(1+2e^{x^1})^2},\end{aligned}$$

$$(H\circ T)_{1414} = -\frac{e^{x^1}(\sigma+\rho)(5+e^{x^1})}{3(1+2e^{x^1})^2},$$

$$(H\circ T)_{2424} = (H\circ T)_{3434} = -\frac{(1+5e^{x^1})(\sigma+\rho)e^{x^1}}{3(1+2e^{x^1})^2}.$$

This leads to the following

Theorem 5.2. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (1+2e^{x^1})[(dx^1)^2 + (dx^3)^2 + (dx^2)^2 - (dx^4)^2],$$

$(i, j = 1, 2, 3, 4)$. Then the behavior of fluids in general relativistic perfect fluid space obeying Einstein's equation is always phantom barrier for each of $R(X,Y)\cdot T = 0$, $E(X,Y)\cdot T = 0$, $\hat{C}(X,Y)\cdot T = 0$, $P(X,Y)\cdot T = 0$ and $H(X,Y)\cdot T = 0$.

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