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Stabilization of the Homotopy Groups of the Moduli Spaces of k-Higgs Bundles

Estabilización de los Grupos de Homotopía de los Espacios Móduli de los k-Fibrados de Higgs

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Abstract. The work of Hausel proves that the Białynicki-Birula stratification of the moduli space of rank two Higgs bundles coincides with its Shatz stratification. He uses that to estimate some homotopy groups of the moduli spaces of k-Higgs bundles of rank two. Unfortunately, those two stratifications do not coincide in general. Here, the objective is to present a different proof of the stabilization of the homotopy groups of $\mathcal{M}^k(2,d)$, and generalize it to $\mathcal{M}^k(3,d)$, the moduli spaces of k-Higgs bundles of degree d, and ranks two and three respectively, over a compact Riemann surface X, using the results from the works of Hausel and Thaddeus, among other tools.

Key words and phrases. Moduli of Higgs Bundles, Variations of Hodge Structures, Vector Bundles.

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Resumen. El trabajo de Hausel prueba que la estratificación de Białynicki-Birula del espacio moduli de fibrados de Higgs de rango dos coincide con su estratificación de Shatz. Él usa este hecho para calcular algunos grupos de homotopía del espacio moduli de k-fibrados de Higgs de rango dos. Desafortunadamente, estas dos estratificaciones no coinciden en general. Aquí, el objetivo es presentar una prueba diferente de la estabilización de los grupos de homotopía de $\mathcal{M}^k(2,d)$, y generalizarla a $\mathcal{M}^k(3,d)$, los espacios moduli de k-fibrados de Higgs de grado d, y rangos dos y tres respectivamente, sobre una superficie de Riemann compacta X, usando los resultados de los trabajos de Hausel y Thaddeus, entre otras herramientas.

Palabras y frases clave. Moduli de Fibrados de Higgs, Variaciones de Estructuras de Hodge, Fibrados Vectoriales.

1. Introduction

In this work, we estimate some homotopy groups of the moduli spaces of k-Higgs bundles $\mathcal{M}^k(r,d)$ over a compact Riemann surface X of genus $g > 2$. This space was first introduced by Hitchin [17]; and then, it was worked by Hausel [14], where he estimated some of the homotopy groups working the particular case of rank two, and denoting $\mathcal{M}^\infty = \lim\limits_{k\to\infty} \mathcal{M}^k$ as the direct limit of the sequence.

The co-prime condition $GCD(r,d) = 1$ implies that $\mathcal{M}^k(r,d)$ is smooth. We shall do the estimate with Higgs bundles of fixed determinant $\det(E) = \Lambda \in \mathcal{J}^d$, where $\mathcal{J}^d$ is the Jacobian of degree d line bundles on X, to ensure that $\mathcal{N}(r,d)$ and $\mathcal{M}(r,d)$ are simply connected. Denote $\mathcal{M}^k_\Lambda$ as the moduli space of k-Higgs bundles with determinant Λ, and $\mathcal{M}^\infty_\Lambda = \lim\limits_{k\to\infty} \mathcal{M}^k_\Lambda$ as the direct limit of these moduli spaces, as before. Hence, the group action $\pi_1(\mathcal{M}^k_\Lambda) \circlearrowleft \pi_n(\mathcal{M}^\infty_\Lambda, \mathcal{M}^k_\Lambda)$ will be trivial.

Hausel [14] estimates the homotopy groups $\pi_n(\mathcal{M}^k(2,1))$ using two main tools: first the coincidence mentioned before between the Białynicki-Birula stratification and the Shatz stratification; and second, the well-behaved embeddings $\mathcal{M}^k(2,1) \hookrightarrow \mathcal{M}^{k+1}(2,1)$. These inclusions are also well-behaved in general for $GCD(r,d) = 1$; nevertheless, those two stratifications above mentioned do not coincide in general (see for instance [11]).

In this paper, our estimate is based on the embeddings

$$\mathcal{M}^k(r,d) \hookrightarrow \mathcal{M}^{k+1}(r,d)$$

and their good behavior, notwithstanding the non-coincidence between stratifications when the rank is $r = 3$. The paper is organized as follows: in section 2 we recall some facts about vector bundles and Higgs bundles; in section 3, we present the cohomology ring $H^n(\mathcal{M}^k)$; in section 4, we discuss the most relevant results about the cohomology and the homotopy of the moduli spaces $\mathcal{M}^k$; finally, in section 5, subsection 5.1, we estimate the homotopy groups of $\mathcal{M}^k$ under the assumption that $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, and hence, in subsection 5.2, we present and prove the main result:

Theorem 1.1. (Corollary 5.14) *Suppose the rank is either $r = 2$ or $r = 3$, and $GCD(r,d) = 1$. Then, for all n exists k_0, depending on n, such that*

$$\pi_j\left(\mathcal{M}^k_\Lambda(r,d)\right) \xrightarrow{\cong} \pi_j\left(\mathcal{M}^\infty_\Lambda(r,d)\right)$$

for all $k \geqslant k_0$ and for all $j \leqslant n-1$.

2. Preliminary definitions

Let X be a compact Riemann surface of genus $g > 2$ and let $K = T^*X$ be the canonical line bundle of X. Note that, algebraically, X is also a nonsingular complex projective algebraic curve.

Definition 2.1. A *Higgs bundle* over X is a pair (E, Φ) where $E \to X$ is a holomorphic vector bundle and $\Phi\colon E \to E \otimes K$ is an endomorphism of E twisted by K, which is called a *Higgs field*. Note that $\Phi \in H^0(X; \mathrm{End}(E) \otimes K)$.

Definition 2.2. For a vector bundle $E \to X$, we denote the *rank* of E by $\mathrm{rk}(E) = r$ and the *degree* of E by $\deg(E) = d$. Then, for any smooth bundle $E \to X$, the *slope* is defined to be

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{d}{r}. \tag{1}$$

A vector bundle $E \to X$ is called *semistable* if $\mu(F) \leqslant \mu(E)$ for any F such that $0 \subsetneq F \subseteq E$. Similarly, a vector bundle $E \to X$ is called *stable* if $\mu(F) < \mu(E)$ for any nonzero proper subbundle $0 \subsetneq F \subsetneq E$. Finally, E is called *polystable* if it is the direct sum of stable subbundles, all of the same slope.

Definition 2.3. A subbundle $F \subset E$ is said to be Φ-*invariant* if $\Phi(F) \subset F \otimes K$. A Higgs bundle is said to be *semistable* [respectively, *stable*] if $\mu(F) \leqslant \mu(E)$ [resp., $\mu(F) < \mu(E)$] for any nonzero Φ-invariant subbundle $F \subseteq E$ [resp., $F \subsetneq E$]. Finally, (E, Φ) is called *polystable* if it is the direct sum of stable Φ-invariant subbundles, all of the same slope.

Fixing the rank $\mathrm{rk}(E) = r$ and the degree $\deg(E) = d$ of a Higgs bundle (E, Φ), the isomorphism classes of polystable bundles are parametrized by a quasi-projective variety: the moduli space $\mathcal{M}(r, d)$. Constructions of this space can be found in the work of Hitchin [17], using gauge theory, or in the work of Nitsure [23], using algebraic geometry methods.

An important feature of $\mathcal{M}(r, d)$ is that it carries an action of $\mathbb{C}^*$: $z \cdot (E, \Phi) = (E, z \cdot \Phi)$. According to Hitchin [17], $(\mathcal{M}, I, \Omega)$ is a Kähler manifold, where I is its complex structure and Ω its corresponding Kähler form. Furthermore, $\mathbb{C}^*$ acts on $\mathcal{M}$ biholomorphically with respect to the complex structure I by the action mentioned above, where the Kähler form Ω is invariant under the induced action $e^{i\theta} \cdot (E, \Phi) = (E, e^{i\theta} \cdot \Phi)$ of the circle $\mathbb{S}^1 \subset \mathbb{C}^*$. Besides, this circle action is Hamiltonian, with proper momentum map $f\colon \mathcal{M} \to \mathbb{R}$ defined by:

$$f(E, \Phi) = \frac{1}{2\pi} \|\Phi\|^2_{L^2} = \frac{i}{2\pi} \int_X \mathrm{tr}(\Phi\Phi^*), \tag{2}$$

where Φ^* is the adjoint of Φ with respect to the hermitian metric on E which provides the Hitchin-Kobayashi correspondence (see Hitchin [17]), and f has finitely many critical values.

There is another important fact mentioned by Hitchin (see the original version in Frankel [8], and its application to Higgs bundles in Hitchin [17]): the critical points of f are exactly the fixed points of the circle action on $\mathcal{M}$.

If $(E,\Phi)=(E,e^{i\theta}\Phi)$ then $\Phi=0$ with critical value $c_0=0$. The corresponding critical submanifold is $F_0=f^{-1}(c_0)=f^{-1}(0)=\mathcal{N}$, the moduli space of semistable bundles. On the other hand, when $\Phi\neq 0$, there is a type of algebraic structure for Higgs bundles introduced by Simpson [24]: a *variation of Hodge structure*, or simply a *VHS*, for a Higgs bundle (E,Φ) is a decomposition:

$$E=\bigoplus_{j=1}^{n}E_j \quad \text{such that} \quad \Phi\colon E_j\to E_{j+1}\otimes K \text{ for } j\leqslant n-1 \quad \text{and} \quad \Phi(E_n)=0. \tag{3}$$

It has been proved by Simpson [25] that the fixed points of the circle action on $\mathcal{M}(r,d)$, and so, the critical points of f, are these VHS, where the critical values $c_\lambda=f(E,\Phi)$ will depend on the degrees d_j of the components $E_j\subset E$. By Morse theory, we can stratify $\mathcal{M}$ in such a way that there is a nonzero critical submanifold $F_\lambda:=f^{-1}(c_\lambda)$ for each nonzero critical value $0\neq c_\lambda=f(E,\Phi)$ where (E,Φ) represents a fixed point of the circle action, or equivalently, a VHS. We then say that (E,Φ) is an $(r_1,\dots,r_n)$-VHS, where $\forall j,\ r_j=\mathrm{rk(E_j)}$.

Definition 2.4. A *holomorphic triple* on X is a triple $T=(E_1,E_2,\phi)$ consisting of two holomorphic vector bundles $E_1\to X$ and $E_2\to X$ and a homomorphism $\phi\colon E_2\to E_1$, i.e., an element $\phi\in H^0(\mathrm{Hom}(E_2,E_1))$.

There are certain notions of σ-degree:

$$\deg_\sigma(T):=\deg(E_1)+\deg(E_2)+\sigma\cdot\mathrm{rk(E_2)},$$

and σ-slope:

$$\mu_\sigma(T):=\frac{\deg_\sigma(T)}{\mathrm{rk(E_1)}+\mathrm{rk(E_2)}}$$

which give rise to notions of σ-stability of triples. The reader may consult the works of Bradlow and García-Prada [5]; Bradlow, García-Prada and Gothen [6]; and Muñoz, Ortega and Vázquez-Gallo [22] for the details.

With this notions, one can construct:

$$\mathcal{N}_\sigma=\mathcal{N}_\sigma(\mathbf{r},\mathbf{d})=\mathcal{N}_\sigma(r_1,r_2,d_1,d_2),$$

the moduli space of σ-polystable triples $T=(E_1,E_2,\phi)$ such that $\mathrm{rk(E_j)}=\mathrm{r_j}$ and $\deg(E_j)=d_j$, and

$$\mathcal{N}_\sigma^s=\mathcal{N}_\sigma^s(\mathbf{r},\mathbf{d}),$$

the moduli space of σ-stable triples, where $(\mathbf{r},\mathbf{d})=(r_1,r_2,d_1,d_2)$ is the type of the triple $T=(E_1,E_2,\phi)$.

We mention the moduli space $\mathcal{N}_\sigma(r_1,r_2,d_1,d_2)$ of σ-stable triples because they are closely related to some of the critical submanifolds F_λ.

Definition 2.5. Fix a point $p \in X$, and let $\mathcal{O}_X(p)$ be the associated line bundle to the divisor $p \in \mathrm{Sym}^1(X) = X$. A *$k$-Higgs bundle* (or *Higgs bundle with poles of order k*) is a pair (E, Φ^k) where:

$$E \xrightarrow{\Phi^k} E \otimes K \otimes \mathcal{O}_X(kp) = E \otimes K(kp)$$

and the morphism $\Phi^k \in H^0\big(X, \mathrm{End}(E) \otimes K(kp)\big)$ is what we call a *Higgs field with poles of order k*. The moduli space of k-Higgs bundles of rank r and degree d is denoted by $\mathcal{M}^k(r,d)$. For simplicity, we will suppose that $\mathrm{GCD}(r,d) = 1$, and so, $\mathcal{M}^k(r,d)$ will be smooth.

There is an embedding

$$\begin{aligned} i_k \colon \mathcal{M}^k(r,d) &\to \mathcal{M}^{k+1}(r,d) \\ \big[(E, \Phi^k)\big] &\longmapsto \big[(E, \Phi^k \otimes s_p)\big] \end{aligned}$$

where $0 \neq s_p \in H^0(X, \mathcal{O}_X(p))$ is a nonzero fixed section of $\mathcal{O}_X(p)$.

All the results mentioned for $\mathcal{M}(r,d)$ hold also for $\mathcal{M}^k(r,d)$.

3. Generators for the Cohomology Ring

According to Hausel and Thaddeus [16, (4.4)], there is a universal family $(\mathbb{E}^k, \boldsymbol{\Phi}^k)$ over $X \times \mathcal{M}^k$ where

$$\begin{cases} \mathbb{E}^k & \to \quad X \times \mathcal{M}^k(r,d) \\ \boldsymbol{\Phi}^k & \in \quad H^0\big(\mathrm{End}(\mathbb{E}^k) \otimes \pi_2^*(K(kp))\big) \end{cases}$$

and from now on, we will refer $(\mathbb{E}^k, \boldsymbol{\Phi}^k)$ as a *universal k-Higgs bundle*. Note that $(\mathbb{E}^k, \boldsymbol{\Phi}^k)$ satisfies the *Universal Property*: in general, for any family $(\mathbb{F}^k, \boldsymbol{\Psi}^k)$ over $X \times M$, there is a morphism $\eta \colon M \to \mathcal{M}^k$ such that $(\mathrm{Id}_X \times \eta)^*(\mathbb{E}^k, \boldsymbol{\Phi}^k) = (\mathbb{F}^k, \boldsymbol{\Psi}^k)$. It means that, for $M = \mathcal{M}^k$ whenever exists $(\mathbb{F}^k, \boldsymbol{\Psi}^k)$ such that

$$(\mathbb{E}^k, \boldsymbol{\Phi}^k)_P \cong (\mathbb{F}^k, \boldsymbol{\Psi}^k)_P \quad \forall P = (E, \Phi^k) \in \mathcal{M}^k(r,d),$$

then, there exists a unique bundle morphism $\xi \colon \mathbb{F}^k \to \mathbb{E}^k$ such that

$$\begin{array}{ccc} \mathbb{F}^k & \overset{\exists! \xi}{\dashrightarrow} & \mathbb{E}^k \\ & {\scriptstyle p_2} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & X \times \mathcal{M}^k(r,d) & \end{array} \tag{4}$$

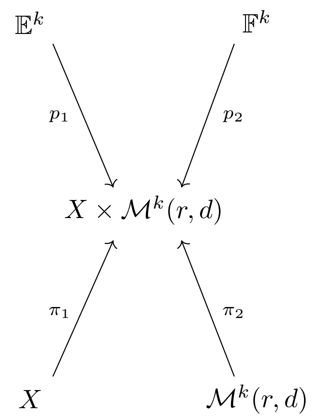
commutes: $p_2 = p_1 \circ \xi$.

The universal bundle extends then to the following: if $(\mathbb{E}^k, \mathbf{\Phi}^k)$ and $(\mathbb{F}^k, \mathbf{\Psi}^k)$ are families of stable k-Higgs bundles parametrized by $\mathcal{M}^k(r,d)$, such that $(\mathbb{E}^k, \mathbf{\Phi}^k)_P \cong (\mathbb{F}^k, \mathbf{\Psi}^k)_P$ for all $P = (E, \Phi^k) \in \mathcal{M}^k(r,d)$, then there is a line bundle $\mathcal{L} \to \mathcal{M}^k(r,d)$ such that

$$(\mathbb{E}^k, \mathbf{\Phi}^k) \cong (\mathbb{F}^k \otimes \pi_2^*(\mathcal{L}), \mathbf{\Psi}^k \otimes \pi_2),$$

where $\pi_2 \colon X \times \mathcal{M}^k(r,d) \to \mathcal{M}^k(r,d)$ is the natural projection and the endomorphisms satisfy $\mathbf{\Phi}^k \cong \mathbf{\Psi}^k \otimes \pi_2(\sigma_P) \cong \mathbf{\Psi}^k$,where σ_P is a section of $X \times \mathcal{M}^k \to \mathcal{M}^k$. For more details, see Hausel and Thaddeus [16, (4.2)].

***Remark* 3.1.** Do not confuse π_2 with p_2 (neither π_1 with p_1); π_j are the natural projections of the cartesian product, while p_j are the bundle surjective maps:



(5)

If we consider the embedding $i_k \colon \mathcal{M}^k(r,d) \to \mathcal{M}^{k+1}(r,d)$ for general rank, we get that:

Proposition 3.2. *Let* $(\mathbb{E}^k, \mathbf{\Phi}^k)$ *be a universal Higgs bundle. Then:*

$$(\mathrm{Id}_X \times i_k)^*(\mathbb{E}^{k+1}) \cong \mathbb{E}^k.$$

***Proof*.** Note that

$$\big(\mathbb{E}^k, \mathbf{\Phi}^k \otimes \pi_1^*(s_p)\big) \to X \times \mathcal{M}^k$$

is a family of $(k+1)$-Higgs bundles on X, where $\pi_1 \colon X \times \mathcal{M}^k \to X$ is the natural projection. So, by the universal property:

$$\big(\mathbb{E}^k, \mathbf{\Phi}^k \otimes \pi_1^*(s_p)\big) = (\mathrm{Id}_X \times i_k)^*\big(\mathbb{E}^{k+1}, \mathbf{\Phi}^{k+1}\big).$$

☑

Consider

$$\operatorname{Vect}(X) := \Big\{ V \to X : V \text{ is a top. vector bundle} \Big\} \Big/ \cong$$

the set of equivalence classes of topological vector bundles taken by isomorphism between them. Define the operation

$$[V] \oplus [W] : = [V \oplus W]$$

and consider the abelian semi-group $\big(\operatorname{Vect}(X), \oplus\big)$. Denote by

$$K^0(X) = K\big(\operatorname{Vect}(X)\big) := \Big\{ [V] - [W] \Big\} \Big/ \sim$$

the abelian K-group of topological vector bundles on X, where

$$[V] - [W] \sim [V \oplus U] - [W \oplus U]$$

for every topological vector bundle $U \to X$.

Let $K^1(X)$ be the odd K-group of X and let

$$K^*(X) = K^0(X) \oplus K^1(X)$$

be the K-ring described by Atiyah [2, Chapter II].

In this case, $K^*(X)$ is torsion free since the Riemann surface X is also a projective algebraic variety. Then, as a consequence of the Künneth Theorem (see for instance Atiyah [2, Corollary 2.7.15.] or [1, Main Theorem]), there is an isomorphism:

$$\begin{array}{c} \big(K^0(X) \otimes K^0(\mathcal{M}^k)\big) \oplus \big(K^1(X) \otimes K^1(\mathcal{M}^k)\big) \\ \Big\downarrow \cong \\ K^0(X \times \mathcal{M}^k) \end{array} \quad . \tag{6}$$

The reader may see Markman [21] for the details. Furthermore, Markman [21] chooses bases $\{x_1, \ ..., \ x_{2g}\} \subset K^1(X)$, and $\{x_{2g+1}, \ x_{2g+2}\} \subset K^0(X)$ to get a total basis

$$\{x_1, \ ..., \ x_{2g}, \ x_{2g+1}, \ x_{2g+2}\} \subset K^*(X) = K^0(X) \oplus K^1(X)$$

and, since there is a universal bundle $\mathbb{E}^k \to X \times \mathcal{M}^k$, we get the Künneth decomposition:

$$[\mathbb{E}^k] = \sum_{j=0}^{2g} x_j \otimes e_j^k$$

where $x_0 \in K^0(X) = \operatorname{span}\{x_{2g+1},\ x_{2g+2}\}$, $e_0^k \in K^0(\mathcal{M}^k)$, $x_j \in K^1(X)$, and $e_j^k \in K^1(\mathcal{M}^k)$ for $j = 1,\ \ldots,\ 2g$. Finally, Markman [21] considers the Chern classes $c_j(e_i^k) \in H^{2j}(\mathcal{M}^k, \mathbb{Z})$ for $e_i^k \in K^*(\mathcal{M}^k)$ and proves the following result:

Theorem 3.3 (Markman [21, Th. 3]). *The cohomology ring $H^*\big(\mathcal{M}^k(r,d), \mathbb{Z}\big)$ is generated by the Chern classes of the Künneth factors of the universal vector bundle.*

4. Preliminary Results

Let $i_k \colon \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding given by the tensorization map of the k-Higgs field $(E, \Phi^k) \longmapsto (E, \Phi^k \otimes s_p)$, where s_p is a fixed nonzero section of L_p. We want to prove that the map

$$\pi_j(i_k) : \pi_j\big(\mathcal{M}^k(r,d)\big) \to \pi_j\big(\mathcal{M}^{k+1}(r,d)\big)$$

stabilizes as $k \to \infty$. But first, we need to present some preliminary results.

Proposition 4.1. *Let $i_k \colon \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding above mentioned. Consider the K-classes $e_i^k \in K(\mathcal{M}^k)$. Then $i_k^*\big(c_j(e_i^{k+1})\big) = c_j(e_i^k)$.*

Proof. By Proposition 3.2, and by the naturality of the Chern classes:

$$\sum_{j=0}^{2g} x_j \otimes e_j^k = [\mathbb{E}^k] = [(\mathrm{Id}_X \times i_k)^*(\mathbb{E}^{k+1})] = \sum_{j=0}^{2g} x_j \otimes i_k^*(e_j^{k+1}).$$

We deduce that $i_k^*\big(e_i^{k+1}\big) = e_i^k$ and hence $i_k^*\big(c_j(e_i^{k+1})\big) = c_j(e_i^k)$. ☑

Corollary 4.2. *Let $i_k \colon \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding above mentioned. Then, the induced cohomology homomorphism $i_k^* \colon H^*(\mathcal{M}^{k+1}, \mathbb{Z}) \twoheadrightarrow H^*(\mathcal{M}^k, \mathbb{Z})$ is surjective.*

Proof. The result is an immediate consequence of Theorem 3.3 and Proposition 4.1. ☑

Definition 4.3. A *gauge transformation* is an automorphism of E. Locally, a gauge transformation $g \in Aut(E)$ is a $C^\infty(E)$-function with values in $GL_r(\mathbb{C})$. A gauge transformation g is called *unitary* if g preserves a hermitian inner product on E. We will denote $\mathcal{G}$ as the group of unitary gauge transformations. Atiyah and Bott [3] denote $\bar{\mathcal{G}}$ as the quotient of $\mathcal{G}$ by its constant central $U(1)$-subgroup. We will follow this notation too. Moreover, denote $B\mathcal{G}$ and $B\bar{\mathcal{G}}$ as the classifying spaces of $\mathcal{G}$ and $\bar{\mathcal{G}}$, respectively.

We get the fibration

$$BU(1) \to B\mathcal{G} \to B\bar{\mathcal{G}}$$

of classifying spaces, which splits actually as the product

$$B\mathcal{G} \cong BU(1) \times B\bar{\mathcal{G}}.$$

Then, the generators of $H^*(B\mathcal{G})$ give generators for $H^*(B\bar{\mathcal{G}})$ and so, $B\bar{\mathcal{G}}$ is a free graded commutative algebra on those generators, since $B\mathcal{G}$ is, and consequently, $B\bar{\mathcal{G}}$ is free of torsion. The reader may see Atiyah and Bott [3, Sec. 9.] and Hausel [14, Chap. 3] for the details.

Let $\mathcal{M}^\infty := \lim\limits_{k\to\infty} \mathcal{M}^k = \bigcup\limits_{k=0}^{\infty} \mathcal{M}^k$ be the direct limit of the spaces $\mathcal{M}^k(r,d)$. Hausel and Thaddeus [16] prove that:

Theorem 4.4 (Hausel and Thaddeus [16, (9.7)])**.** *The classifying space of $\bar{\mathcal{G}}$ is homotopically equivalent to the direct limit of the spaces $\mathcal{M}^k(r,d)$:*

$$B\bar{\mathcal{G}} \simeq \mathcal{M}^\infty = \lim_{k\to\infty} \mathcal{M}^k.$$

Assumption 4.5. Unless otherwise stated, from now on, we will assume that the rank is either $r = 2$ or $r = 3$.

Theorem 4.5. *$H^*\big(\mathcal{M}^k(r,d)\big)$ is torsion free for all k.*

Proof. The proof uses the following result of Frankel [8, Corollary 1]:

$$\forall\lambda \quad F_\lambda^k \quad \text{is torsion free} \quad \Leftrightarrow \mathcal{M}^k \quad \text{is torsion free.}$$

In fact, the result of Frankel is more general. The specific case of moduli spaces of Higgs bundles holds because the proper momentum Hitchin map $f(E,\Phi)$ described in (2) is a perfect Morse-Bott function, since we are taking $\mathrm{GCD}(r,d) = 1$.

In both cases, $r = 2$ and $r = 3$, the moduli space of stable vector bundles corresponds to the first critical submanifold: $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}$, which is indeed torsion free (see Atiyah and Bott [3, Theorem 9.9.]).

When rk(E) = 2, Hitchin notes that the nontrivial critical submanifolds, or $(1,1)$-VHS, are of the form

$$F_{d_1}^k = \left\{ (E,\Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \;\middle|\; \begin{array}{cc} \deg(E_1) = d_1, & \deg(E_2) = d_2, \\ \mathrm{rk}(E_1) = 1, & \mathrm{rk}(E_2) = 1, \\ \multicolumn{2}{c}{\varphi_{21}^k : E_1 \to E_2 \otimes K(kp)} \end{array} \right\}$$

and $F_{d_1}^k$ is isomorphic to the moduli space of σ_H-stable triples $\mathcal{N}_{\sigma_H}(1,1,\bar{d},d_1)$, where σ_H is giving by $\sigma_H = \deg\big(K(kp)\big) = 2g-2+k$ and $\bar{d} = d_2 + 2g - 2 + k$, by the map:

$$(E_1 \otimes E_2, \Phi^k) \mapsto (E_2 \otimes K(kp), E_1, \varphi_{21}^k).$$

Furthermore, by Hitchin [17], $\mathcal{N}_{\sigma_H}(1,1,\bar{d},d_1)$ is isomorphic to the cartesian product $\mathcal{J}^{d_1}(X)\times \mathrm{Sym}^{\bar{d}-d_1}(X)$. Hence:

$$F^k_{d_1}\cong \mathcal{J}^{d_1}(X)\times \mathrm{Sym}^{\bar{d}-d_1}(X)$$

which, by Macdonald [20, (12.3)], is indeed torsion free. When rk(E) = 3, there are three kinds of nontrivial critical submanifolds:

- $(1,2)$-VHS of the form

$$F^k_{d_1}=\left\{(E,\Phi^k)=(E_1\oplus E_2,\begin{pmatrix}0 & 0\\ \varphi^k_{21} & 0\end{pmatrix})\middle|\begin{array}{cc}\deg(E_1)=d_1, & \deg(E_2)=d_2,\\ \mathrm{rk(E_1)}=1, & \mathrm{rk(E_2)}=2,\\ \multicolumn{2}{c}{\varphi^k_{21}:E_1\to E_2\otimes K(kp)}\end{array}\right\}.$$

In this case, there are isomorphisms between the $(1,2)$-VHS and the moduli spaces of triples $F^k_{d_1}\cong \mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$, where $\tilde{d}_1=d_2+2(2g-2+k)$ and $\tilde{d}_2=d_1$, and where the isomorphism is given by a map similar to the above mentioned.

By Muñoz, Ortega, Vázquez-Gallo [22, Theorem 4.8. and Lemma 6.1.], when working with $\mathcal{N}_\sigma(2,1,\tilde{d}_1,\tilde{d}_2)$, they find that either the flip loci $S^+_{\sigma_c}$ is the projectivization of a bundle of rank $r^+=\tilde{d}_1-d_M-\tilde{d}_2$ over

$$\mathcal{J}^{d_M}(X)\times\mathcal{J}^{\tilde{d}_2}(X)\times\mathrm{Sym}^{r^+}(X)$$

where $d_M=\dfrac{\sigma_c+\tilde{d}_1+\tilde{d}_2}{3}\in\mathbb{Z}$, or the flip loci $S^-_{\sigma_c}$ is the projectivization of a bundle of rank $r^-=2d_M-\tilde{d}_1+g-1$ over

$$\mathcal{J}^{d_M}(X)\times\mathcal{J}^{\tilde{d}_2}(X)\times\mathrm{Sym}^{r^+}(X)$$

with $d_M\in\mathbb{Z}$ as above. Hence, by Macdonald [20, (12.3)], the flip loci $S^+_{\sigma_c}$ and $S^-_{\sigma_c}$ are free of torsion for $\sigma_c\in I$. Therefore, $\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ is also torsion free, and so is $F^k_{d_1}$.

The fact that $\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)$ is torsion free since the flip loci are, follows from the next lemma:

Lemma 4.6. *Let M be a complex manifold, and let $\Sigma\subset M$ be a complex submanifold. Let $\tilde{M}$ be the blow-up of M along Σ. Then*

$$H^*(\tilde{M},\mathbb{Z})\cong H^*(M,\mathbb{Z})\oplus H^{*+2}(\Sigma,\mathbb{Z})\oplus\cdots\oplus H^{*+2n-2}(\Sigma,\mathbb{Z})$$

where n is the rank of $N_{\Sigma/M}$, the normal bundle of Σ in M.

Proof. (Lemma 4.6)

Let $E=\mathbb{P}(N_{\Sigma/M})$ be the projectivized normal bundle of Σ in M, sometimes called *exceptional divisor*. The result follows from the fact that the additive cohomology of the blow-up $H^*(\tilde{M},\mathbb{Z})$, can be expressed as:

$$H^*(\tilde{M})\cong \pi^*H^*(M)\oplus H^*(E)/\pi^*H^*(\Sigma)$$

(see for instance Griffiths and Harris [12, Chapter 4.,Section 6.]), and the fact that $H^*(E)$ is a free module over $H^*(\Sigma)$ via the injective map $\pi^*\colon H^*(\Sigma)\to H^*(E)$ with basis

$$1,c,\dots,c^{n-1},$$

where $c\in H^2(E)$ is the first Chern class of the tautological line bundle along the fibres of the projective bundle $E\to\Sigma$ (see the general version at Husemoller [18, Chapter 17.,Theorem 2.5.]). ☑

- $(2,1)$-VHS of the form

$$F^k_{d_2}=\left\{(E,\Phi^k)=(E_2\oplus E_1,\begin{pmatrix}0&0\\ \varphi^k_{21}&0\end{pmatrix})\middle|\begin{array}{c}\deg(E_2)=d_2,\quad \deg(E_1)=d_1,\\ \mathrm{rk}(\mathrm{E}_2)=2,\qquad \mathrm{rk}(\mathrm{E}_1)=1,\\ \varphi^k_{21}:E_2\to E_1\otimes K(kp)\end{array}\right\}.$$

By symmetry, similar results can be obtained using the isomorphisms between the $(2,1)$-VHS and the moduli spaces of triples:

$$F^k_{d_2}\cong\mathcal{N}_{\sigma_H(k)}(1,2,\tilde{d}_1,\tilde{d}_2),$$

and the dual isomorphisms

$$\mathcal{N}_{\sigma_H(k)}(2,1,\tilde{d}_1,\tilde{d}_2)\cong\mathcal{N}_{\sigma_H(k)}(1,2,-\tilde{d}_2,-\tilde{d}_1)$$

between moduli spaces of triples.

- $(1,1,1)$-VHS of the form

$$F^k_{d_1d_2d_3}=\left\{(E,\Phi^k)=(E_1\oplus E_2\oplus E_3,\begin{pmatrix}0&0&0\\ \varphi^k_{21}&0&0\\ 0&\varphi^k_{32}&0\end{pmatrix})\middle|\begin{array}{c}\deg(E_j)=d_j,\\ rk(E_j)=1,\\ \varphi_{ij}:E_j\to E_i\otimes K\end{array}\right\}.$$

Finally, we know that

$$F^k_{d_1d_2d_3}\xrightarrow{\cong}\mathrm{Sym}^{m_1}(X)\times\mathrm{Sym}^{m_2}(X)\times\mathcal{J}^{d_3}(X)$$

$$(E,\Phi^k)\mapsto(\mathrm{div}(\varphi^k_{21}),\mathrm{div}(\varphi^k_{32}),E_3),$$

where $m_i=d_{i+1}-d_i+\sigma_H$, and so, by Macdonald [20, (12.3)] there is nothing to worry about torsion. ☑

Corollary 4.7. *There is an isomorphism*

$$\varprojlim H^*(\mathcal{M}^k,\mathbb{Z})\cong H^*(\mathcal{M}^\infty,\mathbb{Z})\cong H^*(B\bar{\mathcal{G}},\mathbb{Z}).$$

Corollary 4.8. *For each $n \geqslant 0$ there is a k_0, depending on n, such that*

$$i_k^* : H^j(\mathcal{M}^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{M}^k, \mathbb{Z})$$

is an isomorphism for all $k \geqslant k_0$ and for all $j \leqslant n$.

By the Universal Coefficient Theorem for Cohomology (see for instance Hatcher [13, Theorem 3.2. and Corollary 3.3.]), we get

Lemma 4.9. *For each $n \geqslant 0$ there is a k_0, depending on n, such that*

$$H_j(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$$

for all $k \geqslant k_0$ and for all $j \leqslant n$.

Proof. The embedding $i_k \colon \mathcal{M}^k(r,d) \to \mathcal{M}^{k+1}(r,d)$ is injective, and by Corollary 4.2 we know that $i_k^* \colon H^j(\mathcal{M}^k, \mathbb{Z}) \leftarrow H^j(\mathcal{M}^{k+1}, \mathbb{Z})$ is surjective for all k. Hence, by the Universal Coefficient Theorem, we get that the following diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Ext}\big(H_{j-1}(\mathcal{M}^k), \mathbb{Z}\big) & \longrightarrow & H^j(\mathcal{M}^k, \mathbb{Z}) & \longrightarrow & \operatorname{Hom}\big(H_j(\mathcal{M}^k), \mathbb{Z}\big) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle (i_{k*})^*} & & \uparrow{\scriptstyle i_k^*} & & \uparrow{\scriptstyle (i_{k*})^*} & & \\ 0 & \longrightarrow & \operatorname{Ext}\big(H_{j-1}(\mathcal{M}^{k+1}), \mathbb{Z}\big) & \longrightarrow & H^j(\mathcal{M}^{k+1}, \mathbb{Z}) & \longrightarrow & \operatorname{Hom}\big(H_j(\mathcal{M}^{k+1}), \mathbb{Z}\big) & \longrightarrow & 0 \end{array} \tag{7}$$

commutes. By Theorem 4.5 $H^*(\mathcal{M}^k, \mathbb{Z})$ is torsion free, and so, by Corollary 4.8, for all $n \geqslant 0$, there is k_0, depending on n, such that

$$H_j\big(\mathcal{M}^k(r,d), \mathbb{Z}\big) \xrightarrow{\cong} H_j\big(\mathcal{M}^{k+1}(r,d), \mathbb{Z}\big) \xrightarrow{\cong} H_j\big(\mathcal{M}^\infty(r,d), \mathbb{Z}\big)$$

for all $k \geqslant k_0$ and for all $j \leqslant n$. Hence

$$H_j(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$$

for all $k \geqslant k_0$ and for all $j \leqslant n$. ☑

Proposition 4.10. *For general rank r, denoting $\mathcal{M}^k = \mathcal{M}^k(r,d)$ for simplicity, and $\mathcal{N} = \mathcal{N}(r,d)$ as the moduli of stable bundles, the following diagram commutes*

$$\begin{array}{ccc} \pi_1(\mathcal{M}^k) & \xrightarrow{\cong} & \pi_1(\mathcal{M}^{k+1}) \\ \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\ \pi_1(\mathcal{N}) & \xrightarrow[=]{} & \pi_1(\mathcal{N}) \end{array} \tag{8}$$

Proof. It is an immediate consequence of the result proved by Bradlow, García-Prada and Gothen [7, Proposition 3.2.] using Morse theory. ☑

Proposition 4.11. *For all $k \in \mathbb{N}$, there is an isomorphism between the fundamental group of $\mathcal{M}^k$ and the fundamental group of the direct limit of the spaces $\{\mathcal{M}^k(r,d)\}_{k=0}^{\infty}$:*

$$\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^\infty).$$

Proof. Using the generalization of Van Kampen's Theorem presented by Fulton [9], and using the fact that $\mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ are embeddings of *Deformation Neighborhood Retracts* (DNR), *i.e.* every $\mathcal{M}^k(r,d)$ is the image of a map defined on some open neighborhood of itself and homotopic to the identity (see for instance Hausel and Thaddeus [16, (9.1)]), we can conclude that $\pi_1\big(\lim\limits_{k\to\infty} \mathcal{M}^k\big) = \lim\limits_{k\to\infty} \pi_1\big(\mathcal{M}^k\big)$. ☑

***Remark* 4.12.** By Atiyah and Bott [3] we have:

$$\pi_1(\mathcal{N}) \cong H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g},$$

and hence, by Proposition 4.10 and Proposition 4.11,

$$\pi_1(\mathcal{M}^k) \cong \pi_1(\mathcal{M}^\infty) \cong \mathbb{Z}^{2g}.$$

We will need the following version of Hurewicz Theorem, presented by Hatcher [13, Theorem 4.37] (see also James [19]). Hatcher first mentions that, in the relative case when (X,A) is an $(n-1)$-connected pair of path-connected spaces, the kernel of the Hurewicz map

$$h : \pi_n(X,A) \to H_n(X,A;\mathbb{Z})$$

contains the elements of the form $[\gamma][f] - [f]$ for $[\gamma] \in \pi_1(A)$. Hatcher defines $\pi'_n(X,A)$ to be the quotient group of $\pi_n(X,A)$ obtained by factoring out the subgroup generated by the elements of the form $[\gamma][f] - [f]$, or the normal subgroup generated by such elements in the case $n = 2$ when $\pi_2(X,A)$ may not be abelian, then h induces a homomorphism $h' \colon \pi'_n(X,A) \to H_n(X,A;\mathbb{Z})$. The general form of Hurewicz Theorem presented by Hatcher deals with this homomorphism.

Theorem 4.13. (Hurewicz Theorem) *If (X,A) is an $(n-1)$-connected pair of path-connected spaces, with $n \geqslant 2$ and $A \neq \varnothing$, then $h' \colon \pi'_n(X,A) \to H_n(X,A;\mathbb{Z})$ is an isomorphism and $H_j(X,A;\mathbb{Z}) = 0$ for $j \leqslant n-1$.*

Definition 4.14. The *determinant* of a vector bundle $E \to X$ of rank r is a line bundle given by the exterior power of the vector bundle. It gives a natural map of the form:

$$\begin{aligned} \det : \mathcal{N} &\longrightarrow \mathcal{J}^d \\ E \quad &\longmapsto \quad \det(E) = \bigwedge^r E \end{aligned}$$

where $\mathcal{N} = \mathcal{N}(r,d)$ is the moduli space of stable bundles $E \to X$ of rank r and degree d, and $\mathcal{J}^d$ is the Jacobian of X. Fixing a line bundle $\Lambda \to X$, $\Lambda \in \mathcal{J}^d$, the fibre $\mathcal{N}_\Lambda = \mathcal{N}_\Lambda(r,d) := \det^{-1}(\Lambda)$ is the moduli space of stable bundles *with fixed determinant.*

Together with the trace, the determinant allows us to define the map

$$\begin{aligned} \zeta : \mathcal{M}^k(r,d) &\longrightarrow \mathcal{J}^d \times H^0(X, K(kp)) \\ (E,\Phi) \quad &\longmapsto \quad \big(\det(E), \operatorname{tr}(\Phi)\big) \end{aligned}$$

and to consider the fibre $\mathcal{M}^k_\Lambda(r,d) := \zeta^{-1}(\Lambda, 0)$ which is the moduli space of k-Higgs bundles *with fixed determinant and trace zero.*

There is an important result of Atiyah and Bott [3] that is relevant to mention here:

Theorem 4.15 (Atiyah and Bott [3, (9.12)])**.** *The moduli space $\mathcal{N}_\Lambda(r,d)$ of stable bundles of fixed determinant Λ, with* $\mathrm{GCD}(r,d) = 1$*, is simply connected.*

Remark **4.16.** Some of the results mentioned for the moduli space $\mathcal{M}^k(r,d)$ in this section remain valid for the fixed determinant moduli space $\mathcal{M}^k_\Lambda(r,d)$. For instance, Theorem 4.7 holds true also for fixed determinant:

$$\mathcal{M}^\infty_\Lambda(r,d) \simeq B\bar{\mathcal{G}}$$

(see Hausel and Thaddeus [16]). Nevertheless, Corollary 4.2 does not adapt in a straightforward way, as we shall see in subsection 5.2.

The moduli space $\mathcal{M}^k_\Lambda(r,d)$ is simply connected because Proposition 4.10 holds also for fixed determinant k-Higgs bundles. So, $\pi_1(\mathcal{M}^k_\Lambda)$ acts trivially on $\pi_n(\mathcal{M}^\infty_\Lambda, \mathcal{M}^k_\Lambda)$.

5. Main Results

5.1. General Results

Here, we will concern the moduli spaces $\mathcal{M}^k(r,d)$ of k-Higgs bundles, where the results are true under the condition that $\pi_1(\mathcal{M}^k)$ acts trivially on all the higher relative homotopy groups of the pair $\big(\mathcal{M}^\infty, \mathcal{M}^k\big)$. However, we do not know if this condition is true or not.

Lemma 5.1. *If $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, then for all $n \geqslant 0$ exists k_0, depending on n, such that $\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0$ for all $k \geqslant k_0$ and for all $j \leqslant n$.*

Proof. The proof proceeds by induction on $m \in \mathbb{N}$ for $2 \leqslant m \leqslant n$. The first induction step is trivial because

$$\pi_1(\mathcal{N}) = \pi_1(\mathcal{M}) = \pi_1(\mathcal{M}^k) = \pi_1(\mathcal{M}^\infty)$$

by Proposition 4.10. For $m = 2$ we need $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ to be abelian. Consider the sequence

$$\pi_2(\mathcal{M}^\infty) \to \pi_2(\mathcal{M}^\infty, \mathcal{M}^k) \to \pi_1(\mathcal{M}^k) \to \pi_1(\mathcal{M}^\infty) \to \pi_1(\mathcal{M}^\infty, \mathcal{M}^k) \to 0$$

where $\pi_2(\mathcal{M}^\infty) \twoheadrightarrow \pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is surjective, $\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^\infty)$ are isomorphic, and hence $\pi_1(\mathcal{M}^\infty, \mathcal{M}^k) = 0$. So, $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is a quotient of the abelian group $\pi_2(\mathcal{M}^\infty)$, and so it is also abelian.

Finally, suppose that the statement is true for all $j \leqslant m-1$ for $2 \leqslant m \leqslant n$. So, $(\mathcal{M}^\infty, \mathcal{M}^k)$ is $(m-1)$-connected, *i.e.*

$$\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0 \quad \forall j \leqslant m-1.$$

For $m \geqslant 2$, by Hurewicz Theorem 4.13,

$$h' : \pi'_m(\mathcal{M}^\infty, \mathcal{M}^k) \xrightarrow{\quad\cong\quad} H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z})$$

is an isomorphism. By Lemma 4.9, there is an integer k_0, depending on m, such that $H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$ for all $k \geqslant k_0$. Hence, if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then

$$\pi_m(\mathcal{M}^\infty, \mathcal{M}^k) = \pi'_m(\mathcal{M}^\infty, \mathcal{M}^k) = 0$$

finishing the induction process. ☑

Corollary 5.2. *If $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, then for all $n \geqslant 0$ exists k_0,depending on n, such that*

$$\pi_j(\mathcal{M}^k) \xrightarrow{\quad\cong\quad} \pi_j(\mathcal{M}^\infty)$$

for all $k \geqslant k_0$ and for all $j \leqslant n-1$.

5.2. Fixed determinant case

The main goal here, is to avoid the hypothesis of the trivial action of the fundamental group on the relative homotopy group: $\pi_1(\mathcal{M}^k) \circlearrowleft \pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$. So, we want to get the analogue of Lemma 5.1 for $\mathcal{M}^k_\Lambda$, the moduli space of k-Higgs bundles with fixed determinant, since $\mathcal{M}^k_\Lambda$ is simply connected. To do that, we will need the analogue of Corollary 4.2, and then the analogue of Lemma 4.9 also for $\mathcal{M}^k_\Lambda$.

The analogue of Corollary 4.2 for $\mathcal{M}_\Lambda^k$ is not immediate. Note that the group of r-torsion points in the Jacobian:

$$\Gamma = \mathrm{Jac}(r) := \big\{ L \to X \text{ line bundle} : \ L^r \cong \mathcal{O}_X \big\}$$

acts on $\mathcal{M}_\Lambda^k(r,d)$ by tensorization:

$$(E, \Phi^k) \mapsto (E \otimes L, \Phi^k \otimes \mathrm{id}_L).$$

Hence, Γ acts on $H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})$ for all k. This cohomology splits in a Γ-invariant part and in a complement which is called by Hausel and Thaddeus [15] the "*variant part*":

$$H^*(\mathcal{M}_\Lambda^k, \mathbb{Z}) = H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^\Gamma \oplus H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^{var}. \tag{9}$$

This decomposition appears in various cohomology calculations, see e.g., Hitchin [17] for rank two, Gothen [10] for rank three, Hausel [14] also for rank two, Bento [4] for the explicit calculations for rank two and rank three, and Hausel and Thaddeus [15] for general rank.

The analogue of Corollary 4.2 for $\mathcal{M}_\Lambda^k$ will be obtained for each of the pieces in the last direct sum (9) separately:

- For $H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^\Gamma$:

It follows from the corresponding result for $H^*(\mathcal{M}^k, \mathbb{Z})$ because there is a surjection $H^*(\mathcal{M}^k, \mathbb{Z}) \twoheadrightarrow H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^\Gamma$.

Recall that, for general rank r, the moduli space of stable vector bundles corresponds to the first critical submanifold: $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}(r,d)$. The group Γ acts trivially on $H^*(\mathcal{N}, \mathbb{Z})$, and there is a surjection

$$H^*(\mathcal{N}, \mathbb{Z}) \twoheadrightarrow H^*(\mathcal{N}_\Lambda, \mathbb{Z}).$$

The reader may see Atiyah and Bott [3, Prop. 9.7.] for details.

For the rank $r = 2$ case, a nontrivial critical submanifold of $\mathcal{M}_\Lambda^k(2,1)$ is a so-called $(1,1)$-VHS:

$$F_{d_1}^k(\Lambda) = \left\{ (E, \Phi^k) = \left(E_1 \oplus E_2, \left(\begin{smallmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{smallmatrix}\right)\right) \middle| \begin{array}{c} \deg(E_j) = d_j, \quad \mathrm{rk(E_j)} = 1, \\ \varphi_{21}^k : E_1 \to E_2 \otimes K(kp), \\ E_1 E_2 = \Lambda \end{array} \right\},$$

which is a 2^{2g}-covering with covering group the 2-torsion points in the Jacobian $\Gamma \cong (\mathbb{Z}_2)^{2g}$. Hence, the results of Betti numbers presented by Bento [4, Prop. 2.2.3.] let us conclude the following:

Proposition 5.3. *The cohomology map*

$$H^*\big(\mathrm{Sym}^m(X), \mathbb{Z}\big) \to H^*\big(F_{d_1}^k(\Lambda), \mathbb{Z}\big)$$

induced by the Γ*-covering* $F_{d_1}^k(\Lambda) \to \mathrm{Sym}^m(X)$ *where* $m = d_2 - d_1 + 2g - 2 + k$, *is injective, and its image is the* Γ*-invariant subgroup* $H^*\big(F_{d_1}^k(\Lambda), \mathbb{Z}\big)^\Gamma$.

Corollary 5.4. *There exists a surjection*

$$H^*\big(\mathcal{M}^k(2,1),\mathbb{Z}\big) \twoheadrightarrow H^*\big(\mathcal{M}^k_\Lambda(2,1),\mathbb{Z}\big)^\Gamma.$$

When $r=3$, the group of 3-torsion points in the Jacobian looks like $\Gamma \cong (\mathbb{Z}_3)^{2g}$, and the nontrivial critical submanifolds of $\mathcal{M}^k_\Lambda(3,d)$ are VHS either of type $(1,2)$, $(2,1)$ or $(1,1,1)$, where the cohomology of the $(1,2)$ and $(2,1)$ VHS is invariant under the action of Γ, and the $(1,1,1)$-VHS is a 3^{2g}-covering of $\mathrm{Sym}^{m_1}(X)\times\mathrm{Sym}^{m_2}(X)$ with covering group $\Gamma\cong(\mathbb{Z}_3)^{2g}$. Hence:

Proposition 5.5. *There is an equality for cohomology rings*

$$H^*\big(F^k_{d_1}(\Lambda),\mathbb{Z}\big) = H^*\big(F^k_{d_1}(\Lambda),\mathbb{Z}\big)^\Gamma \text{ and } \quad H^*\big(F^k_{d_2}(\Lambda),\mathbb{Z}\big) = H^*\big(F^k_{d_2}(\Lambda),\mathbb{Z}\big)^\Gamma$$

here

$$F^k_{d_1}(\Lambda) = \left\{ (E,\Phi^k) = (E_1\oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi^k_{21} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_1)=d_1, \quad \deg(E_2)=d_2, \\ \mathrm{rk(E_1)}=1, \qquad \mathrm{rk(E_2)}=2, \\ \varphi^k_{21}: E_1 \to E_2\otimes K(kp) \\ E_1E_2=\Lambda \end{array} \right\}$$

and

$$F^k_{d_2}(\Lambda) = \left\{ (E,\Phi^k) = (E_2\oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi^k_{21} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_2)=d_2, \quad \deg(E_1)=d_1, \\ \mathrm{rk(E_2)}=2, \qquad \mathrm{rk(E_1)}=1, \\ \varphi^k_{21}: E_2 \to E_1\otimes K(kp) \\ E_2E_1=\Lambda \end{array} \right\}$$

are the $(1,2)$ *and* $(2,1)$*-VHS of* $\mathcal{M}^k_\Lambda(3,d)$ *respectively, with*

$$\frac{d}{3} \leqslant d_1 \leqslant \frac{d}{3} + \frac{2g-2+k}{2} \text{ and } \quad \frac{2d}{3} \leqslant d_2 \leqslant \frac{2d}{3} + \frac{2g-2+k}{2}.$$

Furthermore:

$$H^*\big(F^k_{m_1m_2}(\Lambda),\mathbb{Z}\big) = H^*\big(F^k_{m_1m_2}(\Lambda),\mathbb{Z}\big)^\Gamma \oplus H^*\big(F^k_{m_1m_2}(\Lambda),\mathbb{Z}\big)^{var}$$

and the cohomology map

$$H^*\big(\mathrm{Sym}^{m_1}(X)\times\mathrm{Sym}^{m_2}(X),\mathbb{Z}\big) \to H^*\big(F^k_{m_1m_2}(\Lambda),\mathbb{Z}\big)$$

induced by the Γ*-covering* $F^k_{m_1m_2}(\Lambda) \to \mathrm{Sym}^{m_1}(X)\times\mathrm{Sym}^{m_2}(X)$ *where*

$$F^k_{m_1m_2}(\Lambda) = \left\{ (E,\Phi^k) = (E_1\oplus E_2\oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi^k_{21} & 0 & 0 \\ 0 & \varphi^k_{32} & 0 \end{pmatrix}) \middle| \begin{array}{c} \deg(E_j)=d_j,\ rk(E_j)=1, \\ \varphi_{ij}: E_j \to E_i\otimes K(kp) \\ E_1E_2E_3=\Lambda \end{array} \right\}$$

is the $(1,1,1)$*-VHS of* $\mathcal{M}^k_\Lambda(3,d)$ *with* $m_j = d_{j+1}-d_j+2g-2+k$*, is injective, and its image is the* Γ*-invariant subgroup* $H^*\big(F^k_{m_1m_2}(\Lambda)\big)^\Gamma$.

Corollary 5.6. *There exists a surjection*

$$H^*\big(\mathcal{M}^k(3,d),\mathbb{Z}\big) \twoheadrightarrow H^*\big(\mathcal{M}^k_\Lambda(3,d),\mathbb{Z}\big)^\Gamma.$$

The reader may see Bento [4], Gothen [10] and also Hausel and Thaddeus [15] for details. Using the results above, we get:

Lemma 5.7. *The induced cohomology homomorphism restricted to the* Γ-*invariant cohomology of the moduli spaces of k-Higgs bundles with fixed determinant* Λ

$$i_k^*\colon H^*(\mathcal{M}^{k+1}_\Lambda(r,d),\mathbb{Z})^\Gamma \twoheadrightarrow H^*(\mathcal{M}^k_\Lambda(r,d),\mathbb{Z})^\Gamma$$

is surjective.

Proof. It is enough to note that the following diagram

$$\begin{array}{ccc} H^*(\mathcal{M}^{k+1},\mathbb{Z}) & \xrightarrow{\;i_k^*\;} & H^*(\mathcal{M}^{k},\mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ H^*(\mathcal{M}^{k+1}_\Lambda,\mathbb{Z})^\Gamma & \underset{i_k^*}{\dashrightarrow} & H^*(\mathcal{M}^{k}_\Lambda,\mathbb{Z})^\Gamma \end{array} \tag{10}$$

commutes, where the top arrow is surjective by Corollary 4.2, and the descending arrows are surjective because of Corollary 5.4 and Corollary 5.6. ☑

- For $H^*(\mathcal{M}_\Lambda,\mathbb{Z})^{var}$:

First, note that with fixed determinant Λ the critical submanifolds of type $(1,1)$ and $(1,1,1)$ are r^{2g}-coverings with covering group $\Gamma \cong (\mathbb{Z}_r)^{2g}$, with $r=2$ or $r=3$ (see Bento [4] Prop. 2.2.1. and Lemma 2.4.4.). Furthermore, when $r=3$ the cohomology of $(1,2)$ and $(2,1)$ critical submanifolds is Γ-invariant. Then, only the cohomology of $(1,1)$-VHS and $(1,1,1)$-VHS split in the Γ-invariant part and the *variant* complement, for rank $r=2$ and $r=3$, respectively. Hence:

$$H^*\big(\mathcal{M}^k_\Lambda(2,1),\mathbb{Z}\big)^{var} = \bigoplus_{d_1>\frac{1}{2}}^{\frac{1+d_k}{2}} H^*\big(F^k_{d_1}(\Lambda)\big)^{var} \text{ and}$$

$$H^*\big(\mathcal{M}^k_\Lambda(3,d),\mathbb{Z}\big)^{var} = \bigoplus_{(m_1,m_2)\in\Omega_{d_k}} H^*\big(F^k_{m_1m_2}(\Lambda),\mathbb{Z}\big)^{var}$$

where $d_k = \deg\big(K \otimes \mathcal{O}_X(kp)\big) = \deg\big(K(kp)\big) = 2g-2+k$, $\frac{1}{2} < d_1 < \frac{1+d_k}{2}$ according to Hitchin [17] for $(1,1)$-VHS in rank two, and $(m_1, m_2) \in \Omega_{d_k}$ where $M_j := E_j^* E_{j+1} K(kp)$, $m_j := \deg(M_j) = d_{j+1} - d_j + d_k$, and the set of indexes

$$\Omega_{d_k} = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \middle| \begin{array}{c} 2m_1 + m_2 < 3d_k \\ m_1 + 2m_2 < 3d_k \\ m_1 + 2m_2 \equiv d(3) \end{array} \right\}$$

for $(1,1,1)$-VHS in rank three is described by Bento [4, Prop. 2.3.9.], Gothen [10, Sec. 3.], Gothen and Zúñiga-Rojas [11, Subsec. 5.1], among others.

There are some results appearing in the work of Bento [4] (Lemma 2.2.4. and Prop. 2.2.5 for $\mathcal{M}_\Lambda^k(2,1)$ and $F_{d_1}^k(\Lambda)$ its $(1,1)$-VHS, and Lemma 2.4.4. and Prop. 2.4.5. for $\mathcal{M}_\Lambda^k(3,d)$ and $F_{m_1 m_2}^k(\Lambda)$ its $(1,1,1)$-VHS) where Bento works with Hitchin pairs twisted by a general line bundle L of degree $\deg(L) = d_L$, and the following results below correspond to the particular case of k-Higgs bundles with $L = K(kp)$, and hence $d_L = d_k = 2g-2+k$.

Lemma 5.8. *Let $F_{d_1}^k(\Lambda)$ be a $(1,1)$-VHS of $\mathcal{M}_\Lambda^k(2,1)$ and let $m = d_2 - d_1 + 2g - 2 + k$. Then*

$$H^j(F_{d_1}^k(\Lambda), \mathbb{Z})^{var} \neq 0 \iff j = m.$$

Proof. See Bento [4, Prop. 2.2.4.]. ☑

Lemma 5.9. *Let $F_{m_1 m_2}^k(\Lambda)$ be a $(1,1,1)$-VHS of $\mathcal{M}_\Lambda^k(3,d)$. Then*

$$H^i\big(F_{m_1 m_2}^k(\Lambda), \mathbb{Z}\big)^{var} \neq 0 \iff i = m_1 + m_2,$$

where $m_j = d_{j+1} - d_j + d_k$.

Proof. See Bento [4, Prop. 2.4.4.]. ☑

Then, in both cases, when $r = 2$ and when $r = 3$, the cohomology groups with integer coefficients are torsion free:

- If $r = 2$, we have just one nonzero component, $H^m(F_{d_1}^k(\Lambda), \mathbb{Z})^{var}$ which is the sum of 2^{2g} copies of $H^m(\mathrm{Sym}^m(X), \mathbb{Z})$, since $F_{d_1}^k(\Lambda) \to \mathrm{Sym}^m(X)$ is a $(\mathbb{Z}_2)^{2g}$-covering.
- Similarly, if $r = 3$, the nonzero component is $H^{m_1+m_2}\big(F_{m_1 m_2}^k(\Lambda), \mathbb{Z}\big)^{var}$ which is the sum of 3^{2g} copies of $H^{m_1+m_2}\big(\mathrm{Sym}^{m_1}(X) \times \mathrm{Sym}^{m_2}(X), \mathbb{Z}\big)$, since
$$F_{m_1 m_2}^k(\Lambda) \to \mathrm{Sym}^{m_1}(X) \times \mathrm{Sym}^{m_2}(X)$$
is a $(\mathbb{Z}_3)^{2g}$-covering.

They are torsion free by Macdonald [20, (12.3)]. The reader may consult Bento [4, Chap. 2] for the details. Hence, we get

Lemma 5.10. *Let $i_k\colon \mathcal{M}_\Lambda^k \hookrightarrow \mathcal{M}_\Lambda^{k+1}$ be the embedding given by the tensorization map $(E, \Phi^k) \mapsto (E, \Phi^k \otimes s_p)$ as above mentioned. Then, the induced cohomology homomorphism*

$$i_k^*\colon H^*(\mathcal{M}_\Lambda^{k+1}, \mathbb{Z})^{var} \twoheadrightarrow H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^{var}$$

is surjective, restricted to the variant complement.

This latter method only works with rank $r = 2$ or $r = 3$, but not in general. The difficulty in calculating $H^*(\mathcal{M}_\Lambda, \mathbb{Z})^{var}$ for general rank is explained also in Hausel and Thaddeus [15].

Finally, we may conclude the following:

Corollary 5.11. *Let $i_k\colon \mathcal{M}_\Lambda^k \hookrightarrow \mathcal{M}_\Lambda^{k+1}$ be the embedding above mentioned. Then, the induced cohomology homomorphism*

$$i_k^*\colon H^*(\mathcal{M}_\Lambda^{k+1}, \mathbb{Z}) \twoheadrightarrow H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})$$

is surjective.

Proof. It is enough to see that the cohomology of $\mathcal{M}_\Lambda^k$ splits in the Γ-invariant part and the *variant* complement:

$$H^*(\mathcal{M}_\Lambda^k, \mathbb{Z}) = H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^\Gamma \oplus H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^{var}$$

and so, the result follows from Lemma 5.7 and Lemma 5.10. ☑

Lemma 5.12. *For all n exists k_0, depending on n, such that*

$$H_j(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k; \mathbb{Z}) = 0$$

for all $k \geqslant k_0$ and for all $j \leqslant n$.

Theorem 5.13. *For all n exists k_0, depending on n, such that*

$$\pi_j(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k) = 0$$

for all $k \geqslant k_0$ and for all $j \leqslant n$.

Proof. The proof is quite similar to the proof of Lemma 5.1, using now Corollary 5.11 and Lemma 5.12, and so, we have a new advantage: $\mathcal{M}_\Lambda^k$ is simply connected, hence the action $\pi_1(\mathcal{M}_\Lambda^k) \circlearrowleft \pi_n(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k)$ is trivial. ☑

Corollary 5.14. *For all n there exists k_0, depending on n, such that*

$$\pi_j(\mathcal{M}_\Lambda^k) \xrightarrow{\cong} \pi_j(\mathcal{M}_\Lambda^\infty)$$

for all $k \geqslant k_0$ and for all $j \leqslant n-1$.

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